

Detecting complex network modularity by dynamical clustering

S. Boccaletti,¹ M. Ivanchenko,² V. Latora,³ A. Pluchino,³ and A. Rapisarda³

¹*CNR-Istituto dei Sistemi Complessi, Via Madonna del Piano, 10, 50019 Sesto Fiorentino (FI), Italy
and the Italian Embassy in Tel Aviv, Trade Tower, 25 Hamered Street, Tel Aviv, Israel*

²*Department of Radiophysics, Nizhny Novgorod University, 23, Gagarin Avenue, 603600 Nizhny Novgorod, Russia*

³*Dipartimento di Fisica e Astronomia, Università di Catania, and INFN Sezione di Catania, Via S. Sofia, 64, 95123 Catania, Italy*

(Received 19 July 2006; published 12 April 2007)

Based on cluster desynchronization properties of phase oscillators, we introduce an efficient method for the detection and identification of modules in complex networks. The performance of the algorithm is tested on computer generated and real-world networks whose modular structure is already known or has been studied by means of other methods. The algorithm attains a high level of precision, especially when the modular units are very mixed and hardly detectable by the other methods, with a computational effort $O(KN)$ on a generic graph with N nodes and K links.

DOI: [10.1103/PhysRevE.75.045102](https://doi.org/10.1103/PhysRevE.75.045102)

PACS number(s): 89.75.Hc, 05.45.Xt, 87.18.Sn, 89.65.Ef

Hierarchical modular structures constitute one of the most important properties of real-world networked systems [1]. For instance, tightly connected groups of nodes in a social network represent individuals belonging to social communities, while modules in cellular and genetic networks are somehow related to functional modules. Consequently, identifying the modular structure of a complex network is a crucial issue in the analysis and understanding of the growth mechanisms and the processes running on top of such networks. Modules (called sometimes community structures in social science) are tightly connected subgraphs of a network, i.e., subsets of nodes within which the network connections are dense, and between which connections are sparser. Nodes, indeed, belonging to such tight-knit modules, constitute units that separately contribute to the collective functioning of the network. For instance, subgroups in social networks often have their own norms, orientations and subcultures, sometimes running counter to the official culture, and are the most important source of a person's identity. Analogously, the presence of subgroups in biological and technological networks is at the basis of their functioning.

The detection of the modular structure of a network is formally equivalent to the classical graph partitioning problem in computer science, which finds many practical applications such as load balancing in parallel computation, circuit partitioning, and telephone network design, and is known to be a NP-complete problem [2]. Therefore, although modules detection in large graphs is potentially very relevant and useful, so far this trial has been seriously hampered by the large associated computational demand. To overcome this limitation, a series of efficient heuristic methods has been proposed over the years. Examples include methods based on spectral analysis [3], the hierarchical clustering methods developed in the context of social networks analysis [4], or methods allowing for multicomunity membership [5]. More recently, Girvan and Newman (GN) have proposed an algorithm that works quite well for general cases [6]. The GN is an iterative divisive method based in finding and removing progressively the edges with the largest betweenness, until the network breaks up into components. The betweenness b_{ij} of an edge connecting nodes i and j is defined as the number of shortest paths between pairs of nodes that

run through that edge [6]. As the few edges lying between modules are expected to be those with the highest betweenness, by removing them recursively a separation of the network into its communities can be found. Therefore, the GN algorithm produces a hierarchy of subdivisions of a network of N nodes, from a single component to N isolated nodes. In order to know which of the divisions is the best one, Girvan and Newman have proposed to look at the maximum of the modularity Q , a quantity measuring the degree of correlation between the probability of having an edge joining two sites and the fact that the sites belong to the same modular unit (see Ref. [6] for the mathematical definition of Q). The GN algorithm runs in $O(K^2N)$ time on an arbitrary graph with K edges and N vertices, or $O(N^3)$ time on a sparse graph. In fact, calculating the betweenness for all edges requires a time of the order of KN [7], since it corresponds to the evaluation of all-shortest-paths (ASP) problem. And the betweenness for all edges needs to be recalculated every time after an edge has been removed (the betweenness recalculation is a fundamental aspect of the GN algorithm) [6]. This restricts the applications to networks of at most a few thousands of vertices with current hardware facilities. Successively, a series of faster methods directly based on the optimization of Q have been proposed [8,9], which allow up to a $O(N \ln^2 N)$ running time for finding modules in sparse graphs.

All the abovementioned methods are based on the structure of a network, meaning that they use solely the information contained in the adjacency matrix $A=\{a_{ij}\}$ (or any equivalent representation of the topology) of the graph. As complementary to such approaches, the authors of Ref. [10] have proposed a method to find modules based on the statistical properties of a system of spins (namely, q -state Potts spins) associated to the nodes of the graphs. In this paper we propose a dynamical clustering (DC) method based on the properties of a dynamical system associated to the graph. DC techniques were initiated by the relevant observation that topological hierarchies can be associated to dynamical time scales in the transient of a synchronization process of coupled oscillators [11]. Although being fast, so far DC methods do not provide in general the same accuracy in the identification of the communities.

Here, we show how to combine topological and dynamical information in order to devise a DC algorithm that is able to identify the modular structure of a graph with a precision competitive with the best techniques based solely on the topology. The method we present is based upon the well-known phenomenon of synchronization clusters of nonidentical phase oscillators [12], each one associated to a node, and interacting through the edges of the graph. Clusters of synchronized oscillators represent an intermediate regime between global phase locking and full absence of synchronization, thus implying a division of the whole graphs into groups of elements that oscillate at the same (average) frequency. The key idea is that, starting from a fully synchronized state of the network, a dynamical change in the weights of the interactions, which retain information on the original betweenness distribution, yields a progressive hierarchical clustering that fully detects modular structures.

For the sake of illustration and without lack of generality, we specify our technique with reference to the so-called opinion changing rate (OCR) model, a continuous-time system of coupled phase oscillators introduced for the modeling of opinion consensus in social networks [13], and representing a variation of the Kuramoto model [14]. Other continuous-time (Kuramoto and Rössler dynamics), and also discrete-time (coupled circle maps) dynamical systems have been investigated and will be reported elsewhere. Given a undirected, unweighted graph with N nodes and K edges, described by the adjacency matrix $A=\{a_{ij}\}$, we associate to each node i ($i=1, \dots, N$) a dynamical variable $x_i(t) \in]-\infty, +\infty[$. The dynamics of each node is governed by

$$\dot{x}_i(t) = \omega_i + \frac{\sigma}{\sum_{j \in \mathcal{N}_i} b_{ij}^{\alpha(t)}} \sum_{j \in \mathcal{N}_i} b_{ij}^{\alpha(t)} \sin(x_j - x_i) \beta e^{-\beta|x_j - x_i|}, \quad (1)$$

where ω_i is the natural frequency of node i (in the numerical simulations the ω_i 's are randomly sorted from a uniform distribution between $\omega_{\min}=0$ and $\omega_{\max}=1$), σ is the coupling strength, and \mathcal{N}_i is the set of nodes adjacent to i , i.e., all nodes j for which $a_{ij}=a_{ji}=1$. The constant parameter β , tuning the exponential factor in the coupling term of Eqs. (1), switches off the interaction when the phase distance between two oscillators exceeds a certain threshold (as usual [13] we fix $\beta=3$). Notice that the interaction between two adjacent nodes i and j is weighted by the term $b_{ij}^{\alpha(t)}/\sum_{j \in \mathcal{N}_i} b_{ij}^{\alpha(t)}$, where b_{ij} is the betweenness of the edge i, j , and $\alpha(t)$ is a time-dependent exponent, such that $\alpha(0)=0$. In Ref. [15] it has been shown that the ability of a dynamical network, as the one in Eqs. (1), to maintain a synchronization state crucially depends on the value of the parameter α . For such a reason, in the DC algorithm to find modular structures, we fix the coupling strength σ equal to an arbitrary value such that the unweighted ($\alpha=0$) network is fully synchronized, and we solve Eqs. (1) with a progressively (stepwise) decreasing value of $\alpha(t)$. Namely, while keeping fixed σ , we consider $\alpha(t_{l+1})\alpha(t_l) - \delta\alpha$ for $t_{l+1} > t > t_l$, where $t_{l+1} - t_l = T \forall l$ (in the following $T=2$), and $\delta\alpha$ is a parameter that will be specified below. As the edges connecting nodes belonging to the same module (to two different modules) have in general small

(large) betweenness, when α decreases from zero, the corresponding interaction strengths on those edges become increasingly enhanced (weakened). Since the network is prepared to be fully synchronized, it has to be expected that, as α decreases, the original synchronization state hierarchically splits into clusters of synchronized elements, according to the hierarchy of modules present in the graph. The individuation of synchronization clusters is made in terms of groups of nodes with the same instantaneous frequency $\dot{x}(t)$. The procedure consists then in monitoring the emerging set of synchronization clusters at each value of $\alpha(t)$. The best division in communities of the graph (the best α value) is individuated by looking at the maximum (as a function of α) of the modularity Q [6].

In order to comparatively evaluate the performance of the algorithm, we have considered, as in Ref. [6], a set of computer generated random graphs constructed in such a way to have a well defined modular structure. All graphs are generated with $N=128$ nodes and $K=1024$ edges. The nodes are divided into four communities, containing 32 nodes each. Pairs of nodes belonging to the same module (to different modules) are linked with probability p_{in} (p_{out}). p_{out} is taken so that the average number z_{out} of edges a node forms with members of other communities can be controlled. While z_{out} can be then varied, p_{in} is chosen so as to maintain a constant total average node degree $\langle k \rangle = 16$. As z_{out} increases, the modular structure of the network becomes therefore weaker and harder to identify. As the real modular structure is here directly imposed by the generation process, the accuracy of the identification method can be assessed by monitoring the fraction p of correctly classified nodes vs z_{out} . In Fig. 1 we report the value of p (averaged over twenty different realizations of the computer generated graphs and of the initial conditions) as a function of z_{out} , for the DC algorithm based on the OCR model of Eqs. (1), with $\sigma=5.0$ and $\delta\alpha=0.1$. The resulting performance (open circles) is comparable to that of the best methods based solely on the topology, such as the GN (full triangles) [6] and the Newman Q -optimization fast algorithm (full squares) [8].

The performance of the DC algorithm considered can be made better by adding a simple modification to the OCR model which further stabilizes the system. Such modification consists in changing in time the natural frequencies ω 's according to the idea of confidence bound introduced by Hegselmann and Krause (HK), in the context of models for opinion formation [16]. Therefore, we will refer to the improved method as the OCR-HK dynamical clustering. The confidence bound is a parameter ϵ which fixes the range of compatibility of the nodes. At each time step, the generic node i , having a dynamical variable $x_i(t)$ and a natural frequency $\omega_i(t)$, checks how many of its neighbors j are compatible, i.e., have a value of the variable $x_j(t)$ falling inside the confidence range $[x_i - \epsilon, x_i + \epsilon]$. Then, at the following step in the numerical integration, we set $\omega_i(t + \Delta t)$, i.e., the node takes the average value of the ω 's of its compatible neighbors at time t . In the OCR-HK, the changes of the $\omega(t)$'s is superimposed to the main dynamical evolution of Eq. (1) and noticeably contributes to stabilize the frequencies of the oscillators according to the correct modular structure of the

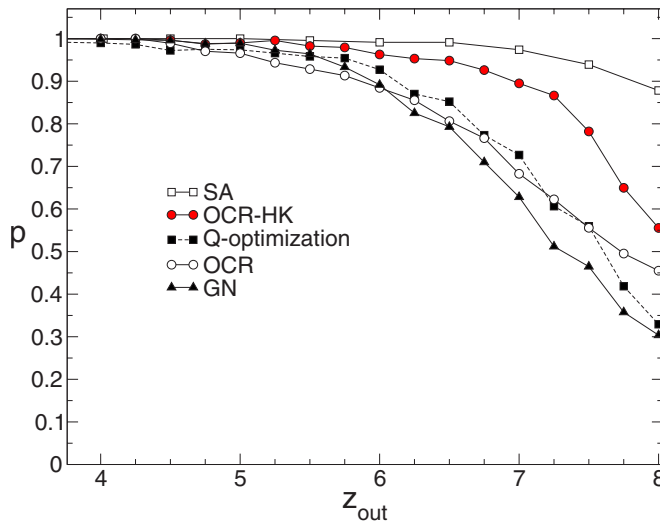


FIG. 1. (Color online) Fraction p of correctly identified nodes as a function of z_{out} (average number of inter-modular edges per node) for computer generated graphs with $N=128$ nodes, four communities and an average degree $\langle k \rangle=16$. The results of DC methods based, respectively, on the OCR (open circles) and the OCR-HK (full circles) models, are compared with three standard methods, two based solely on the topology, such as the GN algorithm (full triangles) [6] and the Newman Q -optimization fast algorithm (full squares) [8], and one based on an evolutionary method, the simulated annealing algorithm [17] (open squares).

network, also reducing the dependence of the algorithm on the initial conditions. The results with computer generated graphs ($\delta\alpha=0.1$) are reported in Fig. 1 as full circles. The improvement in the performance of the OCR-HK method with respect to both the standard OCR and the two topological methods (GN and Q optimization), is evident for $z_{\text{out}} > 5$, and it can be made even larger using for smaller values of $\delta\alpha$. For completeness, we also report the results of an optimization procedure based on a simulated annealing (SA) [17], which is presently the most accurate method available on the market, though very CPU time consuming. As for the computational cost, our algorithm provides an improvement with respect to the majority of other methods (see Table 1 in Ref. [17]). For instance, while iterative topological algorithms [6,9] need to recalculate the betweenness distribution all the times a given edge is removed, in our case that distribution has to be evaluated *only* for the initial graph, as the cluster de-synchronization process itself gives a progressive weakening of the edges with highest betweenness. We have analyzed sparse graphs of size up to $N=16\,384$ and found a scaling law of $O(N^{1.76})$ for the dynamical evolution of the OCR-HK system. However, since the initial calculation of betweenness takes $O(N^2)$ operations, the total CPU time scales as $O(N^2)$ too. Considering that the fastest algorithm on the market [$O(N \ln^2 N)$] is the Q -optimization one [8], which on the other hand is less accurate than the OCR-HK (as shown in Fig. 1), we can conclude that our DC method provides an excellent compromise between accuracy and computational demand [17].

It should be noticed that the proposed method conceptually differs from the dynamical clustering technique intro-

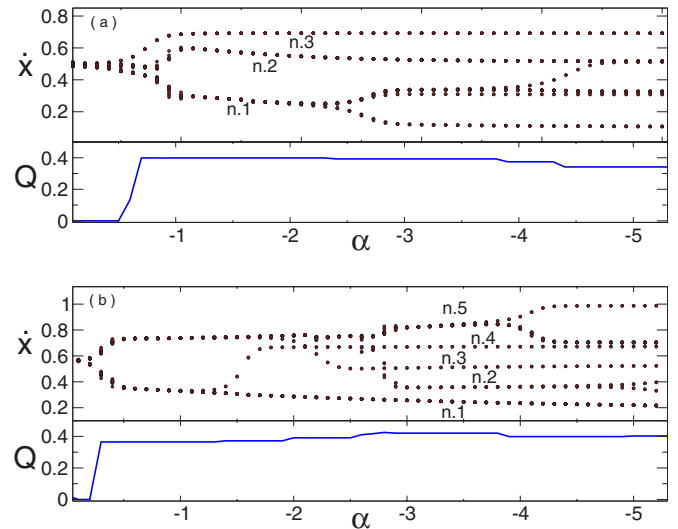


FIG. 2. (Color online) The distribution of instantaneous frequencies and the correspondent modularity Q are reported as a function of α for the OCR-HK model. The application to the karate club network and to the Chesapeake Bay food web are shown in panels (a) and (b), respectively. In both the simulations $\sigma=5.0$, $\delta\alpha=0.1$, and $\epsilon=0.0005$.

duced in Ref. [11]. Indeed, while in Ref. [11] the modular hierarchy of a network was associated to different time scales in the transient dynamics toward a fully synchronized dynamics, here it corresponds to a hierarchical sequence of asymptotically synchronized states, from the initial [$\alpha(0)=0$] full network synchronization, to progressive cluster synchronization as $\alpha(t)$ decreases. A relevant consequence is that our technique is almost fully insensitive to differences in the initial conditions for the phases of the coupled oscillators, as far as the local dynamics is selected to have a unique attractor.

Finally, we tested how the method works on two typical real-world networks: the Zachary karate club network ($N=34$, $K=78$) [18] and the food web of marine organisms living in the Chesapeake Bay ($N=33$, $K=71$) [19,20]. In both cases we have some *a priori* knowledge of the existing modules. In fact, the karate club network is known to split into two smaller communities, whose detailed composition was reported by Zachary [18]. Analogously, the food web organisms contain a main separation in two large communities, according to the distinction between pelagic organisms (living near the surface or at middle depths) and benthic organisms (living near the bottom). As in the previous simulations, we first calculate the set of edge betweenness $\{b_{ij}\}$ and then we integrate numerically Eqs. (1) with the HK modification on the ω 's, with $\delta\alpha=0.1$ and $\sigma=5.0$ (the latter ensures again an initial fully synchronized state at $\alpha=0$).

In Fig. 2 the N instantaneous frequencies \dot{x}_i , and the modularity Q , are plotted as a function of α (i.e., as a function of time) for both the karate club, panel (a), and the food web network, panel (b). In panel (a) the best configuration, with $Q \sim 0.40$, is reached around $-1.0 \gtrsim \alpha \gtrsim -2.5$ and yields a partition of the karate club network into three stable communities that very well describe the real situation. The

largest one, labeled with n.1 in the figure (nodes 9, 10, 29, 31, 15, 16, 19, 21, 23, 32, 33, 34, 24, 25, 26, 27, 28, 30), fully corresponds to one of the two communities reported by Zachary, while the sum of the remaining two communities, labeled as n.2 (nodes 1, 2, 3, 4, 8, 12, 13, 14, 18, 20, 22) and n.3 (nodes 5, 6, 7, 11, 17), corresponds to the second Zachary's module of 16 elements. Notice that cluster n.3 represents a very well connected subset that is frequently recognized as a separated module also by other methods [6]. Moreover, the value of the best modularity found is larger than that of the Zachary partition into two communities ($Q \sim 0.37$). Analogously good performance is obtained for the food web. In panel (b) the highest value of Q , namely, $Q \sim 0.42$, is reached for $-2.8 \geq \alpha \geq -3.8$, yielding a division of the food web into five communities (n.1: nodes 3, 14, 15, 16, 18, 19, 25, 26, 27, 28, 29; n.2: nodes 22, 30, 31, 32; n.3: nodes 5, 6; n.4: nodes 4, 17; n.5: nodes 8, 9, 10, 20, 24, 1, 2, 7, 11, 12, 13, 21, 23, 33) in which, with respect to Refs.

[6,20], the distinction between pelagic and benthic organisms is not only preserved but also improved.

In conclusion, we have introduced an efficient algorithm for the detection and identification of modular structures that attains a very high precision, with a small associated computational effort that scales as $O(N^2)$. Our method, therefore, could be of use for a reliable modules detection in sizable networks (e.g., biological, neural networks), and can contribute to a better understanding of the hierarchical functioning of networked systems in many physical, biological, and technological cases.

The authors are indebted to A. Amann, F.T. Arecchi, M. Chavez, R. López-Ruiz, and Y. Moreno for many helpful discussions on the subject. M.I. acknowledges project Nos. RFBR 05-02-19815-MF-a, 06-02-16499-Á, 06-02-16596-Á. V.L., A.P., and A.R. acknowledge financial support from the PRIN05-MIUR project "Dynamics and Thermodynamics of Systems with Long-Range Interactions."

-
- [1] R. Albert and A.-L. Barabási, *Rev. Mod. Phys.* **74**, 47 (2002); M. E. J. Newman, *SIAM Rev.* **45**, 167 (2003); S. Boccaletti, V. Latora, Y. Moreno, M. Chavez, and D.-U. Hwang, *Phys. Rep.* **424**, 175 (2006).
- [2] M. R. Garey and D. S. Johnson, *Computers and Intractability: A Guide to the Theory of NP-Completeness* (W. H. Freeman, New York, 1979).
- [3] A. Pothen, H. Simon, and K.-P. Liou, *SIAM J. Matrix Anal. Appl.* **11**, 430 (1990).
- [4] S. Wasserman and K. Faust, *Social Networks Analysis* (Cambridge University Press, Cambridge, 1994).
- [5] G. Palla, I. Derenyi, I. Farkas, and T. Vicsek, *Nature (London)* **435**, 814 (2005); I. Derenyi, G. Palla, and T. Vicsek, *Phys. Rev. Lett.* **94**, 160202 (2005).
- [6] M. Girvan and M. E. J. Newman, *Proc. Natl. Acad. Sci. U.S.A.* **99**, 7821 (2002); M. E. J. Newman and M. Girvan, *Phys. Rev. E* **69**, 026113 (2004).
- [7] M. E. J. Newman, *Phys. Rev. E* **64**, 016132 (2001); U. Brandes, *J. Math. Sociol.* **25**, 163 (2001).
- [8] M. E. J. Newman, *Phys. Rev. E* **69**, 066133 (2004).
- [9] A. Clauset, M. E. J. Newman, and C. Moore, *Phys. Rev. E* **70**, 066111 (2004); F. Radicchi, C. Castellano, F. Cecconi, V. Loreto, and D. Parisi, *Proc. Natl. Acad. Sci. U.S.A.* **101**, 2658 (2004); R. Guimerá and L. A. N. Amaral, *Nature (London)* **433**, 895 (2005).
- [10] J. Reichardt and S. Bornholdt, *Phys. Rev. Lett.* **93**, 218701 (2004).
- [11] A. Arenas, A. Díaz-Guilera, and C. J. Pérez-Vicente, *Phys. Rev. Lett.* **96**, 114102 (2006).
- [12] S. Boccaletti, J. Kurths, D. L. Valladares, G. Osipov, and C. S. Zhou, *Phys. Rep.* **366**, 1 (2002).
- [13] A. Pluchino, V. Latora, and A. Rapisarda, *Int. J. Mod. Phys. C* **16**, 515 (2005).
- [14] Y. Kuramoto, *Chemical Oscillations, Waves, and Turbulence* (Springer, Berlin, 1984); S. H. Strogatz, *Physica D* **143**, 1 (2000).
- [15] M. Chavez, D. U. Hwang, A. Amann, H. G. E. Hentschel, and S. Boccaletti, *Phys. Rev. Lett.* **94**, 218701 (2005).
- [16] R. Hegselmann and U. Krause, *J.A.S.S.S.* 5, issue 3, paper 2 (2002).
- [17] L. Danon, A. Diaz-Guilera, J. Duch, and A. Arenas, *J. Stat. Mech.: Theory Exp.* 2005, P09008.
- [18] W. W. Zachary, *J. Anthropol. Res.* **33**, 452 (1977).
- [19] D. Baird and R. E. Ulanowicz, *Ecol. Monogr.* **59**, 329 (1989).
- [20] S. Fortunato, V. Latora, and M. Marchiori, *Phys. Rev. E* **70**, 056104 (2004).