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Glassy dynamics in the HMF model

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Abstract

We discuss the glassy dynamics recently found in the meta-equilibrium quasi-stationary states (QSS) of the HMF model. The relevance of the initial conditions and the connection with Tsallis nonextensive thermostatistics is also addressed.

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Keywords: Hamiltonian dynamics; Long-range interactions; Glassy dynamics

1. Introduction

In this paper we present a brief review of the glassy and anomalous behavior observed in the dynamics of the Hamiltonian mean field (HMF) model [1,2]: a simple XY model of fully coupled inertial spins with ferromagnetic long-range interactions [3–8]. We show, in particular, a more detailed description of the microscopical analogies between the quasi-stationary states (QSS) regime found in the HMF model and the spin glass phase scenario of the Sherrington–Kirkpatrick (SK) infinite-range model [9,10]. We also discuss the importance of the initial conditions in order to observe *dynamical frustration* [2]. The latter is a crucial feature for the emergence of a glassy dynamics, since, a priori, the HMF model is not frustrated. Dynamical frustration is related to the weak-ergodicity breaking phenomenon, typical of glassy systems [11,12] and to other dynamical anomalies, such as superdiffusion and Lévy walks, negative specific heat, vanishing Lyapunov exponents, non-Gaussian velocity pdf's, power-law decaying correlation functions [2,6–8]. This anomalous behavior seems to be linked to the fractal structure of the region of phase space in which the systems remains

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trapped when the dynamics starts sufficiently far from equilibrium. We will show that such a dynamics can be quantitatively characterized by the introduction of a new order parameter, namely the *polarization* p [1]. In the end we will also briefly discuss the links with Tsallis nonextensive thermostatistics scenario [14–16].

2. Glassy phase and nonextensivity in the HMF model

2.1. The model

The HMF model, here considered in its ferromagnetic version, consists of N planar classical spins $\vec{s}_i = (\cos\theta_i, \sin\theta_i)$ interacting through an infinite-range potential [3]. The Hamiltonian is

$$H = K + V = \sum_{i=1}^{N} \frac{p_i^2}{2} + \frac{1}{2N} \sum_{i,j=1}^{N} [1 - \cos(\theta_i - \theta_j)] \,, \tag{1}$$

where $-\pi < \theta_i < \pi$ is the angle of the ith spin and p_i the conjugate variable representing the rotational velocity. Since the modulus of each spin is unitary, we can represent the system of N planar rotating spins as N interacting particles moving on the unit circle. The usual order parameter of the model is the magnetization M:

$$M = \frac{1}{N} \left| \sum_{i=1}^{N} \vec{s}_i \right| \,. \tag{2}$$

The equilibrium solution in the canonical ensemble predicts a second-order phase transition from a low-energy condensed (ferromagnetic) phase with magnetization $M \neq 0$, to a high-energy one (paramagnetic), where the spins are homogeneously oriented on the unit circle and $M = 0$. The *caloric curve*, i.e., the dependence of the energy density $U = E/N$ on the temperature T, is given by $U = T/2 + \frac{1}{2}(1 - M^2)$ [3,4]. The critical point is at energy density $U_c = \frac{3}{4}$, corresponding to a critical temperature $T_c = \frac{1}{2}$ [3].

The dynamics of HMF shows several anomalies before complete equilibration. More precisely, if we adopt the so-called $M1$ initial conditions, i.e., $\theta_i = 0$ for all i ($M(0) = 1$) and velocities uniformly distributed (*water bag*), the results of the simulations, in a special region of energy values ($\frac{1}{2} < U < U_c$), show a disagreement with the canonical prediction for a transient regime whose length depends on the system size N. In such a regime, the system remains trapped in metastable states (QSS) at a temperature lower than the canonical equilibrium one, until it slowly relaxes towards Boltzmann–Gibbs (BG) equilibrium, showing strong memory effects. This transient regime becomes stable if one takes first the infinite size limit and then the infinite time limit [6].

2.2. Glassy dynamics

The observation of these long relaxation times and in particular of aging [2,8] for the QSS was the first indication towards a possible interpretation of this regime

in terms of glassy dynamics. The paradigmatic example of this behavior are spin glasses [12,13]. In the materials that originally were called '*spin glasses*' the randomly distributed magnetic impurities determine a random distribution ('quenched disorder') of ferromagnetic and anti-ferromagnetic interactions among the magnetic spins, thus generating frustration in the lattice. In these systems the impossibility to minimize simultaneously the interaction energies of all the couple of spins leads to a frustrated situation, which determines a very complex energetic landscape in phase space. The latter appears as consisting of large valleys separated by high activation energies. Each valley contains many local minima in which the system, at low temperature, can remain trapped for a very long time. This time grows exponentially with the height of the energy barriers, thus the system shows very slow relaxation, strong memory effects and aging. In an ordinary ferromagnetic phase, where there is only one energy minimum, the application of an external magnetic field gives suddenly rise to a non-zero magnetization. The latter, for a fixed temperature, remains constant until the field is active and then vanishes very rapidly. On the contrary, in the spin glass phase the magnetization shows a strong dependence on the thermal history of the system (aging). After quenching the spin glass below its critical temperature in presence of the external field, the system settles in at a particular magnetization value (*field cooled magnetization*) that does not change instantaneously when the field is switched off, but relaxes to equilibrium very slowly. This relaxation depends on the waiting time spent between the quenching and the elimination of the external field. Such a behavior can be explained within the so-called *weak-ergodicity breaking* framework [11,12]. A very similar situation seems to happen in the QSS regime of the HMF model [1,2]. Within the mean-field framework of the SK model [9,10], the first solvable model of a spin glass system with Gaussian distribution of interactions, it was possible to observe three different phases, namely, paramagnetic (PA), ferromagnetic (FE) and spin glass (SG) phase, depending on the temperature and the parameters of the Gaussian distribution. Each phase is characterized by a different microscopic behavior and a different kind of orientation order. Although today some features of the SK model are considered rather obsolete, its microscopic interpretation of the SG phase can be still considered as representative of a generic glassy-like phase. Thus, in order to get an intuitive picture of the differences between the three phases, let us consider for example a two-dimensional lattice of planar spins, see first column of Fig. 1. This schematic picture describes also the HMF dynamics if one imagines to locate the spins in a square lattice. Now let us take some snapshots of the spin configuration in each of the three phases, see Fig. 2. In the FE phase ($T < T_c$) all the spins results aligned and frozen in their equilibrium position, so it is easy to recognize this phase even by means of snapshots taken for only one particular instant of time. But in this way it would be impossible to distinguish between the PA and the SG phase. In fact in both these phases the instantaneous mutual orientations of the spins are random, in the PA phase ($T > T_c$) due to the high temperature and in the SG phase ($T < T_c$) due to the quenched disorder of the interactions. So we necessarily need to consider a temporal sequence of snapshots in order to discriminate the SG from the PA phase. In the SG phase all the snapshots will be identical with each other, since each spin is frozen and retains the same orientation over very long periods of time. On the other hand, in

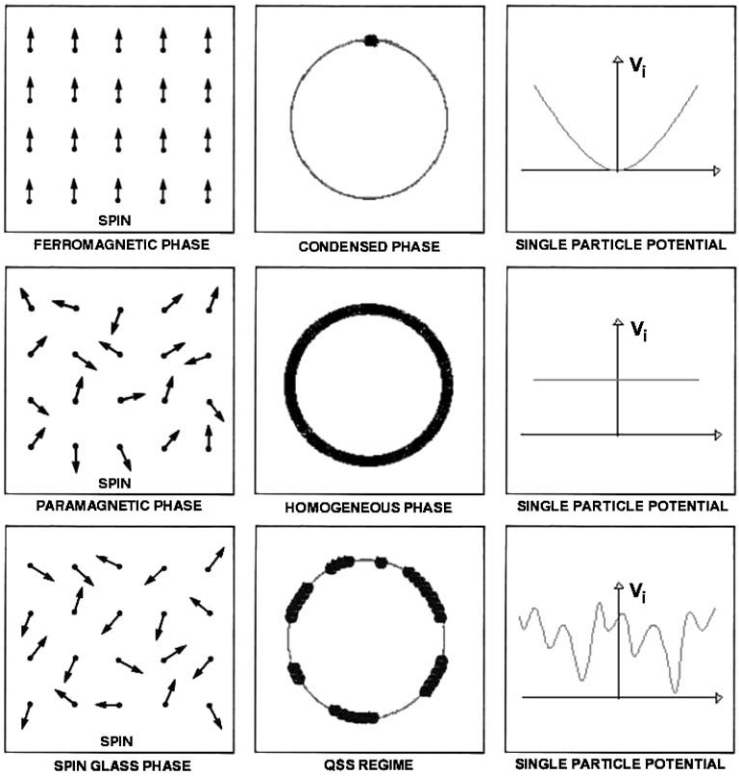

Fig. 1. The figure shows a schematic representation of the three phases of a spin glass system: ferromagnetic phase, paramagnetic phase and spin glass phase. In the first column spins are represented in a two-dimensional lattice. In the second column, in analogy with the HMF model, spins are represented as particles rotating on the unit circle. In the third column we draw the corresponding schematic single-particle potential landscape of the three phases.

the PA phase the orientation of *the same spin* at successive instants of time changes randomly. It appears clearly that the magnetization order parameter, calculated as in Eq. (2) at one instant of time, vanishes in the SG phase just like in the PA one. Therefore, in order to discriminate between spin glass disorder and paramagnetism, one needs an additional order parameter. The latter should take into account the temporal evolution of each spin, in order to measure its degree of freezing. In effect a parameter of this kind, called '*EA order parameter*', was originally proposed in Refs. [9,10], although later it turned out to be inadequate for the mean-field theoretical description of the SG phase [12]. Nevertheless, inspired by the physical meaning of this parameter, we have proposed a new order parameter in the context of the HMF model, the *polarization* p to characterize in a quantitative way the glassy dynamics of the QSS regime [1].

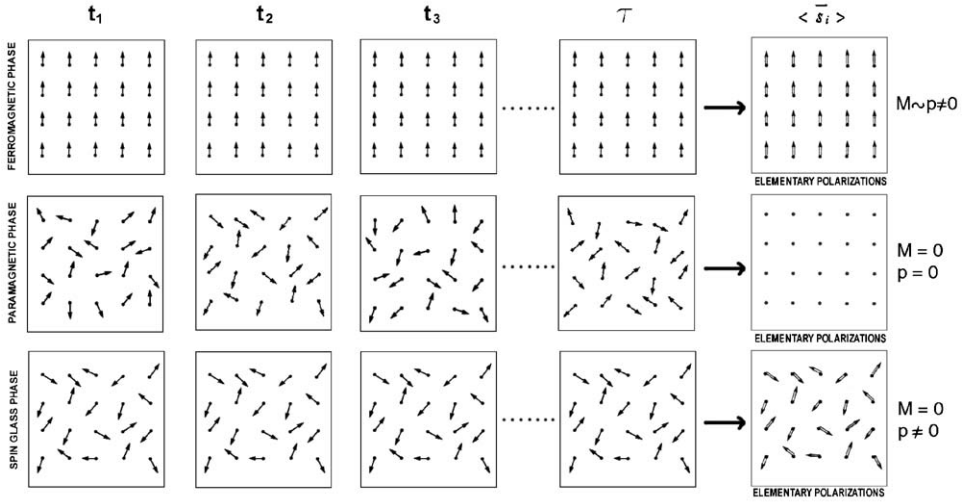

Fig. 2. In this figure we show a temporal sequence of snapshots for each of the three phases of a spin glass. Only comparing the different snapshots in the sequences it is possible to distinguish the paramagnetic phase (where the snapshots change in time), from the spin glass one (where all the snapshots are identical). In the last column we report the elementary polarizations resulting for each phase. By averaging their modulus over all the spins of the lattice we obtain the order parameter p, see text, which allows to discriminate between the three phases.

2.3. The polarization

We define the *elementary polarization* as the temporal average, integrated over an opportune time interval τ, of the successive positions of each elementary spin vector:

$$\langle \vec{s}_i \rangle = \frac{1}{\tau} \int_{t_0}^{t_0+\tau} \vec{s}_i(t)\,\mathrm{d}t, \quad i = 1,\ldots,N\ , \tag{3}$$

being t_0 the initial transient time. Then we further average the modulus of the elementary polarization over the N spin configuration, to finally obtain the *average polarization* p:

$$p = \frac{1}{N} \sum_{i=1}^{N} |\langle \vec{s}_i \rangle|\ . \tag{4}$$

It is easy to see (last column of Fig. 2) that:

(1) in a pure ferromagnetic (condensed) phase each elementary polarization vector coincides with the correspondent spin vector, both being frozen and parallel, then the average polarization p keeps a non zero value equal to the modulus of the average magnetization per spin M;
(2) in a paramagnetic (homogeneous) phase, where the orientation of each spin vector at every time changes in a completely aleatory way, this continuous motion yields a vanishing value for both M and the average polarization;

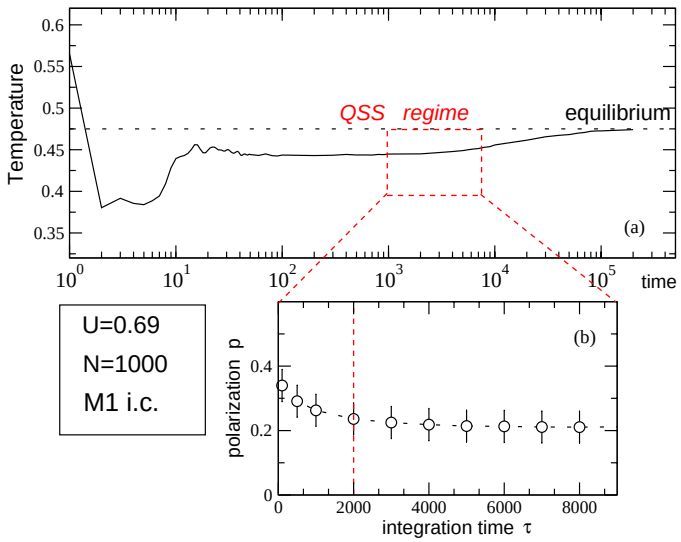

Fig. 3. In the upper panel of the figure we plot the temporal evolution of temperature for the HMF model for $U = 0.69$, $N = 1000$ and $M1$ initial conditions. In the lower panel we show the value of polarization versus the integration time τ (in a linear scale), after a transient time $t_0 = 1000$ time units and for a window of 9000 time units. One can see that for τ greater than 2000—i.e., the standard interval we use for our simulations—the polarization does not change significatively up to the end of the QSS temperature plateau. The values of polarization were averaged over 20 different realizations—the error bars refers to such an average.

(3) in a spin glass phase, where the spatial disorder is random but the dynamics is quenched, while M vanishes as in the PA phase, p gets a nonzero value as in the FE one.

From the numerical simulations, it results that the QSS temperature lies on the extension of the high-temperature line of the caloric curve below T_c [2,6]. This implies that in the QSS regime M vanishes with the size N of the system (more precisely as $N^{-1/6}$), so we have $M \simeq 0$ below the critical temperature, just as in the SG phase of the SK model [9,10]. Thus, the next natural step is to check if the polarization order parameter would remain different from zero in the QSS regime for a growing size of the system. Preliminarly we consider the calculation of p versus the integration time interval τ at $U = 0.69$ and $N = 1000$, after a transient time $t_0 = 1000$. As one can see in Fig. 3, lower panel, the value of the polarization does not change significatively increasing the integration time interval τ beyond $\tau = 2000$, up to the end of the QSS temperature plateau, see upper panel. The same behavior is obtained for greater values of N. In the following, we adopt the time interval $\tau = 2000$ for the calculation of p. Starting the numerical simulations from the usual $M1$ initial conditions, we have found (see the upper part of Fig. 4(a)) that, in the QSS regime, while M goes to zero with the expected scaling, the polarization p does not vanish and remains constant inside the error: $p \sim 0.24$. This finite value of p which characterizes quantitatively a frozen

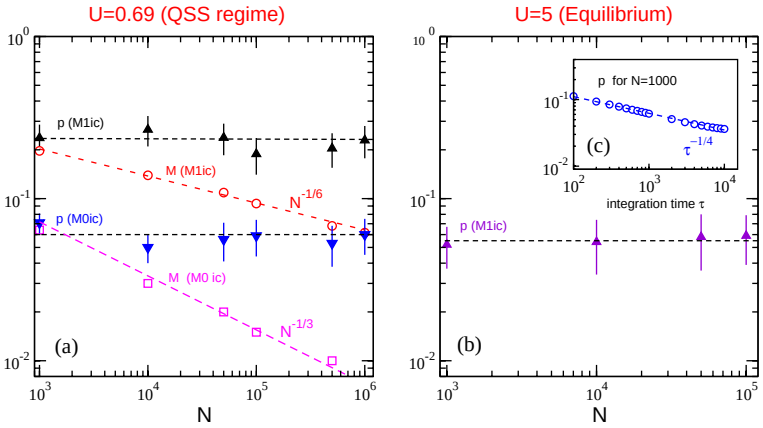

Fig. 4. The figure shows the behavior of p and M with the size of the system. On the left (a) we plot the scaling in the QSS regime at $U = 0.69$ for the two different initial conditions considered in the paper, $M1$ and $M0$. While the magnetization tends to zero going towards the infinite size limit, the polarization remains constant. The polarization p is significatively different from zero only for $M1$ initial conditions, see text for further details. On the right (b), the behavior of p is plotted only for $M1$ initial conditions at the overcritical energy $U = 5$, i.e., in the full paramagnetic (homogeneous) phase, where the system reaches immediately the BG equilibrium. In this case the polarization is very small and almost equal to the case $U = 0.69$ with $M0$ initial conditions. Moreover, in this case, as shown in the inset, at variance with the behavior plotted in Fig. 3 for the QSS regime, the polarization vanishes, as $\tau^{-1/4}$ for $N = 1000$, increasing the integration time interval τ.

dynamics, is due to a 'dynamical frustration' phenomenon [1]: in fact the QSS are characterized by the presence of many clusters of particles appearing and disappearing on the unit circle, see the lower picture in the second column of Fig. 1. Each of them compete with the others trying to capture as many particles as possible in order to relax to the equilibrium configuration with a magnetization $M \sim 0.3$. These results are also in perfect agreement with the observed dynamical correlations in the μ-space [2,6]: as required by the weak-ergodicity breaking hypothesis, during the QSS regime the system lives in a smooth fractal part of the a priori accessible phase space [6], and for N going to infinity it never escapes from that region. So, in the thermodynamic limit, the QSS regime can be considered as a new *glassy phase* of the HMF model. As expected, when the dynamical frustration disappears, i.e., when the system (for N finite) has reached the equilibrium conditions of the condensed phase, we loose any trace of glassy-like dynamics and one obtains values of M and p which are equal everywhere but not zero [1]. Finally, in the full homogeneous phase both M and p vanish, because the spins can rotate freely [1].

2.4. The role of initial conditions

It is important to stress the role of the $M1$ initial condition in order to have weak-ergodicity breaking and glassy behavior. In fact, if we start from initial

conditions with both angles and velocities uniformly distributed (namely $M0$ initial conditions, since $M(t=0)=0$), the QSS regime shows a very different scenario: in fact in this case neither power-law correlation functions nor dynamical structures in the μ-space are present [2]. Such a scenario is consistent with the different value of the polarization calculated in such QSS regime reached from $M0$ initial conditions, see lower part of Fig. 4(a). One can see that in this case the values of p vs. N is constant to a value much smaller than before, i.e., $p \sim 6\times 10^{-2}$. This is also the order of magnitude of the polarization at equilibrium in the full homogeneous phase (for $M1$ initial conditions), see Fig. 4(b). Please note also that here, for fixed N (1000), the value of the polarization vanishes with the integration time interval τ as $\tau^{-1/4}$, see the inset. The intuitive explanation of such a different behavior is quite simple. Starting from $M0$ initial conditions, although we are far from Boltzmann–Gibbs (BG) equilibrium, we do not have the same kind of *kinetic explosion*, as for $M1$ initial conditions, which creates the long-lasting dynamical correlations. In fact, in this case the system is directly put on the QSS plateau at a temperature $T=0.38$ where $M(0)=0$ and thus also the force acting on each spin, proportional to M [6], vanishes since the beginning. For $M0$ initial conditions we do not have any kind of fast quenching from an high temperature phase, at variance with the $M1$ case, and therefore we do not find any glassy-like behavior, dynamical frustration or weak-ergodicity breaking. On the other hand, several other dynamical anomalies observed in connection with the $M1$ case (fractal-like structures in the μ-space, power-law velocity pdf's and correlation functions, Lévy walks and superdiffusion, aging) have not been found for the $M0$ one [2]. This suggests that a connection with Tsallis nonextensive thermodynamics [6,2], exists probably only for the QSS regime obtained starting the system with $M1$ and not with $M0$ initial conditions. The metastable states in this second case ($M0$) have a different microscopic nature and can be probably better interpreted as Vlasov stationary states [17].

2.5. Links to nonextensive thermostatistics

In Ref. [6] we had already found a link of the QSS regime with Tsallis nonextensive thermostatistics, by reproducing the microcanonical non-Gaussian velocity pdf's with a q-exponential curve. However the value of q obtained in that case is rather large and not fully understood. A very interesting progress in that direction has been presented, considering the more appropriate canonical ensemble, by Baldovin [18]. On the other hand, we have recently found that also the power-law decay of correlation functions, from the QSS regime to equilibrium, can be very well explained by q-exponential curves [2,19]. More interesting is the fact that in this case, we obtain $q = 1.65 \pm 0.05$ for the entropic index. In fact this is the value expected from the relationship, derived in Ref. [20], between q and the anomalous diffusion exponent α, i.e., $q = (3\alpha - 2)/\alpha$. In our case we had previously found a value $\alpha = 1.4 \pm 0.2$ for superdiffusion in the QSS regime [5], thus in this respect the nonextensive formalism seems to apply in a consistent way. A more detailed study in this direction is in progress.

3. Conclusions

In this paper we have shown that the metastable quasi-stationary states of the HMF model, obtained from $M1$ initial conditions, can be interpreted as a glassy phase of the system. This phase can be characterized by a new order parameter, the polarization p, which gives a quantitative description of the frozen dynamics. This fact establishes a very interesting and promising relationship between nonextensive systems and glassy ones, which will hopefully lead to new exciting discoveries in the near future.

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