Counting Generalised Dyck Paths

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OUTLINE

Lattice Paths

In Combinatorics, a lattice path is a sequence of points on some regular lattice $\mathbb{Z}^n$.

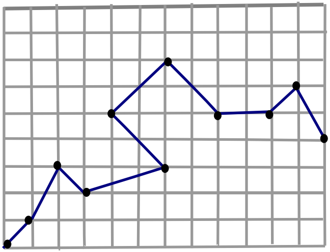

Figure 1: Lattice path on $\mathbb{Z}^2$.

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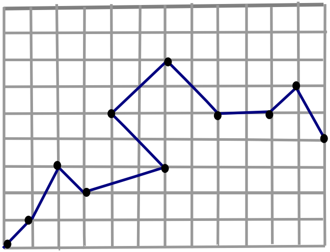

Figure 1: Lattice path on $\mathbb{Z}^2$.

Directed lattice paths have fixed direction of increase, which we choose to be the positive horizontal axis.

Types of Directed Lattice Paths

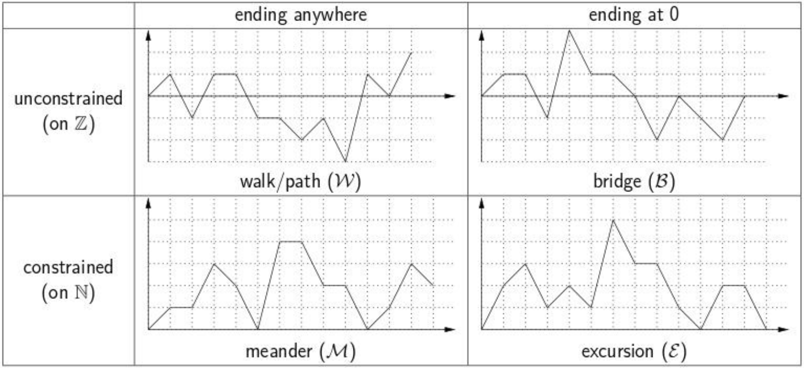

Figure 2: Types of paths (Banderier and Wallner, 2016).

Dyck Paths

A Dyck path is a staircase walk from $(0,0)$ to (n,n) lying below the diagonal $y=x$.

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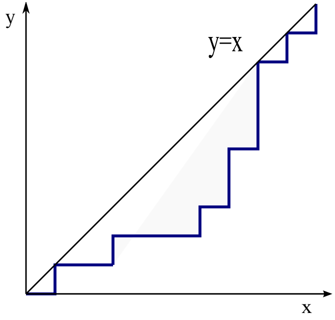

Figure 3: Dyck Path.

Problem Definition

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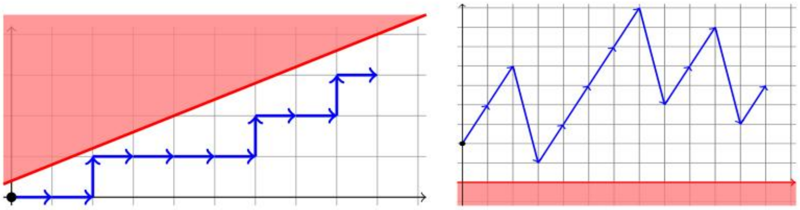

Figure 4: Left: Path below a rational slope 2/5. Right: Bijection to Meanders (Banderier and Wallner, 2015).

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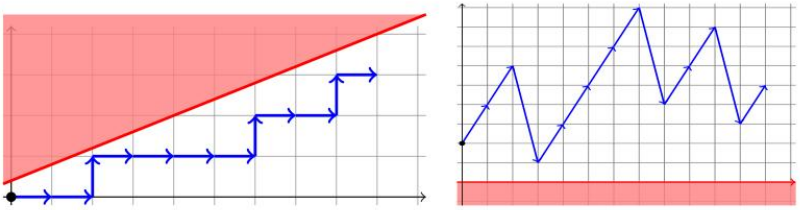

Figure 4: Left: Path below a rational slope 2/5. Right: Bijection to Meanders (Banderier and Wallner, 2015).

Dyck paths below rational slopes are in bijection with lattice paths with more general step sets.

Generalised Dyck Paths

A generalised Dyck path in a slit takes its steps from $S \subset \mathbb{Z}$. It can start and end at any height.

Let $A = S \cap \mathbb{Z}_0^+$ and $B = -(S \backslash A)$ where $\alpha > 0$ is the maximum up step in A and $\beta > 0$ is the maximum down step in B.

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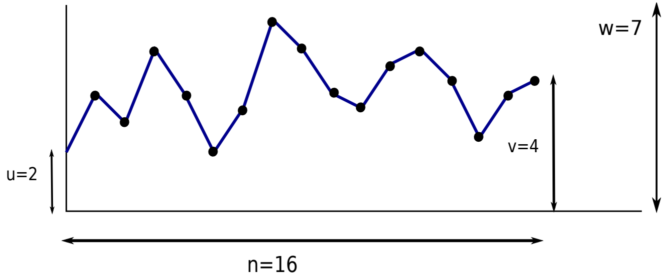

Figure 5: Generalised Dyck path of length $n = 16$ with north-east steps $A = \{1, 3, 4, 5, 6\}$ and south-east steps $B = \{1, 2, 3, 4\}$.

Generating Function

Considering an an n step path in a slit of width w, starting at height u and ending at height v we have the generating function

$$G(t,z) \equiv G_u^{w,\alpha,\beta}(t,z) = \sum_{v=0}^{w} G_{(u,v)}^{w,\alpha,\beta}(t) z^v,$$

where $G_{(u,v)}^{w,\alpha,\beta}(t)$ is the generating function of paths and t is conjugate to the length n of the path.

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Question

What is the number of paths starting at height u and ending at height v with maximum up step α and maximum down step β?

Functional Equation

$$G(t,z) = z^u + t\left(\sum_{a\in A} p_a z^a + \sum_{b\in B}\frac{q_b}{z^b}\right)G(t,z)$$
$$- t\sum_{j=1}^{\infty} z^{w+j}\sum_{a\geq j} p_a G_{(u,w-a+j)}(t) - t\sum_{j=1}^{\infty} z^{-j}\sum_{b\geq j} q_b G_{(u,b-j)}(t)$$

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The kernel $K(t,z)$ of the functional equation is

$$K(t,z) = 1 - t\sum_{a\in A} p_a z^a - t\sum_{b\in B}\frac{q_b}{z^b}.$$

Elementary Symmetric Functions

Using

$$K(t,z) = \xi \prod_{i=1}^{\alpha+\beta} (z - z_i) = \xi \sum_{i=0}^{\alpha+\beta} z^{\alpha+\beta-i} (-1)^i e_i$$

We rewrite our functional equation in terms of elementary symmetric functions instead of weights p_a, q_b and t.

Extracting the coefficients of z^v yields a system of $w+1$ linear equations.

Cramer's Rule

$$\sum_{v=0}^{w}\left(\sum_{i=0}^{\alpha+\beta}(-1)^i e_i G_{(u,v-\alpha+i)}\right) z^v = -\frac{z^u}{tp_\alpha}$$

We evaluate the equation for $v = 0 \cdots w$.

Matrix Equation

$$\begin{bmatrix} (-1)^\alpha e_\alpha & (-1)^{\alpha+1} e_{\alpha+1} & \cdots & (-1)^{\alpha+\beta} e_{\alpha+\beta} & \cdots & 0 \\ (-1)^{\alpha-1} e_{\alpha-1} & (-1)^\alpha e_\alpha & \cdots & (-1)^{\alpha+\beta-1} e_{\alpha+\beta-1} & \cdots & 0 \\ (-1)^{\alpha-2} e_{\alpha-2} & (-1)^{\alpha-1} e_{\alpha-1} & \cdots & (-1)^{\alpha+\beta-2} e_{\alpha+\beta-2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ e_0 & -e_1 & \cdots & (-1)^\beta e_\beta & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & (-1)^\alpha e_\alpha \end{bmatrix} \begin{bmatrix} G_{(u,0)} \\ G_{(u,1)} \\ G_{(u,2)} \\ \vdots \\ G_{(u,w)} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ \frac{-1}{tp_\alpha} \\ \vdots \\ 0 \end{bmatrix}$$

As per Cramer's rule

$$G_{(u,v)} = \frac{|A_{(u,v)}|}{|A|}$$

where

$$A = \begin{bmatrix} e_\alpha & e_{\alpha+1} & e_{\alpha+2} & \cdots & e_{\alpha+\beta} & \cdots & 0 \\ e_{\alpha-1} & e_\alpha & e_{\alpha+1} & \cdots & e_{\alpha+\beta-1} & \cdots & 0 \\ e_{\alpha-2} & e_{\alpha-1} & e_\alpha & \cdots & e_{\alpha+\beta-2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ e_0 & e_1 & e_2 & \cdots & e_\beta & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & e_\alpha \end{bmatrix}$$

Using similarity transformation we remove negative signs from the alternate entries.

Jacobi Trudi Identities

The second Jacobi-Trudi formula expresses the Schur polynomial as a determinant in terms of the elementary symmetric polynomials,

Schur polynomial as determinant

$$s_\lambda = det(e_{\lambda'_i+j-i})_{i,j=1}^{l(\lambda')} = \begin{vmatrix} e_{\lambda'_1} & e_{\lambda'_1+1} & e_{\lambda'_1+2} & \cdots & e_{\lambda'_1+l-1} \\ e_{\lambda'_2-1} & e_{\lambda'_2} & e_{\lambda'_2+1} & \cdots & e_{\lambda'_1+l-2} \\ e_{\lambda'_3-2} & e_{\lambda'_3-1} & e_{\lambda'_3} & \cdots & e_{\lambda'_1+l-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ e_{\lambda'_l-l+1} & e_{\lambda'_l-l+2} & e_{\lambda'_l-l+3} & \cdots & e_{\lambda'_l} \end{vmatrix}$$

where λ' is the conjugate partition.

We need to evaluate $|A|$, which is done by comparison with the second Jacobi-Trudi formula. The conjugate partition λ' is

$$\lambda' = (\overbrace{\alpha, \alpha, \cdots, \alpha}^{\text{w+1}})$$

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Determinant of A

$$|A| = S_{(w+1^{\alpha}, 0^{\beta})}(z_1, z_2, \cdots, z_{\alpha+\beta}).$$

Next we need to find $A_{(u,v)}$ by replacing the column v of matrix A with B with single non zero entry $-\frac{1}{tp_\alpha}$.

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$$A_{(u,v)} = \begin{bmatrix} e_\alpha & e_{\alpha+1} & e_{\alpha+2} & \cdots & e_{\alpha+v-1} & 0 & \cdots & e_{\alpha+\beta} & \cdots & 0 \\ e_{\alpha-1} & e_\alpha & e_{\alpha+1} & \cdots & e_{\alpha+v-2} & 0 & \cdots & e_{\alpha+\beta-1} & \cdots & 0 \\ e_{\alpha-2} & e_{\alpha-1} & e_\alpha & \cdots & e_{\alpha+v-3} & 0 & \cdots & e_{\alpha+\beta-2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ e_{\alpha-u+1} & e_{\alpha-u+2} & e_{\alpha-u+3} & \cdots & e_{\alpha+v-u} & 0 & \cdots & e_{\alpha+\beta-u+1} & \cdots & 0 \\ e_{\alpha-u} & e_{\alpha-u+1} & e_{\alpha-u+2} & \cdots & e_{\alpha+v-u-1} & -\frac{1}{tp_\alpha} & \cdots & e_{\alpha+\beta-u} & \cdots & 0 \\ e_{\alpha-u-1} & e_{\alpha-u} & e_{\alpha-u+1} & \cdots & e_{\alpha+v-u-2} & 0 & \cdots & e_{\alpha+\beta-u-1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ e_0 & e_1 & e_2 & \cdots & e_{v-1} & 0 & \cdots & e_\beta & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \cdots & e_\alpha \end{bmatrix}_{[w+1]}$$

To evaluate the determinant we expand the matrix by v^{th} column. This gives us a new matrix of size w as follows

$$
-\frac{1}{tp_\alpha}\begin{vmatrix}
e_\alpha & e_{\alpha+1} & e_{\alpha+2} & \cdots & e_{\alpha+v-1} & e_{\alpha+v+1} & \cdots & e_{\alpha+\beta} & \cdots & 0 \\
e_{\alpha-1} & e_\alpha & e_{\alpha+1} & \cdots & e_{\alpha+v-2} & e_{\alpha+v} & \cdots & e_{\alpha+\beta-1} & \cdots & 0 \\
e_{\alpha-2} & e_{\alpha-1} & e_\alpha & \cdots & e_{\alpha+v-3} & e_{\alpha+v-1} & \cdots & e_{\alpha+\beta-2} & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
e_{\alpha-u+1} & e_{\alpha-u+2} & e_{\alpha-u+3} & \cdots & e_{\alpha+v-u} & e_{\alpha+v-u+2} & \cdots & e_{\alpha+\beta-u+1} & \cdots & 0 \\
e_{\alpha-u-1} & e_{\alpha-u} & e_{\alpha-u+1} & \cdots & e_{\alpha+v-u-2} & e_{\alpha+v-u-1} & \cdots & e_{\alpha+\beta-u-1} & \cdots & 0 \\
e_{\alpha-u-2} & e_{\alpha-u-1} & e_{\alpha-u} & \cdots & e_{\alpha+v-u-3} & e_{\alpha+v-u-1} & \cdots & e_{\alpha+\beta-u-2} & \cdots & 0 \\
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e_0 & e_1 & e_2 & \cdots & e_{v-1} & e_{v+1} & \cdots & e_\beta & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \cdots & e_\alpha
\end{vmatrix}
$$

The corresponding Jacobi-Trudi identity expressing Skew Schur polynomials as determinant is

Skew Schur polynomial as determinant

$$s_{\lambda'/\mu'} = \begin{vmatrix} e_{\lambda'_1-\mu'_1} & e_{\lambda'_1-\mu'_2+1} & e_{\lambda'_1-\mu'_3+2} & \cdots & e_{\lambda'_1-\mu'_l+l-1} \\ e_{\lambda'_2-\mu'_1-1} & e_{\lambda'_2-\mu'_2} & e_{\lambda'_2-\mu'_3+1} & \cdots & e_{\lambda'_1-\mu'_l+l-2} \\ e_{\lambda'_3-\mu'_1-2} & e_{\lambda'_3-\mu'_2-1} & e_{\lambda'_3-\mu'_3} & \cdots & e_{\lambda'_1-\mu'_l+l-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ e_{\lambda'_l-\mu'_1-l+1} & e_{\lambda'_l-\mu'_2-l+2} & e_{\lambda'_l-\mu'_3-l+3} & \cdots & e_{\lambda'_l-\mu'_l} \end{vmatrix}$$

Comparing the matrix with the Jacobi identity for skew functions, we get the conjugate partitions

Conjugate Partitions

$$\lambda' = (\overbrace{\alpha+1, \alpha+1, \cdots, \alpha+1}^{\text{u}}, \overbrace{\alpha, \alpha, \cdots, \alpha}^{\text{w-u}})$$

and

$$\mu' = (\overbrace{1, 1, \cdots, 1}^{\text{v}}, \overbrace{0, 0, \cdots, 0}^{\text{w-v}})$$

The partitions λ and μ are given by

Partitions

$$\lambda = (w^{\alpha}, u, 0^{\beta-1})$$

and

$$\mu = (v, 0^{\alpha+\beta-1})$$

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$$\lambda = (w^{\alpha}, u, 0^{\beta-1})$$

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Skew Schur Function

$$S_{(w^{\alpha},u,0^{\beta-1})/(v,0^{\alpha+\beta-1})}(z_1, z_2, \cdots, z_{\alpha+\beta})$$

We have

$$|A_{(u,v)}| = -\frac{1}{tp_\alpha} S_{(w^\alpha,u,0^{\beta-1})/(v,0^{\alpha+\beta-1})}(z_1, z_2, \cdots, z_{\alpha+\beta})$$

and

We have

$$|A_{(u,v)}| = -\frac{1}{tp_\alpha} S_{(w^\alpha,u,0^{\beta-1})/(v,0^{\alpha+\beta-1})}(z_1, z_2, \cdots, z_{\alpha+\beta})$$

and

$$|A| = S_{(w+1^\alpha,0^\beta)}(z_1, z_2, \cdots, z_{\alpha+\beta}).$$

Result

Theorem

The generating function of generalised Dyck paths in a slit of width w is

$$G_{(u,v)}(t) = -\frac{1}{tp_\alpha} \frac{S_{(w^\alpha,u,0^{\beta-1})/(v,0^{\alpha+\beta-1})}(z)}{S_{(w+1^\alpha,0^\beta)}(z)}.$$

Here z are the roots of kernel $1 - t\sum_{a\in A} p_a z^a - t\sum_{b\in B} \frac{q_b}{z^b}$.

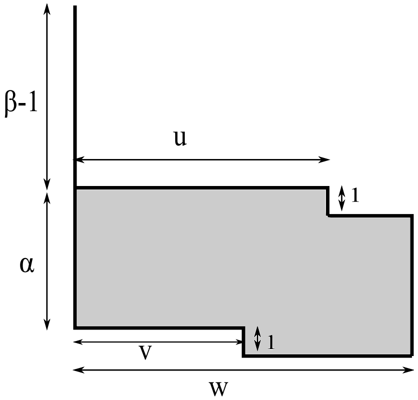

Figure 6: Schur diagram for the skew Schur function.

Result

Schur functions form a linear basis for symmetric polynomials. An equivalent result in terms of Schur functions is as follows:

Theorem

The generating function of generalised Dyck paths in a slit of width w is

$$-\frac{1}{tp_\alpha}\frac{\sum_{l=0}^{min(u,v,w-u,w-v)} S_{(w^{\alpha-1},w-(v-u)_+-l,(u-v)_++l,0^{\beta-1})}(z)}{S_{(w+1^\alpha,0^\beta)}(z)}.$$

where z are the roots of kernel $0 = 1 - t\sum_{a\in A} p_a z^a - t\sum_{b\in B} \frac{q_b}{z^b}$.

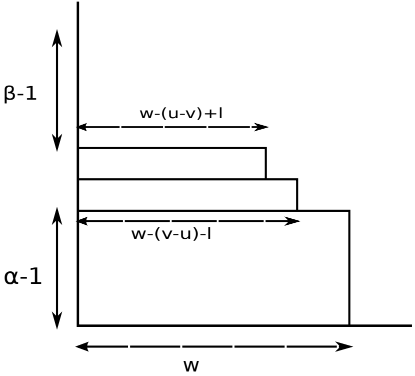

Figure 7: Schur diagram for the terms in summation.

Proof

The proof is based on

Lemma

Let λ/ν be a partition with ν a horizontal strip of size v. Then

$$S_{\lambda/\nu} = \sum_{\mu} S_{\mu}$$

where the sum runs over all partitions μ such than λ/μ is a horizontal strip of size v.

Thank
You.