Cluster Approximation for the Farey Fraction Spin Chain

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Joint work with

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Overview

- Definition of the Farey fraction spin chain
- Connection with other models, in particular
  - Number theoretical spin chain
- Thermodynamics, phase transition
- Coupling to external field
  - Rigorous results
  - Renormalization group analysis
  - Cluster approximation
- Dynamical systems analysis
  - Connection with Farey map
  - Piecewise linear map $\Rightarrow$ “correct” cluster approximation
- Full phase diagram in cluster approximation

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Farey Fraction Spin Chain - Definition

- Chain of N spins $\vec{\sigma} = \{\sigma_i\}_{i=1}^N$ with $\sigma_i \in \{\uparrow, \downarrow\}$
- Associate with each spin $\sigma_i \in \{\uparrow, \downarrow\}$ a matrix

$$A_\uparrow = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad A_\downarrow = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

- Energy of a configuration $\vec{\sigma}$

$$E_N(\vec{\sigma}) = \log \mathrm{Tr}\Big(\prod_i A_{\sigma_i}\Big)$$

- Partition function

$$Z_N(\beta) = \sum_{\vec{\sigma}} e^{-\beta E_N(\vec{\sigma})} = \sum_{\vec{\sigma}} \Big[\mathrm{Tr}\Big(\prod_i A_{\sigma_i}\Big)\Big]^{-\beta}$$

- Thermodynamic limit

$$-\beta f(\beta) = \lim_{N\to\infty} \frac{1}{N} \log Z_N(\beta)$$

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Farey Fraction Spin Chain - Motivation

- Number theoretical spin chain (Knauf, 1993)
  - Statistical mechanics interpretation of the Riemann ζ-function
  - Geodesic flow in the hyperbolic upper half plane $\mathbb{H}$
    thermodynamic formalism $\Rightarrow$ energy of a spin chain
- Farey Fraction spin chain (Kleban and Özlük, 1998)
  - Energy symmetric under spin flip, translation, inversion
    $\Rightarrow$ more “physical” model
- Farey tree model (Feigenbaum, Procaccia, Tel, 1989)
  - “Farey tree” instead of all Farey fractions
- Transfer operator of Farey map (Feigenbaum, 1988)
  - Complete spectral analysis (TP, 2003)

$\boxed{\text{All models have the same free energy}}$

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Number Theoretical Spin Chain

- $Z(s) = \zeta(s-1)/\zeta(s)$ has the Dirichlet series

$$Z(s) = \sum_{n=1}^{\infty} \frac{\varphi(n)}{n^s} \quad (\Re(s) > 2)$$

with Euler totient function $\varphi(n) = |\{j \in \{1, \dots, n\} | \gcd(j, n) = 1\}|$

- Define spin chain energies $H_N(\vec{\sigma}) = \log h_N(\vec{\sigma})$ via $h_0 = 1$ and

$$h_{N+1}(\vec{\sigma}, \uparrow) = h_N(\vec{\sigma}) \quad \text{and} \quad h_{N+1}(\vec{\sigma}, \downarrow) = h_N(\vec{\sigma}) + h_N(\vec{\sigma}^c)$$

- The partition function

$$Z_N^{(K)}(s) = \sum_{\vec{\sigma}} e^{-sH_N(\vec{\sigma})}$$

gives in the thermodynamic limit

$$Z(s) = \lim_{N \to \infty} Z_N^{(K)}(s) \quad (\Re(s) > 2)$$

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Connection to Farey Fraction Spin Chain

- Pascal's triangle with memory: $h_N(\sigma)$ in lexicographic ordering

N=0	1																(1)
N=1	1								2								(1)
N=2	1				3				2				3				(1)
N=3	1		4		3		5		2		5		3		4		(1)
N=4	1	5	4	7	3	8	5	7	2	7	5	8	3	7	4	5	(1)

- Denominators of the set F_N of modified Farey fractions of order N:

$$F_3 = \left\{ \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1} \right\}$$

- Connection with Farey fraction spin chain

$$\lim_{N\to\infty} \frac{1}{N} \log Z_N(\beta = s) = \lim_{N\to\infty} \frac{1}{N} \log Z_N^{(K)}(s) \quad (s \in \mathbb{R})$$

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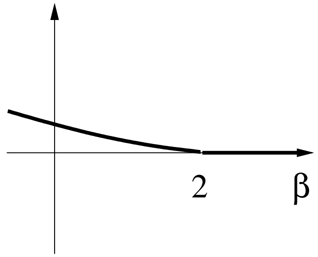
Phase Transition

Mathematically and physically interesting system

- One-dimensional spin chain with phase transition at $\beta_c = 2$
- For $-\beta f(\beta) = \lim_{N\to\infty} \frac{1}{N} \log Z_N(\beta)$ we have

  Fiala et al (2003) using results from TP (1992)

  - $-\beta f(\beta)$ analytic in $\beta < \beta_c$
  - $-\beta f(\beta) \sim \frac{\beta_c - \beta}{-\log(\beta_c - \beta)}$ as $\beta \to \beta_c^-$
  - $-\beta f(\beta) = 0 \qquad \forall \beta \geq \beta_c$



- Necessarily long-range interactions
- High temperature state is paramagnetic
- Low temperature state is completely ordered, no thermal effects
- The phase transition is second-order, but the magnetization jumps at β_c from saturation to zero (first-order like)

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Farey Fraction Spin Chain with Field

In the TDL both spin chains are equivalent, so why bother?

- Number theoretic spin chain mathematically more fundamental (connection to ζ-function)
- Farey fraction spin chain has physical symmetries typical for spin systems
  - Translation: $E_N(\sigma_1, \sigma_2, \ldots, \sigma_N) = E_N(\sigma_2, \ldots, \sigma_N, \sigma_1)$
  - Inversion: $E_N(\sigma_1, \sigma_2, \ldots, \sigma_N) = E_N(\sigma_N, \ldots, \sigma_2, \sigma_1)$
  - Spin flip: $E_N(\sigma_1, \sigma_2, \ldots, \sigma_N) = E_N(\sigma_1^c, \sigma_2^c, \ldots, \sigma_N^c)$
- Natural generalization: coupling to external magnetic field h
  - Add energy proportional to difference between $\uparrow$ and $\downarrow$ spins

$$E_N(\vec{\sigma}, h) = E_N(\vec{\sigma}) + h \sum_i \big(\chi_\uparrow(\sigma_i) - \chi_\downarrow(\sigma_i)\big)$$

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A Rigorous Result

For $\beta > 2$ and $h \neq 0$, the system is fully magnetized:

Consider free energy $$-\beta f(\beta, h) = \lim_{N\to\infty} \frac{1}{N} \log Z_N(\beta, h)$$

- $\beta > \beta_c = 2$:
  - For $h > 0$ we have
    $$2^{-\beta} e^{\beta h N} < Z_N(\beta, h) < Z_N(\beta, 0) e^{\beta h N}$$
  - It follows
    $$h \leq -f(\beta, h) \leq -f(\beta, 0) + h$$
    with $f(\beta, 0) = 0$
- Thus
  $$\boxed{f(\beta, h) = -|h| \text{ for } \beta > 2}$$

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Renormalization Group Analysis

Fiala and Kleban (2004)

- Mean field expansion $f_{MF} = a + btM^2 + uM^4 - ghM + \ldots$
- Two relevant fields $t = 1 - \beta/\beta_c$ and h, one marginal field u
- RG transformation for singular part $f_s(t, h, u)$
- Result for high-temperature phase

$$f_s(t,h,u) \sim \left|\frac{t}{t_0}\right| \left(\frac{x}{y_t} u \log \frac{t_0}{t}\right)^{-1} a - \frac{h^2}{t} \left(\frac{x}{y_t} u \log \frac{t_0}{t}\right) \frac{3g^2}{16b}$$

(x, y_t are scaling exponents)

- Combine with low-temperature result to get phase boundary

$$\boxed{-|h| \sim t/\log t}$$

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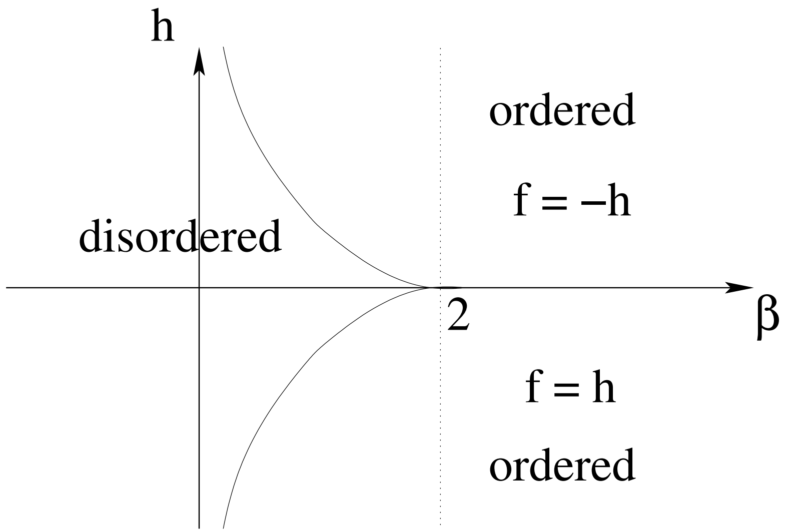
Phase Diagram from RG



- Disordered phase, small field: $t = 1 - \beta/\beta_c$

$$f(\beta, h) \sim a\frac{t}{\log t} - b\frac{h^2 \log t}{t}$$

- Phase boundary, $h_c = |h| = -f$:

$$h_c(\beta) \sim -a\frac{t}{\log t}$$

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Spin Cluster Representation

- Observation: $A_\uparrow^n = \left(\begin{smallmatrix}1 & n\\ 0 & 1\end{smallmatrix}\right)$, $A_\downarrow^n = \left(\begin{smallmatrix}1 & 0\\ n & 1\end{smallmatrix}\right)$ leads to ground state energy
$$\log \mathrm{Tr}\left(A_\uparrow^n\right) = \log \mathrm{Tr}\left(A_\downarrow^n\right) = \log 2$$
- Describe excited states by a sequence of $2K$ clusters of size n_k
$$\underbrace{\uparrow\cdots\uparrow}_{n_1}\underbrace{\downarrow\cdots\downarrow}_{n_2}\underbrace{\uparrow\cdots\uparrow}_{n_3}\downarrow\cdots\uparrow\underbrace{\downarrow\cdots\downarrow}_{n_{2K}}$$
- Using $A_\downarrow = SA_\uparrow S^{-1}$ with $S = \left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right)$ we write
$$\mathrm{Tr}\Big(\prod_i A_{\sigma_i}\Big) = \mathrm{Tr}\Big(\prod_k M_{n_k}\Big)\ , \qquad M_n = A_\uparrow^n S = \left(\begin{smallmatrix}0 & 1\\ 1 & n\end{smallmatrix}\right)$$
- n_k large
$$\log \mathrm{Tr}\Big(\prod_i A_{\sigma_i}\Big) \approx \sum_k \log \mathrm{Tr}\left(M_{n_k}\right) = \sum_k \log n_k$$
- Cluster approximation $\boxed{E_N \approx \sum_k \epsilon_{n_k}}$

Connection with Dynamical Systems

- $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{R})$ acts on upper half plane $\mathbb{H} = \{z \in \mathbb{C} | \Im(z) > 0\}$ by Möbius transformations:

$$z \mapsto \hat{M}(z) = \frac{az+b}{cz+d}$$

- Modular group Γ:

$$\Gamma = \mathrm{PSL}(2,\mathbb{Z}) = \mathrm{SL}(2,\mathbb{Z})/\{\pm\mathbf{1}\}$$

- For deeper analysis consider hyperbolic metric on $\mathbb{H}$
  - $\hat{M}$ leaves $\Delta = y^2(\partial_x^2 + \partial_y^2)$ invariant
  - Scattering in modular domain $\Gamma\backslash\mathbb{H}$
  - Geodesic flow along arcs in $\mathbb{H}$
- Here it suffices to consider transformations on $[0,1] \subset \mathbb{R} = \partial\mathbb{H}$

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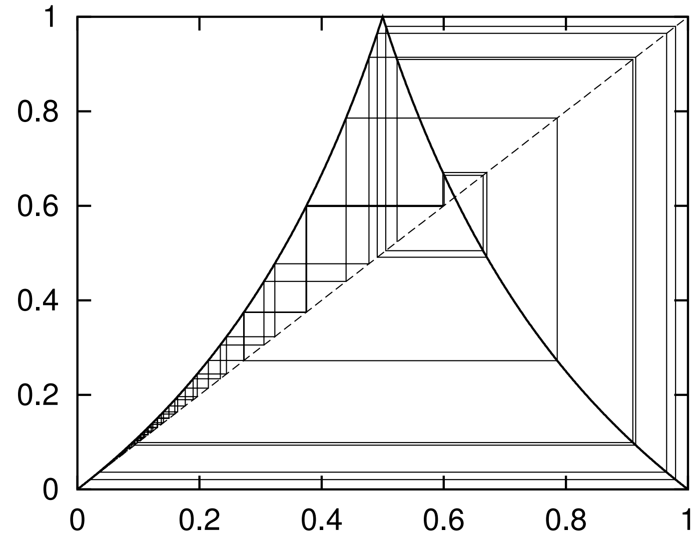
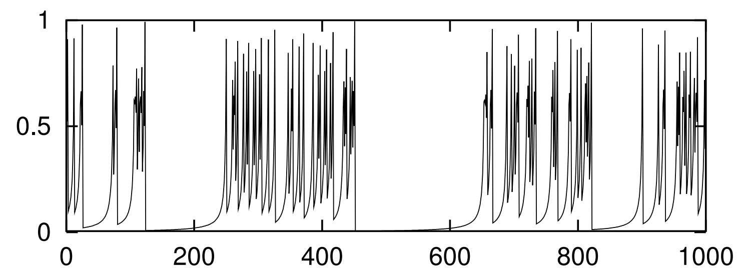
Farey Map

- Farey map on $[0,1]$

$$f(x) = \begin{cases} \frac{x}{1-x}, & x \le 1/2 \\ \frac{1-x}{x}, & 1/2 < x \end{cases}$$



- Toy model for intermittency
  - “intermittent” left branch
  - “chaotic” right branch



- $f'(0) = 1$: almost expanding (non-uniformly expanding)

- Invariant density $\rho(x) = 1/x$ not normalizable

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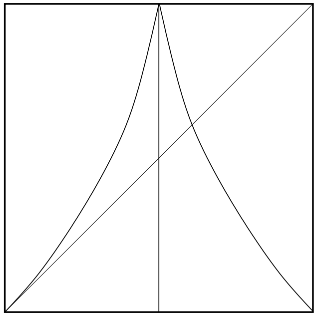
Farey Operator

- Transfer operator



$$\mathcal{L}\varphi(x) = \sum_{f(y)=x} |f'(y)|^{-\beta}\varphi(y)$$
$$= \frac{1}{(1+x)^{2\beta}}\varphi\left(\frac{x}{1+x}\right) + \frac{1}{(1+x)^{2\beta}}\varphi\left(\frac{1}{1+x}\right)$$

- Transformations correspond to matrices

$$L_0 = \left(\begin{smallmatrix}1 & 0\\ 1 & 1\end{smallmatrix}\right) \quad \text{and} \quad L_1 = \left(\begin{smallmatrix}0 & 1\\ 1 & 1\end{smallmatrix}\right)$$

- Notice that with $P = \left(\begin{smallmatrix}0 & 0\\ 0 & 1\end{smallmatrix}\right)$

$$\mathcal{L}_\beta^N \mathbf{1}(0) = \sum_{\vec{\tau}\in\{0,1\}^N} \left[\mathrm{Tr}\big(P\prod_i L_{\tau_i}\big)\right]^{-2\beta} = 2Z_{N-1}^{(K)}(2\beta)$$

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Farey Operator

- Observe

$$L_0^{n-1} L_1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & n \end{smallmatrix}\right) = M_n$$

- Similar cluster expansions for $Z_N(\beta)$ and $\mathcal{L}_\beta^N \mathbf{1}(0)$
  - For $Z_N(\beta)$, the sum involves two groundstates and clusters with $\sum_{k=1}^{2K} n_k = N$ of multiplicity $2n_1$
  - For $\mathcal{L}_\beta^N \mathbf{1}(0)$, the sum involves clusters with $\sum_{k=1}^{K} n_k = N+1$
- We can relate the asymptotic behavior of $Z_N(\beta)$ to the spectral properties of the transfer operator $\mathcal{L}_\beta$
- In the thermodynamic limit

$$\lim_{N\to\infty} \frac{1}{N} \log Z_N(2\beta) = \lim_{N\to\infty} \frac{1}{N} \log \mathcal{L}_\beta^N \mathbf{1}(0) = \log r(\mathcal{L}_\beta) \quad (\beta \in \mathbb{R})$$

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First Return Map

Interpretation of

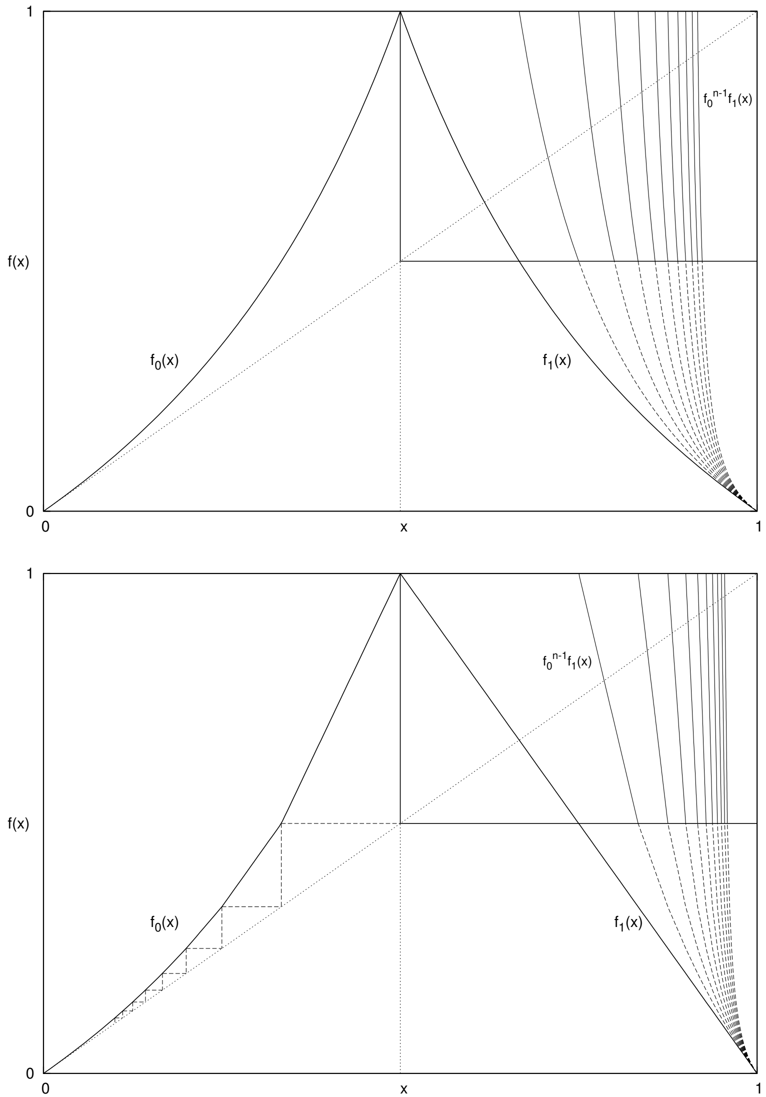
$$M_n = L_0^{n-1} L_1 = M_n$$

- Consider first return map on $[1/2, 1]$
- Branches $g_n = f^n|_{[1/2,1]} = {f_0}^{n-1} \circ f_1$
- Conjugate to Gauss map

$$x \mapsto 1/x \mod 1$$

Linearize first return map

- Branches $|\tilde{g}_n'| = n(n+1)$



Cluster Approximation for Farey Map

- Approximating with the linearized map we find

$$\mathcal{L}_\beta^N \mathbf{1}(0) \approx \tilde{\mathcal{L}}_\beta^N \mathbf{1}(0) = \sum_{\sum n_k = N+1} \prod_{k=1}^{K} e^{-\beta \epsilon_{n_k}}$$

with effective cluster interactions $\epsilon_n = \log n(n+1)$

- For the generating function $G(z,\beta) = \sum_{N=0}^{\infty} z^N \mathcal{L}_\beta^N \mathbf{1}(0)$ we find

$$G(z,\beta) \approx \frac{1}{z} \sum_{K=1}^{\infty} \Big(\sum_{n=1}^{\infty} z^n e^{-\beta \epsilon_n} \Big)^K = \frac{1}{z} \frac{\Lambda(z,\beta)}{1-\Lambda(z,\beta)}$$

with cluster generating function $\Lambda(z,\beta) = \sum_{n=1}^{\infty} z^n e^{-\beta \epsilon_n}$

- Singularities of $G(z,\beta) \Rightarrow$ asymptotics of $\mathcal{L}_\beta^N \mathbf{1}(0)$

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Farey Fraction Spin Chain without Field

- Using the effective cluster interactions

$$\epsilon_n = \frac{1}{2} \log n(n+1)$$

we define the cluster generating function

$$\Lambda(z,\beta) = \sum_{n=1}^{\infty} z^n e^{-\beta\epsilon_n}$$

- Identifying $z_c = e^{\beta f}$ with $f = f(\beta)$ we obtain

$$\boxed{\Lambda(e^{\beta f},\beta) = 1 \quad (\beta < 2)}$$

- The singularity of $\Lambda(z,\beta)$ at $z_c = 1$ implies

$$f = 0 \quad (\beta \geq 2)$$

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Farey Fraction Spin Chain with Field

- A slightly more involved calculation leads to an implicit expression for $f = f(\beta, h)$

$$\boxed{\Lambda(e^{\beta(f-h)}, \beta)\Lambda(e^{\beta(f+h)}, \beta) = 1}$$

- The phase boundary is reached for $e^{\beta(f\pm h)} = 1$, i.e.

$$f = -|h|$$

  and $h_c(\beta)$ satisfies

$$\boxed{\Lambda(1, \beta)\Lambda(e^{-2\beta|h_c|}, \beta) = 1}$$

- The RG result is confirmed by an asymptotic analysis of

$$\Lambda(z, \beta) = \sum_{n=1}^{\infty} \frac{z^n}{[n(n+1)]^{\beta/2}} \sim 1 + Ct + (1-z)\log(1-z)$$

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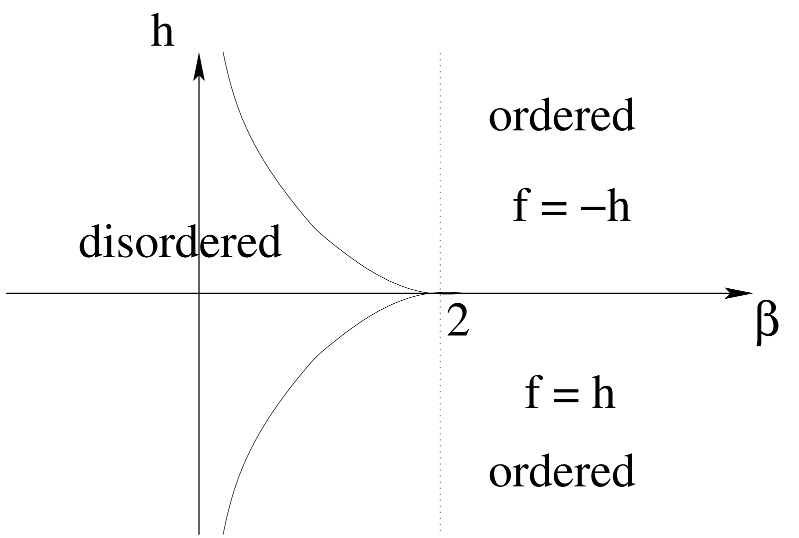
Phase Diagram revisited



- Disordered phase, small field: $t = 1 - \beta/\beta_c$

$$f(\beta, h) \sim \frac{C}{\beta_c} \frac{t}{\log t} - \frac{\beta_c}{2C} \frac{h^2}{t} \quad \text{where } |h| \ll |t/\log t|$$

- Phase boundary, $h_c = |h| = -f$:

$$h_c(\beta) \sim -\frac{C}{\beta_c} \frac{t}{\log t}$$

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Summary and Outlook

- Summary
  - Farey fraction spin chain and other models
  - Coupling to external field
  - Rigorous results, RG calculation
  - Dynamical System Interpretation $\Rightarrow$ cluster approximation
  - Leads to explicit expressions for free energy
- Outlook
  - Utilize first return map without approximation
  - Possibility of rigorous work using transfer operators and operator relations directly

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