The combinatorics of the leading root of Ramanujan's (and related) functions

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Combinatorics, Algebra, and More
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Topic Outline

Outline

Ramanujan's Function

$$A_q(x) = \sum_{n=0}^{\infty} \frac{q^{n^2}(-x)^n}{(q;q)_n} = \sum_{n=0}^{\infty} \frac{q^{n^2}(-x)^n}{(1-q)(1-q^2)\dots(1-q^n)}$$

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Rogers-Ramanujan Identities

$$A_q(-1) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})}$$

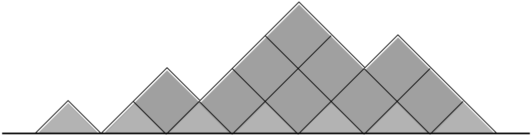
and

$$A_q(-q) = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+2})(1-q^{5n+3})}$$

Related to partitions of integers into parts mod 5

Area-weighed Dyck paths

Count Dyck paths with respect to steps and enclosed area



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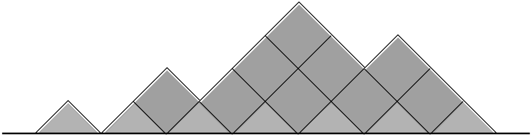

Generating function

$$G(x,q)=\sum_{m,n} c_{m,n}x^n q^m=\frac{A_q(x)}{A_q(x/q)}$$

x counts pairs of up/down steps, q counts enclosed area

Ramanujan's Lost Notebook, page 57

$$\sum_{n=0}^{\infty} \frac{q^{n^2}(-x)^n}{(q;q)_n} = \prod_{n=1}^{\infty} \left(1 - \frac{xq^{2n-1}}{1 - q^n y_1 - q^{2n} y_2 - q^{3n} y_3 - \dots}\right)$$

where

$$(t;q)_n = (1-t)(1-tq)(1-tq^2)\dots(1-tq^{n-1})$$

with y_1, y_2, y_3, y_4 explicitly given

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with y_1, y_2, y_3, y_4 explicitly given

More precisely,

$$y_1 = \frac{1}{(1-q)\psi^2(q)}, \quad y_2 = 0$$

$$y_3 = \frac{q+q^3}{(1-q)(1-q^2)(1-q^3)\psi^2(q)} - \frac{\sum_{n=0}^{\infty} \frac{(2n+1)q^{2n+1}}{1-q^{2n+1}}}{(1-q)^3\psi^6(q)}, \quad y_4 = y_1 y_3$$

and

$$\psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}$$

The roots of $A_q(x)$

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$$= \prod_{n=0}^{\infty} \left(1 - \frac{x}{x_n(q)}\right)$$

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$$= \prod_{n=0}^{\infty}\left(1 - \frac{x}{x_n(q)}\right)$$

- Roots are positive, real, and simple (Al-Salam and Ismail, 1983)
- Ramanujan's expansion is an asymptotic series (Andrews, 2005)
- Relation to Stieltjes-Wigert polynomials (Andrews, 2005)
- Integral equation for roots (Ismail and Zhang, 2007)
- Combinatorial interpretation of y_k (Huber, 2008, and Huber and Yee, 2010)

The aim of this talk

$$\sum_{n=0}^{\infty} \frac{q^{n^2}(-x)^n}{(q;q)_n} = \prod_{n=0}^{\infty}\left(1 - \frac{x}{x_n(q)}\right)$$

$$qx_0(-q) = 1 + q + q^2 + 2q^3 + 4q^4 + 8q^5 + 16q^6 + 33q^7 + 70q^8 + 151q^9 + \ldots$$

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Goal

A combinatorial interpretation of the coefficients of the leading root

Painlevé Airy Function

$$\mathrm{Ai}_q(x) = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}(-x)^n}{(q^2;q^2)_n} = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}(-x)^n}{(1-q^2)(1-q^4)\dots(1-q^{2n})}$$

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$$x_0(q) = 1 + q + q^2 + 2q^3 + 3q^4 + 6q^5 + 12q^6 + 25q^7 + 54q^8 + 120q^9 + \dots$$

The coefficients of the leading root also seem to be positive integers

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Connection Formula (Morita, 2011)

$$A_{q^2}\left(-\frac{q^3}{x^2}\right) = \frac{1}{(q;q)_\infty(-1;q)_\infty}\left\{\Theta\left(-\frac{x}{q},q\right)\mathrm{Ai}_q(-x) + \Theta\left(\frac{x}{q},q\right)\mathrm{Ai}_q(x)\right\}$$

with Theta Function

$$\Theta(x,q) = \sum_{n=-\infty}^{\infty} q^{\binom{n}{2}}(-x)^n$$

Partial Theta Function

The Theta Function has roots $x_k(q) = q^k \quad k \in \mathbb{Z}$

$$\Theta(x,q) = \sum_{n=-\infty}^{\infty} q^{\binom{n}{2}}(-x)^n = (q;q)_\infty (x;q)_\infty (q/x;q)_\infty$$

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The Partial Theta Function

$$\Theta_0(x,q) = \sum_{n=0}^{\infty} q^{\binom{n}{2}}(-x)^n$$

does not admit a "nice" product formula, but

$$x_0(q) = 1+q+2q^2+4q^3+9q^4+21q^5+52q^6+133q^7+351q^8+948q^9+\dots$$

The coefficients of the leading root are positive integers (Sokal, 2012)

Outline

1. q-Airy (and related) Functions
2. Identities for the Leading Roots
   - The Leading Roots
   - Key Identities
   - Positivity
3. Combinatorics
4. Outlook

The Leading Roots

Partial Theta Function $\Theta_0(x,q) = \sum_{n=0}^{\infty} q^{\binom{n}{2}}(-x)^n$

$\Theta_0(x,q) = 0$

$x = 1 + q + 2q^2 + 4q^3 + 9q^4 + 21q^5 + 52q^6 + 133q^7 + 351q^8 + 948q^9 + \ldots$

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Ramanujan Function $A_q(x) = \sum_{n=0}^{\infty} q^{n^2}(-x)^n/(q;q)_n$

$$A_{-q}(x/q) = 0$$

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Key Identities

$\Theta_0(x,q)$ satisfies (Andrews and Warnaar, 2007)

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$\mathrm{Ai}_q(x)$ satisfies (Gessel and Stanton, 1983)

$$\mathrm{Ai}_q(x) = (x;q)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2+\binom{n}{2}} x^n}{(q^2;q^2)_n (x;q)_n}$$

Identities for the Roots

Partial Theta Function (Sokal, 2012)

$\Theta_0(x, q) = 0$ if
$$x = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2} x^n}{(q;q)_n (qx;q)_{n-1}}$$

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Positivity

- Letting $x^{(0)} = 1$ and iterating

$$x^{(N+1)} = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q;q)_n (qx^{(N)};q)_{n-1}}$$

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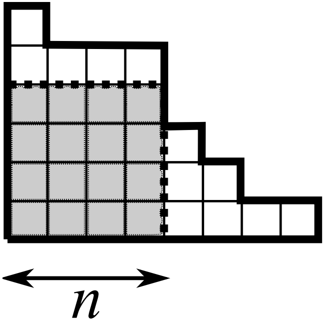
Is there an underlying combinatorial structure?

Outline

Ferrers Diagrams



a Ferrers diagram with Durfee square of size n

Why Ferrers Diagrams?

The generating function $G(x,y,q)$ of Ferrers diagrams with n-th largest row having length n for some positive integer n, enumerated with respect to width (x), height (y), and total area (q), is given by

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Compare with

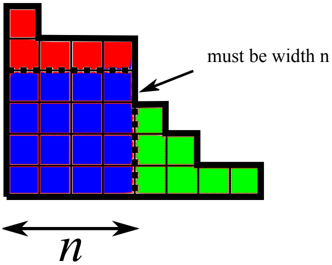
$$x = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2} x^n}{(q; q)_n (qx; q)_{n-1}}$$

to get

$$x = 1 + G(x, 1, q)$$

Enumerating Ferrers Diagrams

$$G(x,y,q) = \sum_{n=1}^{\infty} (xy)^n q^{n^2} \frac{1}{(yq;q)_n} \frac{1}{(xq;q)_{n-1}}$$



The sum is over all sizes n of Durfee squares

Trees Decorated with Ferrers Diagrams

The functional equation

$$x = 1 + G(x, 1, q)$$

admits a combinatorial interpretation using the "theory of species"

Theorem (TP, 2012)

Let F_q be the species of Ferrers diagrams with n-th largest row having length n for some integer n, weighted by area (q), with size given by the width of the Ferrers diagram, augmented by the 'empty polyomino'.
Then x enumerates F_q-enriched rooted trees (trees decorated such that the out-degree of the vertex matches the width of the Ferrers diagram) with respect to the total area of the Ferrers diagrams at the vertices of the tree.

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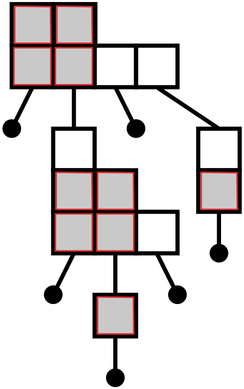

Tree with area 15, contributing q^{15} to the Partial Theta Function root

The Same Trees, But Different Area Weights

Partial Theta Function (TP, 2012)

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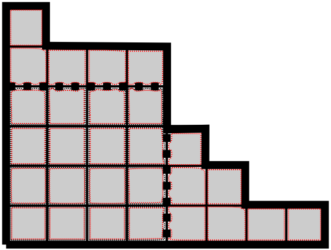

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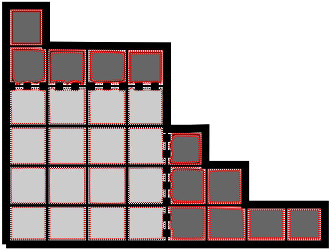

Count dark area twice

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Painlevé Airy Function (TP, 2013)

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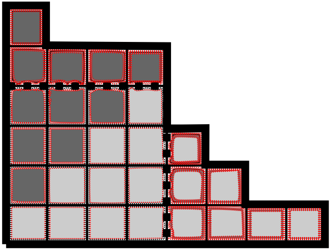

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Why are Θ, A_q, Ai_q special?

Can we understand and generalise?

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$$\sum_{n=0}^{N} \frac{(-1)^n q^{\binom{n+1}{2}}}{(q;q)_n (q;q)_{N-n}} = 1$$

follows from the q-binomial theorem

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These look similar, but their proofs have nothing in common!

Higher Roots for $A_q(x)$

Let $x = y/q^{2m}$ in

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$A_{-q}(y/q^{2m+1}) = 0$ if

$$y = 1 + \frac{(-1)^m q^m}{\sum_{n=0}^{m} \frac{(-1)^n y^n q^n}{(q^2;q^2)_n} \prod_{k=1}^{m-n} (y - q^{2k})} \sum_{n=1}^{\infty} \frac{q^{n^2} y^{m+n}}{(q^2;q^2)_{m+n} (q^2 y;q^2)_{n-1}}$$

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Numerically, the m-th root seems to satisfy

$$\frac{1}{q^{2m+1}} (1 + (-1)^m q^{m+1} (\text{positive terms}))$$

Acknowledgement

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Thank you, Peter!