Observing emerging bifurcations in complex systems

Jan Sieber

University of Exeter (UK)

Outline

- two didactic examples
  - agent-based simulation — traders
  - disease spreading on network
- definition of macroscopic unstable equilibria and bifurcations
- folds & Hopf bifurcations

Agent-based simulation — traders

[Siettos *et al*, EPL 2012]

- N traders, buying & selling
- each trader k has internal state s_k, evolving

$$s_{k,\mathrm{new}} = \begin{cases} \mathrm{e}^{-\gamma\Delta t} s_k + p_k^+ \varepsilon^+ - p_k^- \varepsilon^- & \text{if } |s_k| < 1 \\ 0 \quad \text{reset} & \text{if } s_k \geq \ \ 1 \Leftarrow \text{buy} \\ 0 \quad \text{reset} & \text{if } s_k \leq -1 \Leftarrow \text{sell} \end{cases}$$

- $p_k^\pm$ random number of news $\sim \mathrm{Pois}(n_\pm(1+gR_\pm))\Delta t$
- $R_\pm$ avg rate of buys/sells over past period T
- g gain
- $\varepsilon^\pm$ jump size
- $n_\pm$ rate of good/bad news other than buys/sells

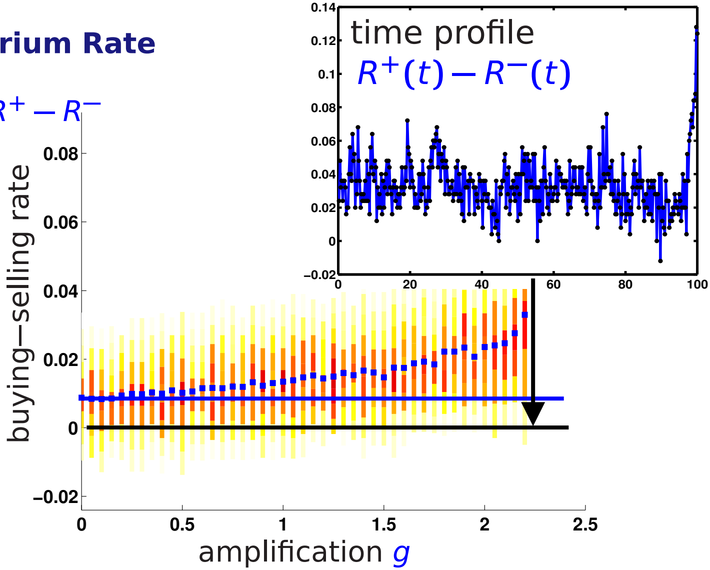
Agent-based simulation — traders

⇒Matlab animation

Example: collective behaviour of agents

Equilibrium Rate

$R = R^{+} - R^{-}$



Definition of equilibrium?

- positive feedback
  - others buying ⇒good news ⇒buy
  - others selling ⇒bad news ⇒sell
- stochastic system has stationary density with (in projection) 3 well-separated local maxima

  stable equilibria:
  - everyone buys as fast as possible
  - everyone sells as fast as possible
  - balance
- Balance $R_{eq} = R^{+}_{eq} - R^{+}_{eq}$: in the long run $\text{mean}\, R = R_{eq}$:

$$R_{eq} = \lim_{t \to \infty} \text{mean}_{s \in [t, t+T]} R(s)$$

(mean of conditional stationary density)

UNIVERSITY OF EXETER

Proposed definition of (unstable) equilibrium

Include feedback loop:

$$\text{bias } p = [\varepsilon^+ - \varepsilon^-](t) = k[R(t) - R_{\text{ref}}]$$

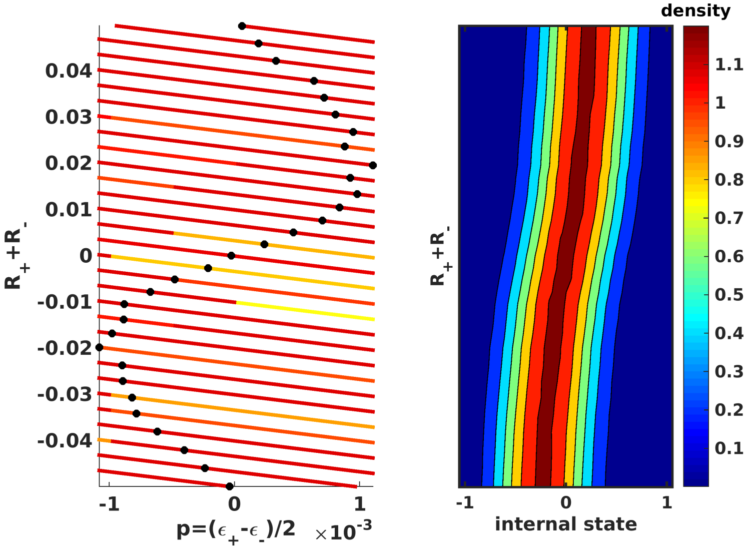
R_{ref} is equilibrium if

$$R_{\text{ref}} = \lim_{t\to\infty} \text{mean}_{s\in[t,t+T]} R(s)$$

- The long-time mean of feedback loop input is zero.
- For large numbers N of traders:
  - $E[R(t) - R_{\text{ref}}]^2 \rightarrow_{N\to\infty,t\to\infty} 0$
  - Resulting equilibrium R_{ref} independent of choice of feedback loop for $N \to \infty$.

UNIVERSITY OF EXETER

Bifurcation diagram in bias parameter



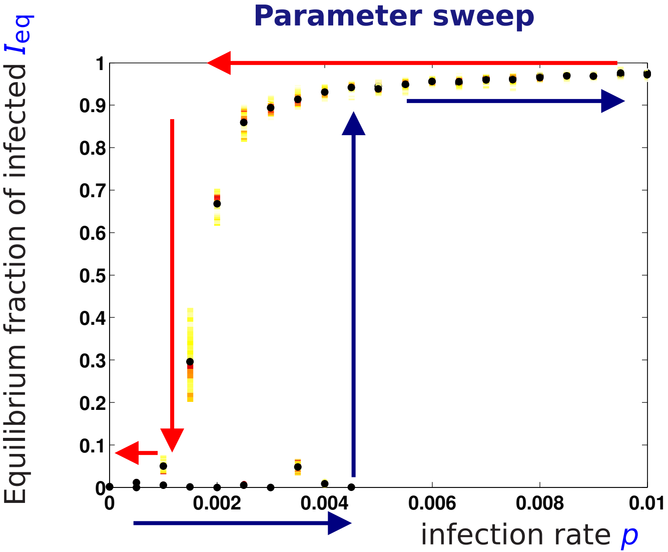
2nd example — disease spreading on network

[Gross *et al*, PRL 2006]

- network with N nodes (individuals) with state either S (susceptible) or I (infected)
- kN links (initially random, $k \sim 10$)
- at every step:
  - I individual recovers with probability r
  - infection travels along SI link (infects S node) with probability p
  - SI link is rewired (keep S node, replace I node by random other S node) with probability w
- system has parameter range where disease-free and endemic equilibrium coexist

2nd example — disease spreading on network

[Gross *et al*, PRL 2006]



Proposed definition of (unstable) equilibrium

- Choose reference fraction of infected I_{ref}
- at every step:

  if $I < I_{\mathrm{ref}}$, infect $I_{\mathrm{ref}} - I$ individuals along SI links
  if $I > I_{\mathrm{ref}}$, "cure" $I - I_{\mathrm{ref}}$ individuals

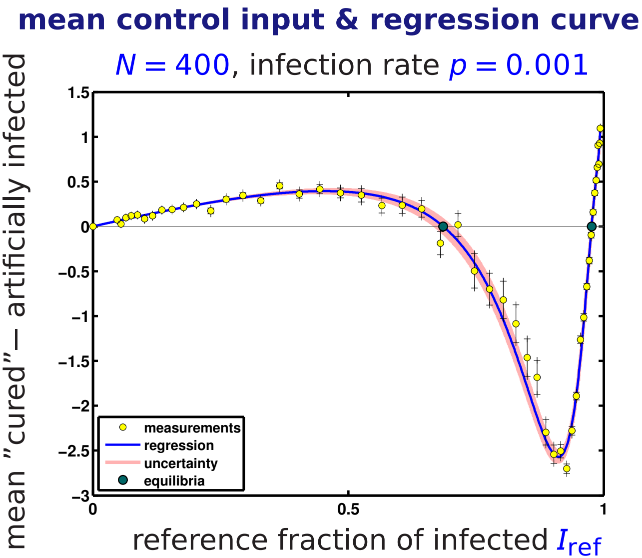
- I_{ref} is equilibrium value if

  mean artificically cured = mean artificially infected

  after transients have settled

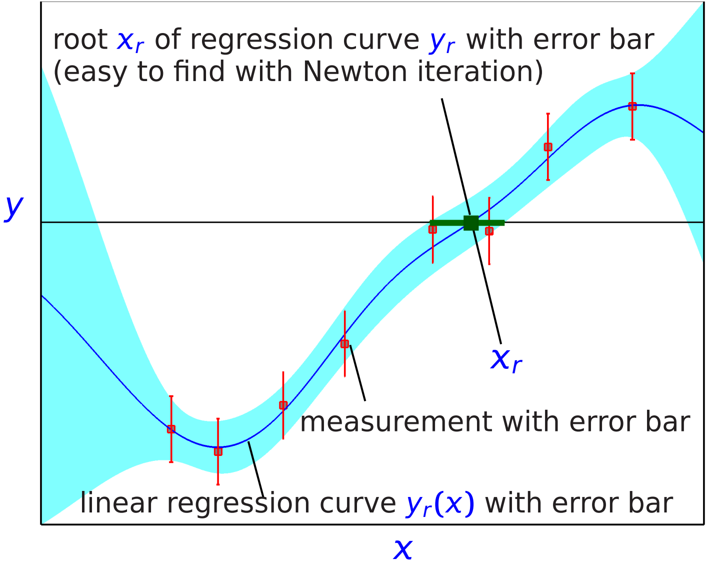
UNIVERSITY OF EXETER

Proposed definition of (unstable) equilibrium



Newton iteration & continuation with uncertainty

Linear regression with Gaussian process



Procedure for continuation with uncertainty

1. find roots (bifurcations) of regression curve $y_r(x)$
2. determine where to measure next:
   - x for which measurement $y_r(x) \pm \sigma_r(x)$ would minimize error bar of root for updated y_r, or
   - x where measurement $y_r(x) + \sigma_r(x)$ changes root the most

   Both are nonlinear optimization problems on current regression curve y_r (cheap in principle).
3. optimal new x not necessary, only sensible x
4. stop if expected effect on x is not worth additional measurement.

Example – traders fold continuation

2 system parameters:

- bias $\varepsilon^+ - \varepsilon^-$ (also control input)
- self-referentialness g

⇒ 2 base variables: R_{ref}, g.

Run with feedback: bias $= [\varepsilon^+ - \varepsilon^-](t) = k[R(t) - R_{\mathrm{ref}}]$

after transients, read off

- $\mathrm{mean}[\varepsilon^+ - \varepsilon^-]$,
- $R_{\mathrm{eq}} = \mathrm{mean}\, R$

⇒equilibrium surface in space $(g, \varepsilon^+ - \varepsilon^-, R_{\mathrm{eq}})$

fold condition: $$\frac{\partial R_{\mathrm{eq}}}{\partial R_{\mathrm{ref}}}(R_{\mathrm{ref}}, g) = 1$$

UNIVERSITY OF EXETER

Matlab demo

Example – traders fold continuation

Comments

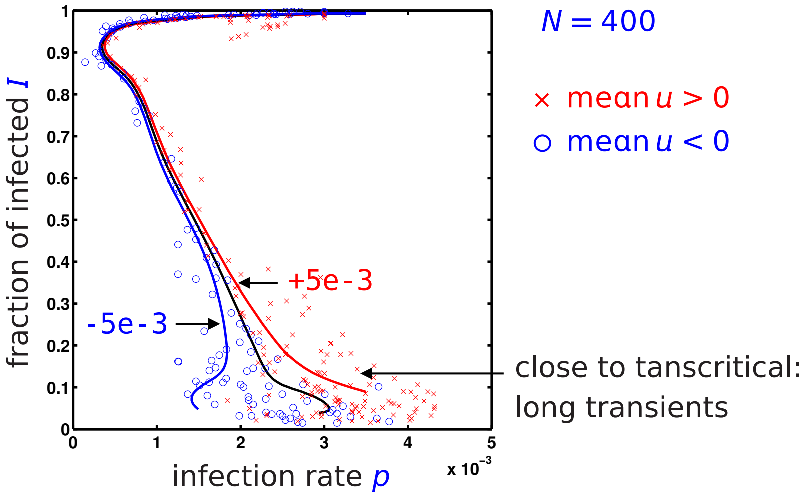
- during continuation regression surface always evaluated near boundary

  $\Rightarrow$ results less accurate (larger uncertainty)
  $\Rightarrow$ large correlation parameter

- at end standard continuation for entire regression surface

  $\Rightarrow$ more accurate (interpolation)
  $\Rightarrow$ small correlation parameter

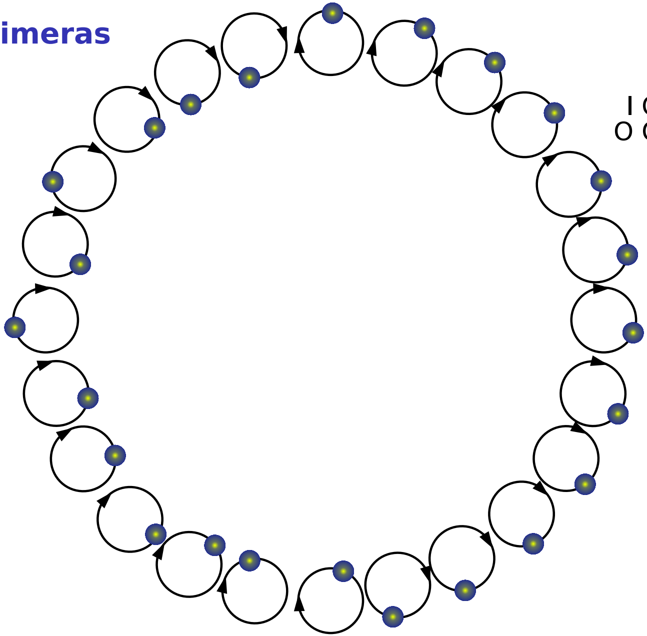
Disease on network equilibrium continuation



Oscillations

Ring of nonlocally coupled phase oscillators

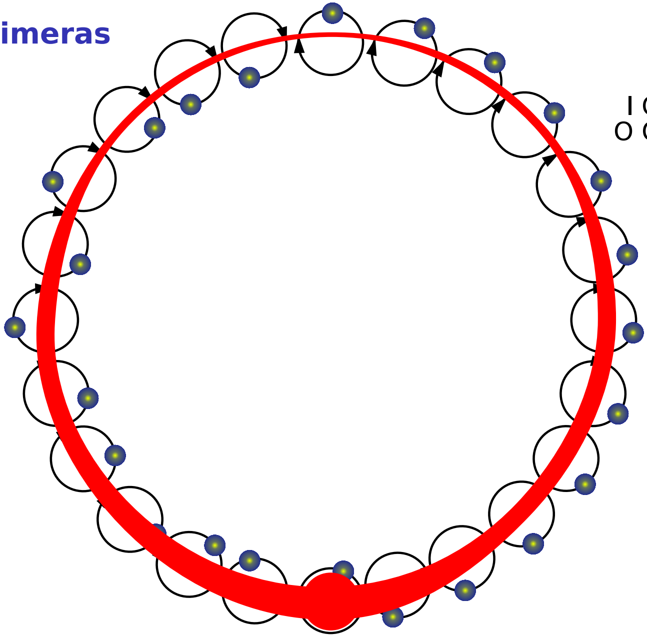
Chimeras



Abrams
Strogatz
Laing
I Omelchenko
O Omelchenko
Zakharova
Wolfrum
Schoell
...

UNIVERSITY OF EXETER

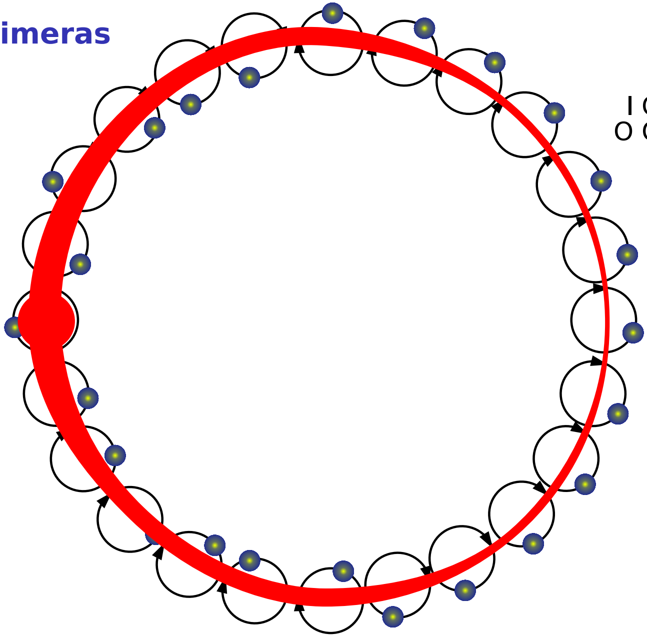
Chimeras



Abrams
Strogatz
Laing
I Omelchenko
O Omelchenko
Zakharova
Wolfrum
Schoell
...

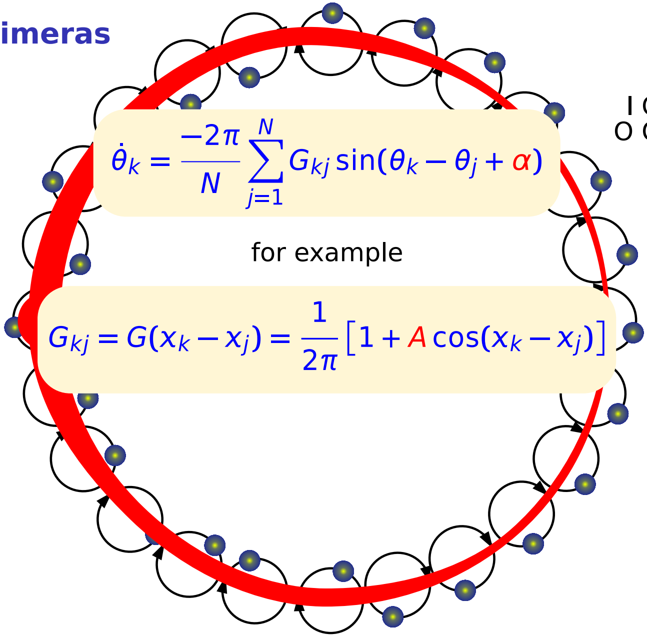
University of Exeter

Chimeras



Abrams
Strogatz
Laing
I Omelchenko
O Omelchenko
Zakharova
Wolfrum
Schoell
...

Chimeras



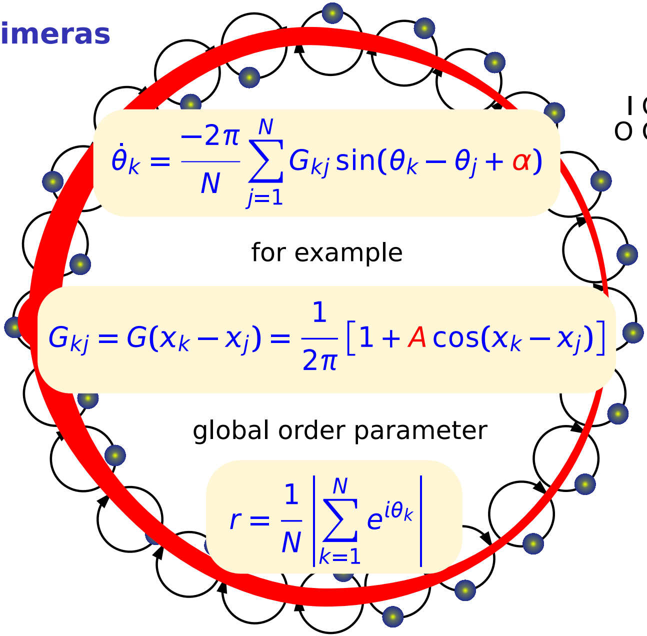
$$\dot{\theta}_k = \frac{-2\pi}{N} \sum_{j=1}^{N} G_{kj} \sin(\theta_k - \theta_j + \alpha)$$

for example

$$G_{kj} = G(x_k - x_j) = \frac{1}{2\pi}\left[1 + A\cos(x_k - x_j)\right]$$

Abrams
Strogatz
Laing
I Omelchenko
O Omelchenko
Zakharova
Wolfrum
Schoell
...

Chimeras



$$\dot{\theta}_k = \frac{-2\pi}{N}\sum_{j=1}^{N} G_{kj}\sin(\theta_k - \theta_j + \alpha)$$

for example

$$G_{kj} = G(x_k - x_j) = \frac{1}{2\pi}\left[1 + A\cos(x_k - x_j)\right]$$

global order parameter

$$r = \frac{1}{N}\left|\sum_{k=1}^{N} e^{i\theta_k}\right|$$

Abrams
Strogatz
Laing
I Omelchenko
O Omelchenko
Zakharova
Wolfrum
Schoell
...

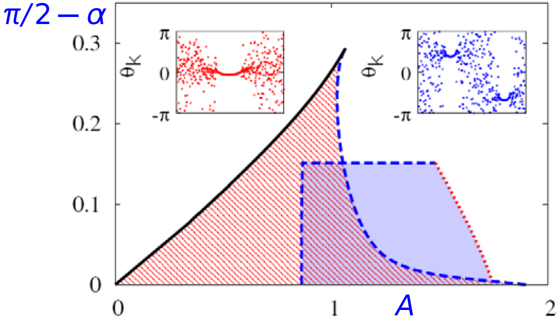
Continuum limit (Ott-Antonsen)

Bifurcation diagram for continuum limit

chimera=rotating wave

OE Omel'chenko, Nonlinearity 2013

coupling function $G(x) = \frac{1}{2\pi}[1 + A\cos x]$

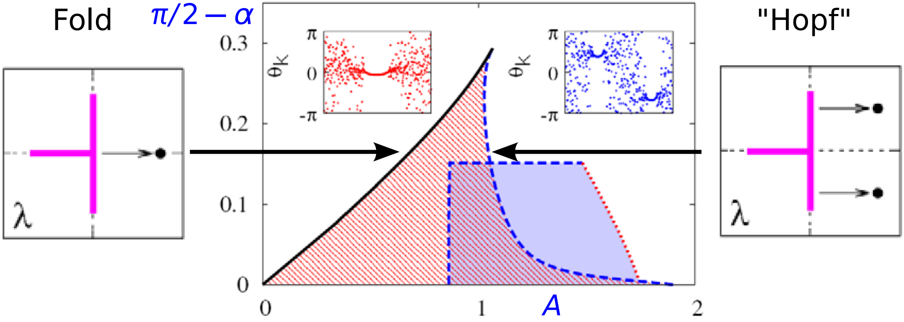


Continuum limit (Ott-Antonsen)

Bifurcation diagram for continuum limit
chimera=rotating wave

OE Omel'chenko,
Nonlinearity 2013

coupling function $G(x) = \frac{1}{2\pi}[1 + A\cos x]$



Conclusion

- Stabilizing feedback loop makes it possible to define macroscopic
  - equilibria (stable/unstable)
  - periodic orbits (stable/unstable)
  - fold, Hopf, pitchfork, period-doubling bifurcations

  in a computable manner.
- Examples studied until now
  - interaction of traders
  - disease spread on network
  - chimeras on rings of phase oscillators