# *Four Ways of Counting Incidence Matrices*

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## *What are Incidence Matrices?*

# Incidence Matrices

- How many zero-one Matrices are there with exactly $n$ ones?

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- No zero rows or columns allowed (for finite answer)
- Counting Function $F_{ijkl}(n)$, $i, j, k, l \in \{0, 1\}$
  - $i(k) = 0$: count matrices modulo row (column) permutations
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- Identify transposed matrices $\Rightarrow$ 4 additional cases, $\Phi_{ij}(n)$

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# Examples

- Incidence matrices with two ones, $n = 2$

$$I_1 = \begin{pmatrix} 1 & 1 \end{pmatrix} \;, \quad I_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \;, \quad I_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \;, \quad I_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

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  - $I_1$ has repeated columns, $I_2$ has repeated rows
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- For example,

$$F_{0000}(2) = 1 \,, \quad F_{1010}(2) = 2 \,, \quad F_{0101}(2) = 3 \,, \quad F_{1111}(2) = 4 \,,$$

$$\Phi_{00}(2) = 1 \,, \quad \Phi_{10}(2) = 2 \,, \quad \Phi_{11}(2) = 3$$

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## Some Enumeration Data

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| $F_{0000}(n)$ | 1 | 1 | 2 | 4 | 7 | 16 | | | |
| $F_{0010}(n)$ | 1 | 1 | 3 | 11 | 40 | 174 | | | |
| $F_{1010}(n)$ | 1 | 2 | 10 | 72 | 624 | 6522 | | | |
| $F_{0001}(n)$ | 1 | 2 | 4 | 9 | 18 | 44 | | | |
| $F_{0011}(n)$ | 1 | 2 | 7 | 28 | 134 | 729 | 4408 | 29256 | 210710 |
| $F_{1001}(n)$ | 1 | 2 | 6 | 20 | 73 | 315 | | | |
| $F_{1011}(n)$ | 1 | 3 | 17 | 129 | 1227 | 14123 | | | |
| $F_{0101}(n)$ | 1 | 3 | 6 | 16 | 34 | 90 | 211 | 558 | 1430 |
| $F_{0111}(n)$ | 1 | 3 | 10 | 41 | 192 | 1025 | 6087 | 39754 | 282241 |
| $F_{1111}(n)$ | 1 | 4 | 24 | 196 | 2016 | 24976 | 361792 | 5997872 | 111969552 |
| $\Phi_{00}(n)$ | 1 | 1 | 2 | 3 | 5 | 11 | | | |
| $\Phi_{10}(n)$ | 1 | 2 | 8 | 44 | 340 | 3368 | | | |
| $\Phi_{01}(n)$ | 1 | 2 | 4 | 10 | 20 | 50 | | | |
| $\Phi_{11}(n)$ | 1 | 3 | 15 | 108 | 1045 | 12639 | 181553 | 3001997 | 55999767 |

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## *Why are Incidence Matrices interesting?*

# Incidence matrices count ...

- ... hypergraphs by weight $n$
  - vertex set $\{x_1, \ldots, x_r\}$
  - (hyper)-edges $E_1, \ldots, E_s$
  - *weight* of the hypergraph: sum of the cardinalities of the edges

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- Choose $i, k$ to get labelled/unlabelled vertices/edges, choose $j, l$ to get repeated/no repeated vertices/edges
- Examples
  - $F_{0101}(n)$ counts simple hypergraphs with no isolated vertices, up to isomorphism
  - $F_{1101}(n)$ is the number of (vertex)-labelled hypergraphs

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# Incidence matrices count ...

- ... bipartite graphs by number of edges $n$
  - partition vertices into sets $\{r_1, \ldots, r_r\}$ and $\{c_1, \ldots, c_s\}$

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- Choose $i, k$ to get labelled/unlabelled vertex sets ($j = l = 1$)
- Examples
  - $F_{0101}(n)$ counts unlabelled bipartite graphs with distinguished bipartite block
  - $F_{1111}(n)$ counts labelled bipartite graphs with a distinguished bipartite block
  - $\Phi_{01}(n)$ counts unlabelled bipartite graphs with a distinguished bipartition

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  - let $S$ be the symmetric group acting on $\mathbb{Q}$
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- First consider the number of orbits of $S$ and $A$ on $n$-element subsets of $\mathbb{Q}$
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  - therefore the number of orbits of $S$ or $A$ is $1$

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  - both $S$ and $A$ carry any $n$-element subset into any other
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- Now consider the number of orbits of $S \times S$, $S \times A$, and $A \times A$ on $n$-element subsets of $\mathbb{Q} \times \mathbb{Q}$. Then (details later)
  - $F_{0101}(n)$ is the number of orbits of $S \times S$
  - $F_{1101}(n)$ is the number of orbits of $A \times S$
  - $F_{1111}(n)$ is the number of orbits of $A \times A$

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# *Incidence matrices count ...*

- ... binary block designs
  - *block design*: set of $n$ *plots* carrying two partitions $\{B_1, \ldots, B_r\}$ and $\{B'_1, \ldots, B'_s\}$
  - *binary*: the intersection of two blocks $B_i \cap B'_j$ contains at most one plot

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- Examples
  - $F_{0101}(n)$ counts binary block designs
  - $F_{1111}(n)$ counts labelled binary block designs
  - $\Phi_{01}(n)$ counts binary block designs up to duality

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***Focus on*** $F(n) = F_{1111}(n)$

## **Counting** $F(n)$

- $m_{ij}$ the number of $i \times j$ incidence matrices with $n$ ones

$$\sum_{i \leq k} \sum_{j \leq l} \binom{k}{i} \binom{l}{j} m_{ij} = \binom{kl}{n}$$

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- $F(n)$ is given as 4-fold sum

$$F(n) = \sum_{k \le n} \sum_{l \le n} m_{kl}$$

## Bounds and Asymptotics

- Upper bound

$$F(n) \leq \binom{n^2}{n} \sim \frac{1}{\sqrt{2\pi n}}(n\mathrm{e})^n$$

(choose $n$ ones in $n \times n$ matrix, remove zero rows/cols)

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(permutation matrices)

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$\boxed{\text{Four methods} \Longrightarrow \text{four proofs}}$

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- $P(n)$ are known as *ordered Bell numbers*, their EGF is

$$\sum_{n=0}^{\infty} \frac{P(n)}{n!} z^n = \frac{1}{2 - e^z} \quad \Rightarrow \quad P(n) \sim \frac{n!}{2} \left( \frac{1}{\log 2} \right)^{n+1}$$

# Proof 1 (ctd.)

- Calculation of $p$
  - event $D_{ij} = \{i$ and $j$ are in the same block for each preorder$\}$
  - labelled binary block design if no such $D_{ij}$ occurs

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  - one can show that for each $r \ge 0$

$$\mathbb{E}[W(W-1)\cdots(W-r+1)] \sim [(\log 2)^2/2]^r$$

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  - thus $W$ converges weakly to Poisson$[(\log 2)^2/2]$

$$\mathbb{P}(W = 0) \sim e^{-\frac{1}{2}(\log 2)^2}$$

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implies

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# *Counting Orbits of Permutation Groups*

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- Denote
  - $F_n^*(G)$ the number of orbits on ordered $n$-tuples
  - $F_n(G)$ the number of orbits on ordered $n$-tuples of distinct elements
  - $f_n(G)$ the number of orbits on $n$-element subsets

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- Examples
  - $G = S$ symmetric group acting on a countably infinite set $X$

$$f_n(S) = 1 \qquad F_n(S) = 1 \qquad F^*_n(S) = B(n) \quad \text{Bell numbers}$$

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  - $G = A$ group of all order preserving permutations on $X = \mathbb{Q}$

$$f_n(A) = 1 \qquad F_n(A) = n! \qquad F_n^*(S) = P(n) \quad \text{ordered Bell numbers}$$

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## Combinatorics of Orbit Counting

- Stirling transforms

$$F_n^*(G) = \sum_{k=1}^{n} S(n,k) F_k(G) \quad \text{and} \quad F_n(G) = \sum_{k=1}^{n} s(n,k) F_k^*(G)$$

where $s(n,k)$ and $S(n,k)$ are Stirling numbers of the first and second kind

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  - $G = A$ group of all order preserving permutations on $X = \mathbb{Q}$

$$F_n^*(A) = \sum_{k=1}^{n} S(n,k) F_k(S) = \sum_{k=1}^{n} S(n,k) k! = P(n)$$

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- Apply to $A \times A$ acting on $\mathbb{Q} \times \mathbb{Q}$
  - $F_n^*(A \times A) = P(n)^2$
  - $F_n(A \times A) = \sum_{k=1}^n s(n,k) F_k^*(A \times A) = \sum_{k=1}^n s(n,k) P(k)^2$
  - $f_n(A \times A) = F_n(A \times A)/n!$

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  - $F_n(A \times A) = \sum_{k=1}^n s(n,k) F_k^*(A \times A) = \sum_{k=1}^n s(n,k) P(k)^2$
  - $f_n(A \times A) = F_n(A \times A)/n!$
- For $n$ distinct elements $(x_1, y_1), \ldots, (x_n, y_n)$ of $\mathbb{Q} \times \mathbb{Q}$, form incidence matrix by indexing rows and columns by the sets $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$

$$F(n) = f_n(A \times A) = \frac{1}{n!} \sum_{k=1}^{n} s(n,k) P(k)^2$$

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## Proof 2 (ctd.)

- Approximate $P(n)$ by $P(n) \sim \frac{n!}{2}\left(\frac{1}{\log 2}\right)^{n+1}$ to get

$$F(n) = \frac{1}{n!}\sum_{k=1}^{n} s(n,k)P(k)^2 \sim \frac{1}{4n!}\sum_{k=1}^{n} s(n,k)(k!)^2(\log 2)^{-2n-2}$$

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- This is $e^{-\frac{1}{2}(\log 2)^2}$ times the contribution of the identity ($k = n$)

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# Proof 3: A Simpler Identity

- Reconsider $F(n) = \sum_{k \leq n} \sum_{l \leq n} m_{kl}$ with

$$\sum_{i \leq k} \sum_{j \leq l} \binom{k}{i} \binom{l}{j} m_{ij} = \binom{kl}{n}$$

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# *Proof 3 (ctd.)*

- The sum

$$F(n) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{1}{2^{k+l+2}} \binom{kl}{n}$$

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$$\begin{aligned} F(n) &\sim \frac{1}{4n!} \int dk \int dl \, \frac{(kl)^n}{2^{k+l}} e^{-\frac{n^2}{2kl}} \\ &= \frac{n^{2n+2}}{4n!} \int d\kappa \int d\lambda \, e^{n(\log \kappa - \kappa \log 2)} e^{n(\log \lambda - \lambda \log 2)} e^{-\frac{1}{2\kappa\lambda}} \end{aligned}$$

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- Standard saddle point asymptotics using $w(\kappa) = \log\kappa - \kappa\log 2$
  - $w'(\kappa_s) = 0$ implies saddle $\kappa_s = 1/\log 2$

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## Proof 3 (ctd.)

- Use a Gaussian approximation around $\kappa_s = \lambda_s = 1/\log 2$ to get

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# *Proof 4: Calculus of Residues*

- Rewrite

$$m_{kl} = \sum_{i \le k} \sum_{j \le l} (-1)^{k+l-i-j} \binom{k}{i} \binom{l}{j} \binom{ij}{n}$$

using

$$\mathrm{Res}(\Gamma(s),\ s = -m) = \frac{(-1)^m}{m!}$$

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- This leads to

$$m_{kl}(n) = \frac{k!l!}{n!} \mathrm{Res}\left( \frac{(st)_n}{(s)_{k+1}(t)_{l+1}};\ s = \infty,\ t = \infty \right)$$

where $(x)_m = x(x-1)\ldots(x-m+1)$ is the falling factorial

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## Proof 4 (ctd.)

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- Do a saddle point analysis of the contour integral ...
- For fixed $\sigma, \tau > 0$

$$m_{k,l}(n) \sim \frac{n^{2n}}{n!} e^{nw(\sigma)} v(\sigma) e^{nw(\tau)} v(\tau) e^{-\frac{1}{2\sigma\tau}}$$

with

$$\begin{aligned} w(x) &= x(1-e^{-1/x})\log(1-e^{-1/x}) + \log x - e^{-1/x} \\ v(x) &= \sqrt{x(1-e^{-1/x})/(x(1-e^{-1/x}) - e^{-1/x})} \end{aligned}$$

and $k = n\sigma(1-e^{-1/\sigma})$, $l = n\tau(1-e^{-1/\tau})$

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- $F(n) = \sum_{k \le n} \sum_{l \le n} m_{kl}$ is dominated by $k_s = l_s = n/2\log 2$

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# *Summary*

- Mathematics of incidence matrices
- Counting functions $F_{ijkl}(n)$, $\Phi_{ij}(n)$
- Asymptotics of $F(n) = F_{1111}(n)$

$$F(n) \sim \frac{n!}{4} e^{-(\log 2)^2/2} \frac{1}{(\log 2)^{2n+2}}$$

  Four proofs

  - probabilistic, binary block designs
  - product action of permutation groups
  - “simple” identity, gaussian approximation
  - residues and saddle point method
- Many open problems for other $F_{ijkl}(n)$, $\Phi_{ij}(n)$

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**The End**