A tight race between deterministic and stochastic dynamics of RPS-model

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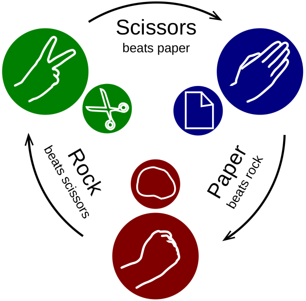
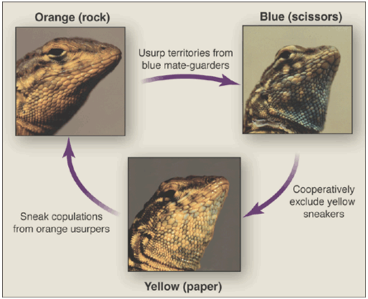
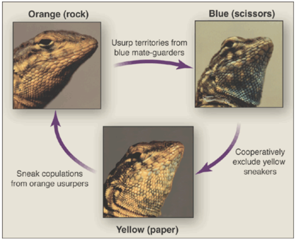
Overview

Cyclic Dominance (Rock-Paper-Scissors)–

1. widely exists in nature, eg. Biology, Chemistry
2. describes the interactions between species







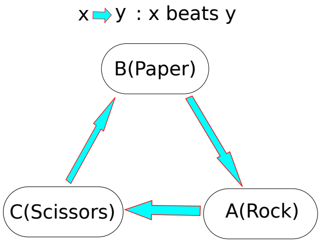
Analysis Method –

1. Deterministic: Continuous and infinite.
2. Stochastic: Discrete and finite

Main work –

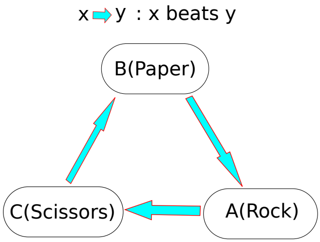
1. Agreement and disagreement of RPS-model with the two methods.
2. Noise slows down the evolution of cyclic dominance.

- RPS simplest model



2.1 Rock-Paper-Scissors Game

- RPS simplest model



- Payoff matrix

$$P = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \end{array}$$

2.1 Rock-Paper-Scissors Game

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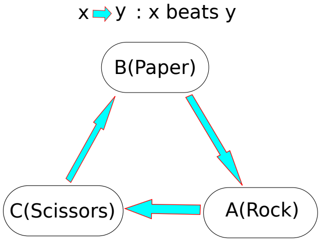

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- Differential equations for deterministic study

$$\dot{a} = a(c - b),$$
$$\dot{b} = b(a - c),$$
$$\dot{c} = c(b - a),$$

and $a(t) + b(t) + c(t) = 1$.

2.1 Rock-Paper-Scissors Game

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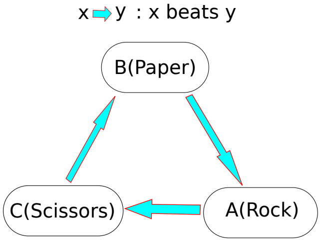

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- Differential equations for deterministic study

$$\dot{a} = a(c - b),$$
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and $a(t) + b(t) + c(t) = 1$.

- Chemical reactions for stochastic simulation

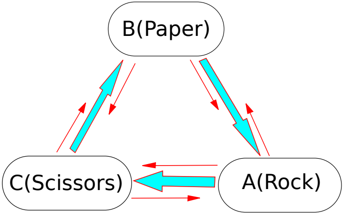
$$A + B \xrightarrow{1} B + B,$$
$$B + C \xrightarrow{1} C + C,$$
$$C + A \xrightarrow{1} A + A,$$

and $A + B + C = N$, N is fixed

Update the simplest RPS-model

- RPS-model with mutation and unbalanced payoff

x ⇨ y : x beats y
x → y : x mutates into y



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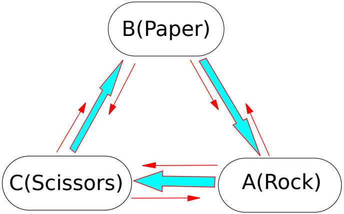

- Payoff matrix

$$P = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 0 & -\beta-1 & 1 \\ 1 & 0 & -\beta-1 \\ -\beta-1 & 1 & 0 \end{pmatrix} \end{array} + \text{Mutations}$$

Update the simplest RPS-model

- Ordinary Differential Equations

$$\dot{a} = a[c - (1+\beta)b + \beta(ab + bc + ac)] + \mu(b + c - 2a),$$
$$\dot{b} = b[a - (1+\beta)c + \beta(ab + bc + ac)] + \mu(c + a - 2b),$$
$$\dot{c} = c[b - (1+\beta)a + \beta(ab + bc + ac)] + \mu(a + b - 2c).$$

with $\beta > 0$ and $\mu > 0$, μ is mutation rate.

Update the simplest RPS-model

- Ordinary Differential Equations

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with $\beta > 0$ and $\mu > 0$, μ is mutation rate.

- Chemical reactions for stochastic simulations

$$A + B \xrightarrow{1} B + B, \quad B + C \xrightarrow{1} C + C, \quad C + A \xrightarrow{1} A + A,$$
$$A + B + B \xrightarrow{\beta} B + B + B, \quad A \xrightarrow{\mu} B, \quad A \xrightarrow{\mu} C,$$
$$A + A + C \xrightarrow{\beta} A + A + A, \quad B \xrightarrow{\mu} C, \quad B \xrightarrow{\mu} A,$$
$$B + C + C \xrightarrow{\beta} C + C + C, \quad C \xrightarrow{\mu} A, \quad C \xrightarrow{\mu} B.$$

Jacobian Matrix of ODEs

1. The only interior equilibrium:

$$x^* = (a^*, b^*, c^*) = (1/3, 1/3, 1/3)$$

2. Jacobian matrix around the equilibrium

$$J^* = \begin{pmatrix} -\frac{1}{3} - 3\mu & -\frac{2}{3} - \frac{1}{3}\beta \\ \frac{2}{3} + \frac{1}{3}\beta & \frac{1}{3} + \frac{1}{3}\beta - 3\mu \end{pmatrix}$$

3. The critical value of μ:

$$\mu_c = \frac{\beta}{18}.$$

4. When $\mu < \mu_c$, Hopf bifurcation happens.

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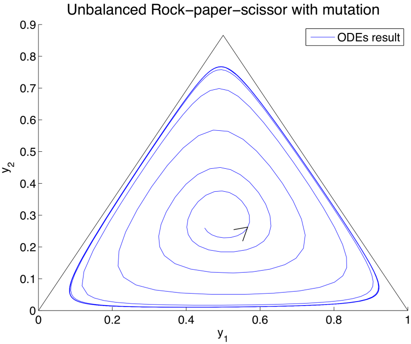
Formation of a robust cycle - limit cycle

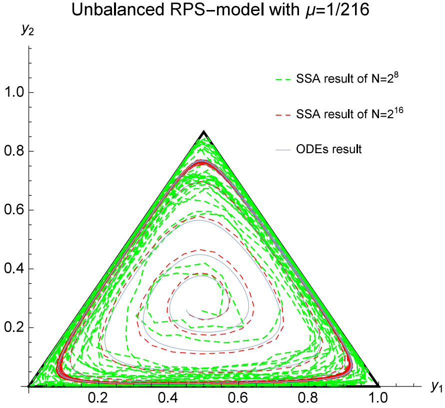
- Figure: $\beta = \frac{1}{2}$, $\mu = \frac{1}{216} < \mu_c = \frac{1}{36}$, $y_1 = a + \frac{1}{2}b$, $y_2 = \frac{\sqrt{3}}{2}b$.



- The coordinates transformation makes the flow spiraling outwards visible.

Comparison with numerical solution to ODEs

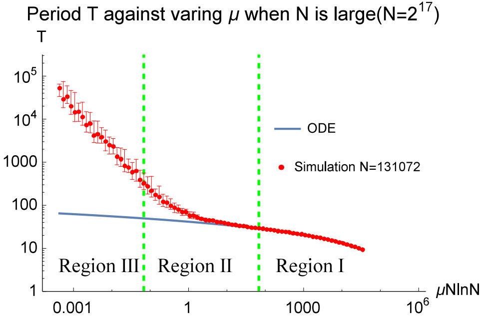
- Figure: $\beta = \frac{1}{2}$, $\mu = \frac{1}{216}$. N is total of individuals in simulation.



- Behave differently. SSA - Stochastic Simulation Algorithm.
- Size N controls the level of randomness.

Average period of these quasi-periodic cycles

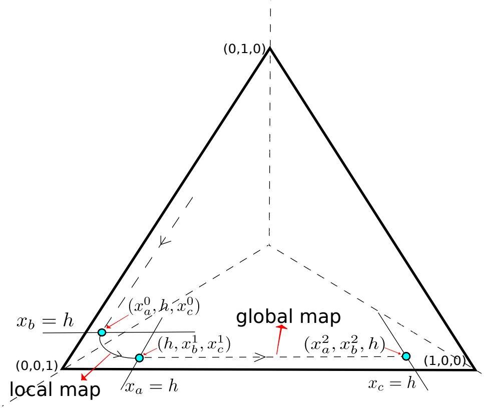
- An interesting comparison:



- **There is a RACE between stochastic dynamic and deterministic dynamic!**

Determinsitic dynamic governs the game

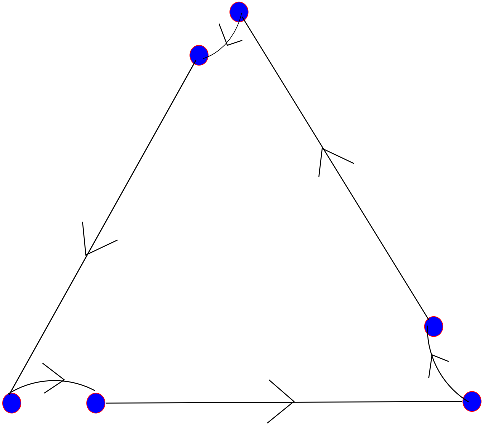
- Compose local and global map:



- $T_{\mathrm{ODE}} \propto -\ln \mu$ is proved theoretically.

Stochastic dynamic prevails the opponent

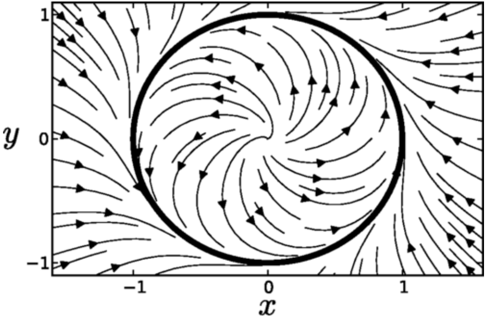
- The cycle in this region looks like:



- It is a 1-D death-birth process.
- When $\mu \ll \frac{1}{N \ln N}$, $T_{\mathrm{SSA}} \propto \frac{1}{N\mu}$.

A tight race between the two dynamics

- Explanation: the idea of asymptotic phase (the picture, cited from J. M. Newby) and SDE.



- When $\mu \sim \frac{1}{N \ln N}$, $T_{\mathsf{SDE}} \approx -3 \ln \mu + \mathcal{C} \frac{3}{N\mu}$.

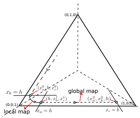
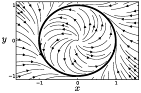
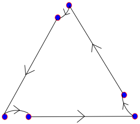
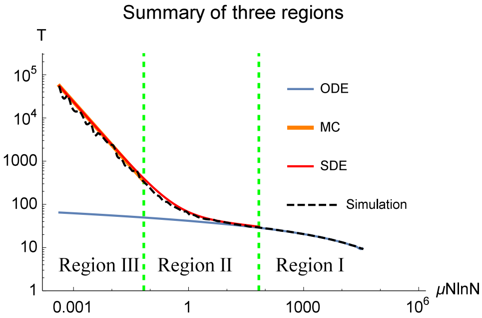
Region	Winner	Methods	Figure
Region I $\frac{1}{N \ln N} \ll \mu < \mu_c$	Deterministic $T_{\text{ODE}} \propto -\ln \mu$	composition of local and global maps	
Region II $\mu \sim \frac{1}{N \ln N}$	Even $T_{\text{SDE}} \approx -3 \ln \mu + \mathcal{C} \frac{3}{N\mu}$	asymptotic phase and SDE	
Region III $0 < \mu \ll \frac{1}{N \ln N}$	Stochastic $T_{\text{SSA}} \propto \frac{1}{N\mu}$	Markov Chain	

Table: Different result of the race in different region.

A tight race between the two dynamics

- Summarise three theoretical analysis, and draw the following figure:



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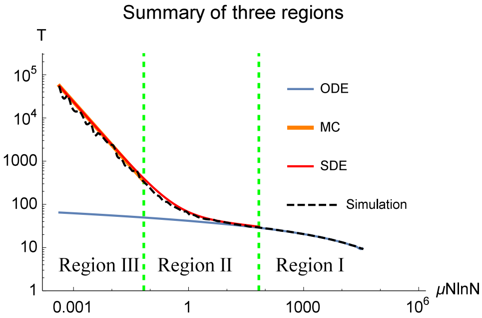

- Finally, the result of the match is revealed. Thank you for your attention!