Spectral Analysis of Transfer Operators associated with Intermittency

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Transfer Operators for Interval Maps

- $f : [0, 1] \to [0, 1]$ piecewise monotone, analytic
(minimal requirement: piecewise $C^{1+\epsilon}$)
- Ruelle-Perron-Frobenius operator

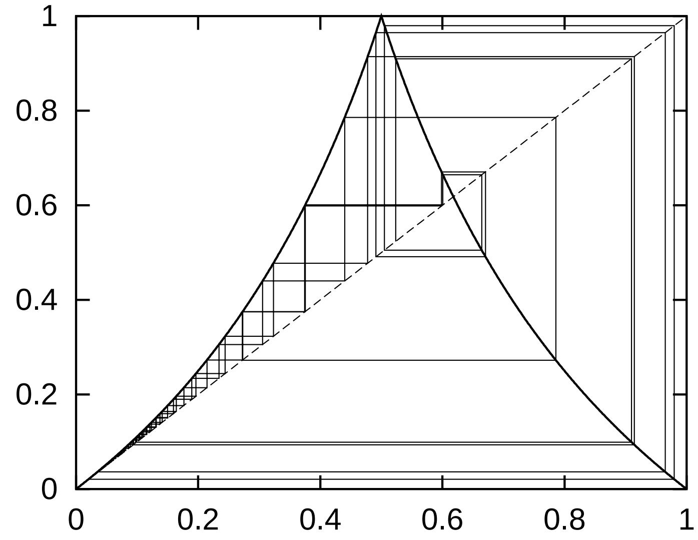
$$\mathcal{P}\varphi(x) = \sum_{f(y)=x} |f'(y)|^{-\beta} \varphi(y)$$

- $\beta = 1$: Perron-Frobenius operator, $\mathcal{P}\rho = \rho$, $\mathcal{P}^*\mu_L = \mu_L$
ρ invariant density, eigenfunction to maximal eigenvalue $\lambda = 1$
- Expanding map ($|f'| \geq c > 1$) $\Rightarrow$ spectral gap
- Spectral gap $\Rightarrow$ exponential decay of correlations

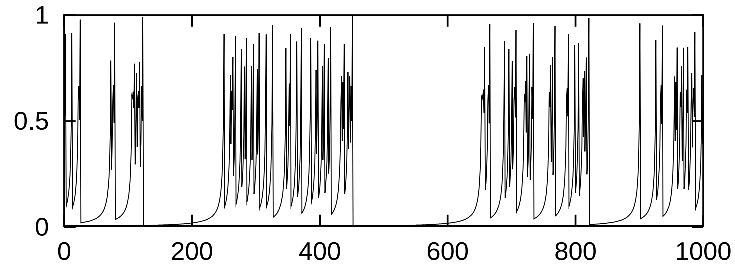
Farey Map

- Farey map on $[0, 1]$

$$f(x) = \begin{cases} \frac{x}{1-x}, & x \le 1/2 \\ \frac{1-x}{x}, & 1/2 < x \end{cases}$$



- Toy model for intermittency
  - “intermittent” left branch
  - “chaotic” right branch

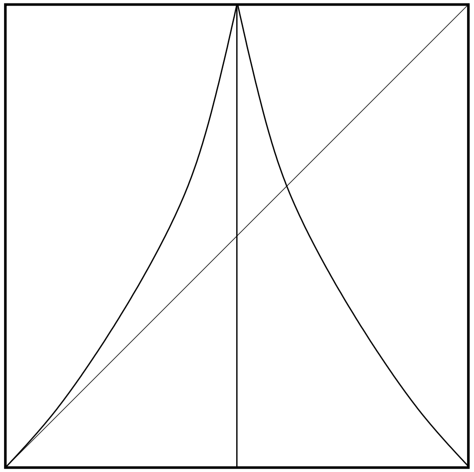


- $f'(0) = 1$: almost expanding (non-uniformly expanding)
- Invariant density $\rho(x) = 1/x$ not normalizable

Farey Operator

- RPF Operator

$$\begin{aligned}\mathcal{P}\varphi(x) &= \sum_{f(y)=x} |f'(y)|^{-\beta}\varphi(y) \\ &= \mathcal{P}_0\varphi(x) + \mathcal{P}_1\varphi(x)\end{aligned}$$



with

$$\mathcal{P}_0\varphi(x) = \frac{1}{(1+x)^{2\beta}}\varphi\left(\frac{x}{1+x}\right)$$ “intermittent part”

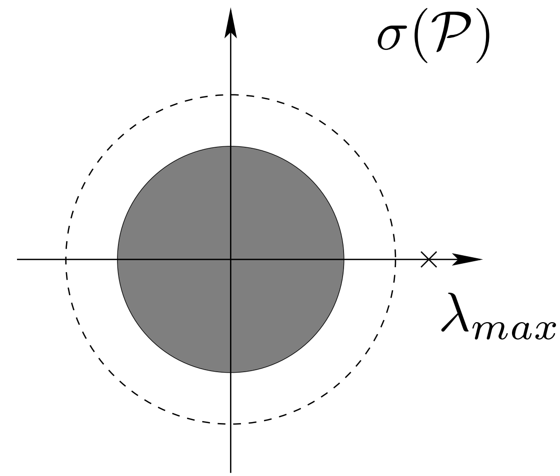
and

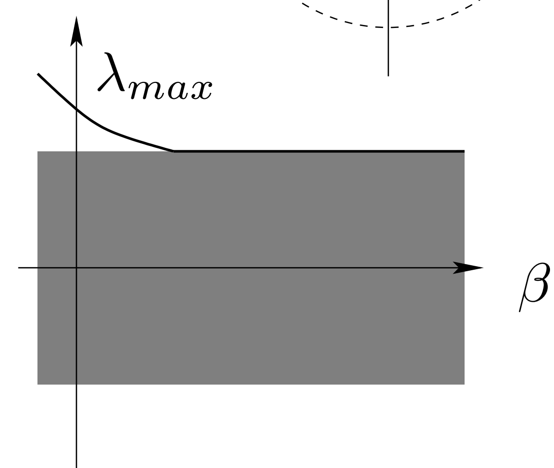
$$\mathcal{P}_1\varphi(x) = \frac{1}{(1+x)^{2\beta}}\varphi\left(\frac{1}{1+x}\right)$$ “chaotic remainder”

Banach Space Approach

$BV[0,1]$ Banach space of functions on $[0,1]$ with bounded variation.
Theorem 1 (Prellberg, Slawny, 1991): $\mathcal{P}$ acting on $BV[0,1]$ is a bounded operator with

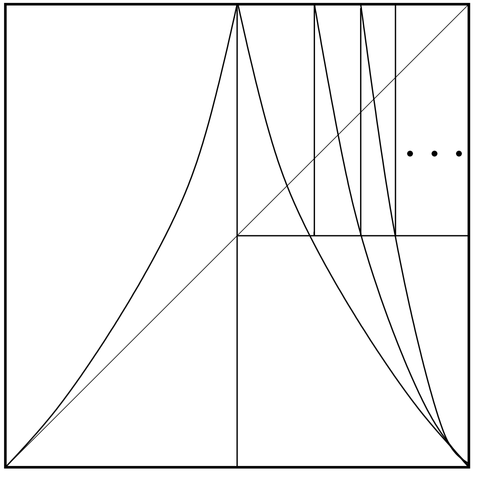
- $r_{ess}(\mathcal{P}) = 1 \qquad \forall \beta \in \mathbb{R}$
- $r(\mathcal{P}) = 1 \qquad \forall \beta \geq 1$
- $r(\mathcal{P}) = \lambda_{max}(\beta) > 1 \qquad \forall \beta < 1$
- $\lambda_{max}(\beta)$ analytic in $\beta < 1$
- $\lambda_{max}(\beta) \sim \exp\left[C \frac{1-\beta}{-\log(1-\beta)}\right]$ as $\beta \to 1^-$





Method: Inducing away from indifferent fixed point

- Consider first return map on $[1/2, 1]$
- Branches $g_n = f^n|_{[1/2,1]} = {f_0}^{n-1} \circ f_1$
- Associated transfer operator must keep track of n



$$\begin{aligned}\mathcal{M}_z\varphi(x) &= \sum_{n=1}^{\infty} z^n |(g_n^{-1})'(x)|^\beta \varphi \circ (g_n^{-1})(x) \\ &= \sum_{n=1}^{\infty} \frac{z^n}{(1+nx)^{2\beta}} \varphi\left(1 - \frac{x}{1+nx}\right)\end{aligned}$$

- Proof of Theorem 1 uses operator relations between $\mathcal{M}_z$ and $\mathcal{P}$

Banach Space Approach (ctd.)

- Drawback: $BV[0,1]$ is too large
- Remark 1: induced map is expanding, branches are analytic $\Rightarrow$ $\mathcal{M}_z$ defines a nuclear operator on a suitable space of analytic functions
- Remark 2: for $z = 1$, $\mathcal{M}_1$ is related to transfer operator of Gauss map known to be self-adjoint on suitable Hilbert space (Mayer)

Wish: find Hilbert space on which $\mathcal{P}$ is self-adjoint

Progress using results of Mayer, Isola, Rugh, $\ldots$

Self-Adjointness

Theorem 2 (Prellberg, 2002): $\mathcal{P}$ is self-adjoint on the Hilbert space

$$\mathcal{H} = \left\{ \varphi(x) = x^{-2\beta} \int_0^\infty e^{-s\frac{1-x}{x}} \psi(s) d\mu(s) : \psi \in L^2(\mathbb{R}_+, \mu) \right\}$$

with $\mu(s) = e^{-s} s^{2\beta-1}$ and appropriate induced inner product.
Proof: $\mathcal{P} = \mathcal{K}\mathcal{S}\mathcal{K}^{-1}$ where $\mathcal{S}$ acting on $L^2(\mathbb{R}_+, \mu)$ is given by

$$\mathcal{S}\psi(s) = e^{-s}\psi(s) + \int_0^\infty K(s,t)\psi(t) d\mu(t)$$

with $K(s,t) = (st)^{\frac{1}{2}-\beta} J_{2\beta-1}(2\sqrt{st})$ and $\mathcal{K}$ is given by

$$\mathcal{K}\varphi(x) = x^{-2\beta} \int_0^\infty e^{-s\frac{1-x}{x}} \psi(s) d\mu(s)$$

Spectral Properties

Theorem 3 (Prellberg, 2002): $\mathcal{P}$ acting on $\mathcal{H}$ has spectrum $\sigma = \sigma_c \cup \sigma_p$ where

- $\sigma_c = [0, 1]$
- $\sigma_p \backslash \{0\}$ isolated eigenvalues of finite multiplicity
- 0 is an eigenvalue of infinite multiplicity

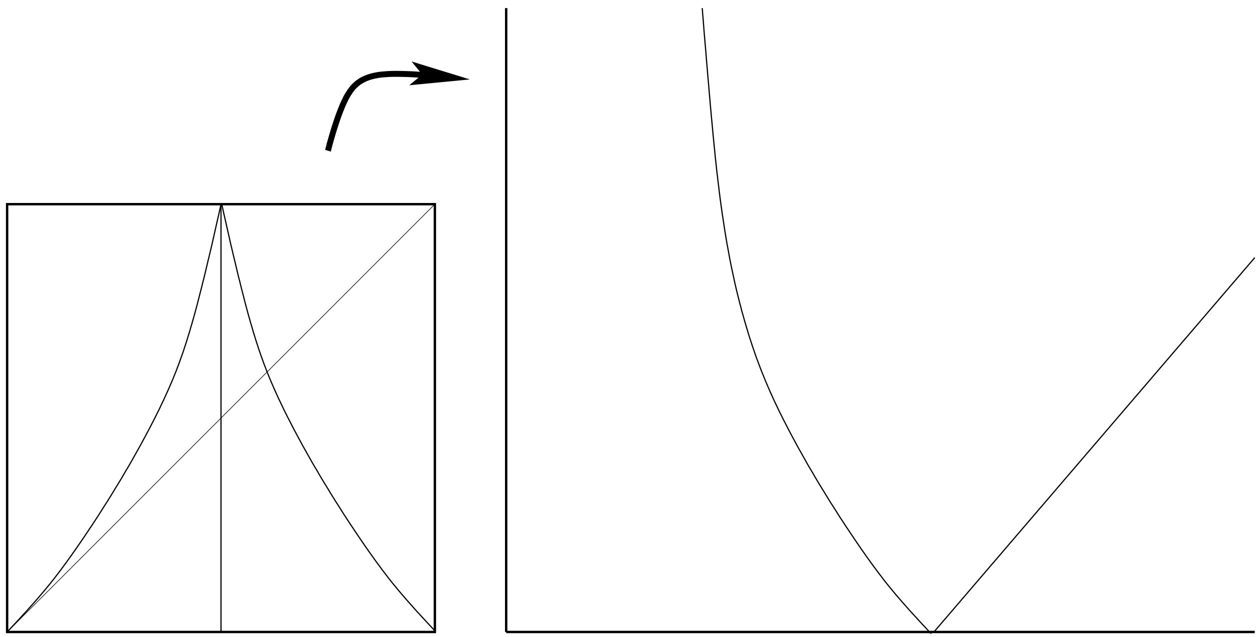
Proof: utilizes $\mathcal{P} = \mathcal{K}\mathcal{S}\mathcal{K}^{-1}$

- $\mathcal{S}_0\psi(x) = e^{-s}\psi(s)$ multiplication operator with $\sigma_c = [0, 1]$
- $\mathcal{S}$ is compact perturbation of $\mathcal{S}_0$
- Operator relations between $\mathcal{M}_z$ and $\mathcal{P}$
- $\varphi(x) = -\varphi(1-x) \quad \Rightarrow \quad \mathcal{P}\varphi = 0$

Understanding Theorem 2

First look at $\mathcal{P}_0$, then consider $\mathcal{P}_1$ as a perturbation

- Formal conjugation of the Farey map with $\mathcal{C}\varphi(x) = \varphi\left(\frac{1-x}{x}\right)$



$P_0 = \mathcal{C}\mathcal{Q}_0\mathcal{C}^{-1}$ $\qquad$ $\mathcal{Q}_0\varphi(x) = \varphi(x-1)$ shift

Understanding Theorem 2 (ctd.)

- Conjugation of $\mathcal{Q}$ with generalized Laplace transform

$$\mathcal{L}\psi(x) = \int_0^\infty \exp(-sx)\psi(s)d\mu(s)$$

  leads to multiplication operator $\mathcal{S}_0\psi(x) = e^{-s}\psi(s)$
- On $L^2(\mathbb{R}_+, \mu)$, $\mathcal{S}_0$ has continuous spectrum $[0, 1]$

This shows
Lemma: On $\mathcal{CL}L^2(\mathbb{R}_+, \mu)$, $\mathcal{P}_0$ has spectrum $\sigma(\mathcal{P}_0) = \sigma_c(\mathcal{P}_0) = [0, 1]$

- Is there a measure μ for which $\mathcal{S} = \mathcal{S}_0 + \mathcal{S}_1$ (and therefore $\mathcal{P}$) is self-adjoint?

Understanding Theorem 2 (ctd.)

- Consider the transfer operator for $h(x) = (ax+b)/(cx+d)$ with $\sigma = ad - bc = \pm 1$

$$\mathcal{H}\varphi(x) = |(h^{-1})'(x)|^{\beta}\varphi \circ h^{-1}(x) = (a-cx)^{-2\beta}\varphi\left(\frac{dx-b}{a-cx}\right)$$

- Conjugation with a generalized Laplace transform $\mathcal{L}$ gives

$$\mathcal{L}^{-1}\mathcal{H}\mathcal{L}\psi(s) = \int_0^\infty K(s,t)\psi(t)d\mu(t)$$

with kernel

$$\mu(s)K(s,t) = \frac{1}{c}\exp\frac{as+td}{c}\left(\frac{s}{t}\right)^{\beta-\frac{1}{2}} Z_{2\beta-1}\left(\frac{2}{c}\sqrt{st}\right)$$

where $Z_\nu(u) = I_\nu(u)$ for $\sigma = 1$ and $Z_\nu(u) = J_\nu(u)$ for $\sigma = -1$

Understanding Theorem 2 (ctd.)

- Specialize to $\mathcal{H} = \mathcal{P}_1$, i.e. $a = -1$, $b = 1$, $c = 1$, $d = 0$:

$$\mu(s)K(s,t) = e^{-s}\left(\frac{s}{t}\right)^{\beta-\frac{1}{2}} J_{2\beta-1}(2\sqrt{st})$$

- The kernel is symmetric for $\mu(s) = e^{-s}s^{2\beta-1}$, and we obtain

$$K(s,t) = (st)^{\frac{1}{2}-\beta} J_{2\beta-1}(2\sqrt{st})$$

This proves Theorem 2

Understanding Theorem 3

- We reintroduce $\mathcal{M}_z$ abstractly by defining for $z \in \mathbb{C} - [1, \infty)$

$$\mathcal{M}_z = (1 - z\mathcal{P}_0)^{-1} z\mathcal{P}_1$$

- Formal expansion in z gives the transfer operator associated with the induced map g on $[1/2, 1]$

$$\mathcal{M}_z = \sum_{n=1}^{\infty} z^n {\mathcal{P}_0}^{n-1} \mathcal{P}_1$$

- Lemma: $\mathcal{M}_z$ is nuclear on $\mathcal{CL}L^2(\mathbb{R}_+, \mu)$
Proof: (a) $|z| < 1$: sum of nuclear operators
(b) analytic continuation (Rugh)

Understanding Theorem 3 (ctd.)

- We have the key identity (for $z \in \mathbb{C} - [1, \infty)$)

$$1 - z\mathcal{P} = (1 - z\mathcal{P}_0)(1 - \mathcal{M}_z)$$

- Lemma:
  - $\sigma_c(\mathcal{P}) = \sigma_c(\mathcal{P}_0) = [0, 1]$
  - $\lambda = z^{-1}$ is eigenvalue of $\mathcal{P} \Leftrightarrow 1$ is eigenvalue of $\mathcal{M}_z$

Theorem 3 follows from considering the solutions to $1 \in \sigma(\mathcal{M}_z)$ using nuclearity and analytic perturbation theory

Numerical Analysis of the Spectrum

- $\sigma_c(\mathcal{P})$ is known
- $\sigma_p(\mathcal{P})$ can be determined by solving $1 \in \sigma(\mathcal{M}_z)$ numerically
- Eigenvalues of $\mathcal{M}_z$ are (branches of) analytic functions in z, $\lambda_n(z)$
- Solve $\lambda_n(z) = 1$ to get eigenvalue $\lambda = 1/z$ of $\mathcal{P}$
- Self-adjointness of $\mathcal{P}$: only real z needed!

Preliminary work (Dodds 1993):

considered $z \in [-1, 1]$ (i.e. real λ outside the unit disc)

Numerical Analysis (ctd.)

Computation of the spectrum of $\mathcal{M}_z$:

- $\mathcal{M}_z$ acts on analytic functions $\Rightarrow$ expand in power series
- Expansion around $x = 1$ gives matrix elements

$$M_z^{n,m} = \sum_{k=0}^{n} (-1)^{n-k} \binom{-2\beta - m}{k} \binom{-2\beta - k}{n-k} \left(\frac{1}{z} \mathrm{Li}_{2\beta+m+k}(z) - 1 \right)$$

with dilogarithm $\mathrm{Li}_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s}$

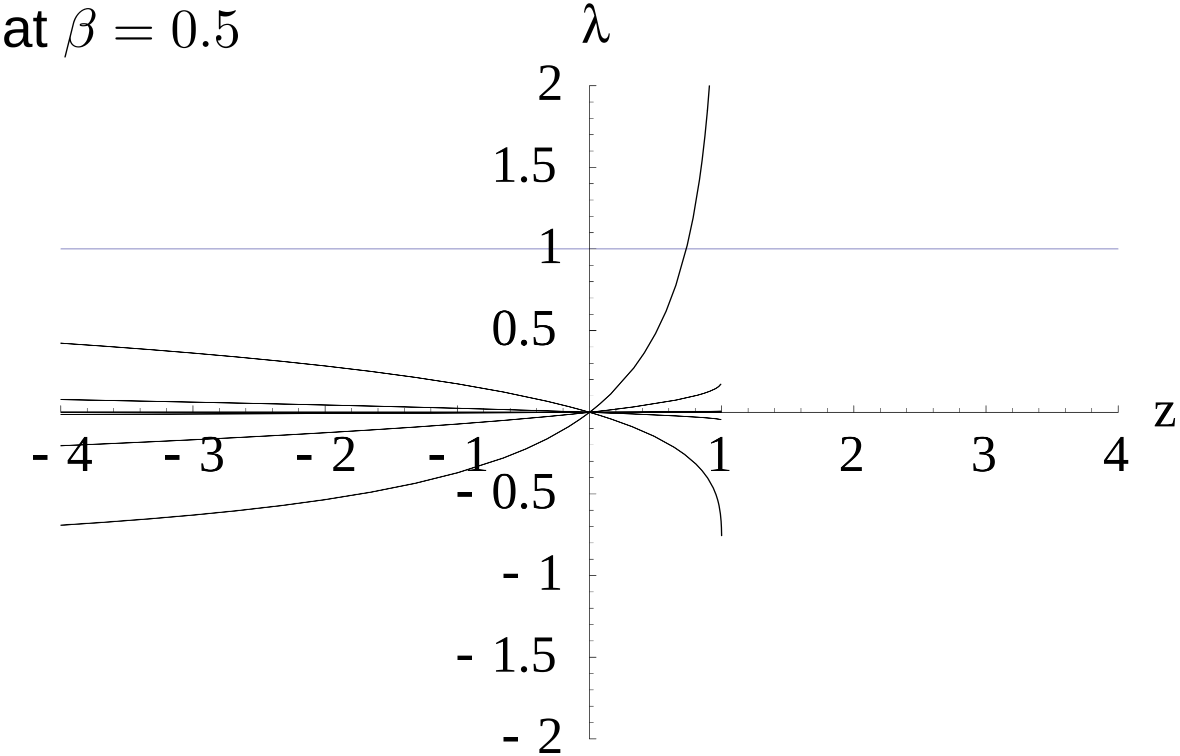
- Approximate $\mathcal{M}_z$ with truncated operators $\mathcal{M}_z^{(N)}$ acting on a subspace of polynomials of at most degree N

Related work: Dodds 1993; Daudé, Flajolet, Vallée 1997

Lhote 2002: Rigorous computation of a leading eigenvalue

Numerical Analysis (ctd.)

$\sigma(\mathcal{M}_z)$ at $\beta = 0.5$

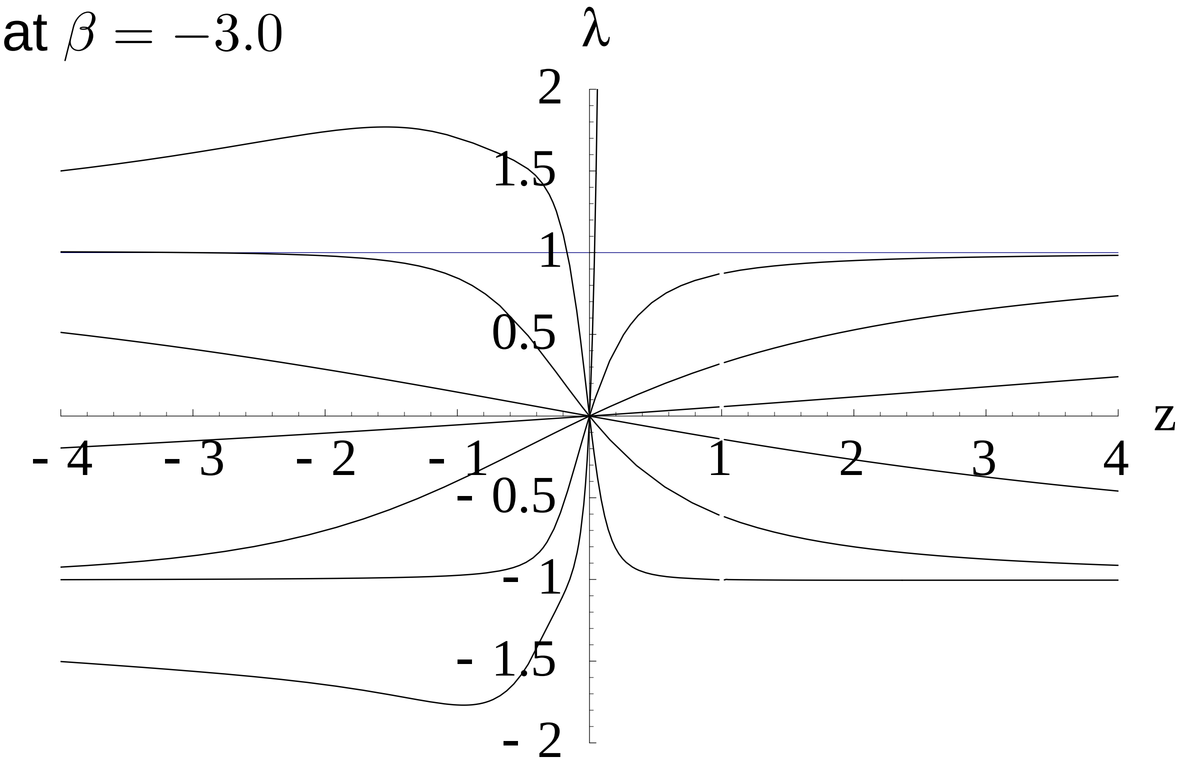


leading eigenvalue diverges at $z = 1$

analytic continuation has non-vanishing imaginary part along the cut

Numerical Analysis (ctd.)

$\sigma(\mathcal{M}_z)$ at $\beta = -3.0$



analytic continuation beyond $z = 1$ possible

$\lambda = 1$ at $z = -2.971$, $z = -0.168$, $z = 0.038$, and $z = 13.101$

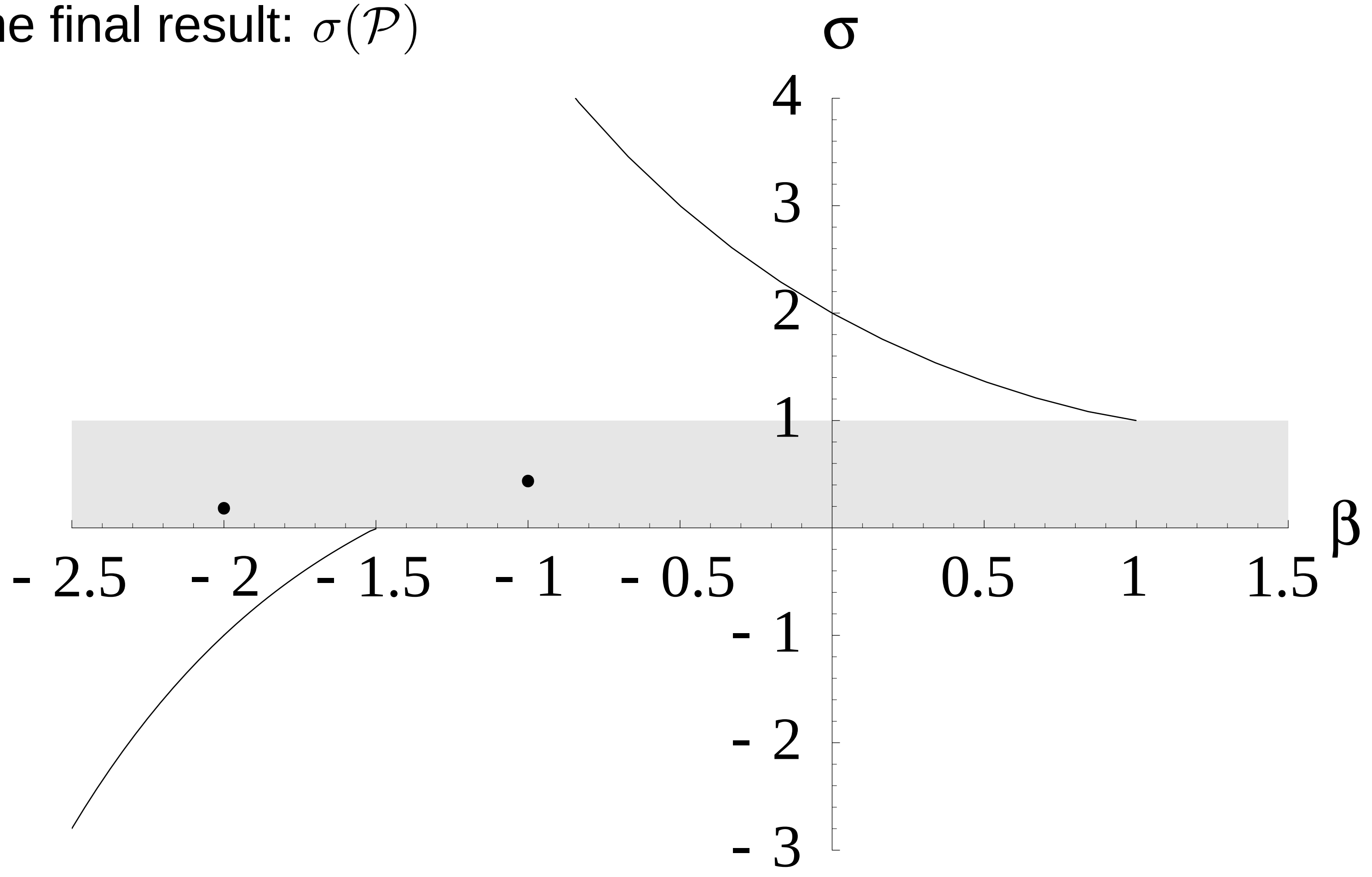
Numerical Analysis (ctd.)

Numerical observation:

- When continuing eigenvalue branches to the cut $[1, \infty)$, the result is real *only* for $\beta = -N/2$
- For $\beta \neq -N/2$, the analytic extension has a non-vanishing jump in the imaginary part along the cut
- Therefore $\lambda_n(z) = 1$ has no solutions for $z > 1$ unless $\beta = -N/2$
- Eigenvalues of $\mathcal{P}$ embedded in the continuous spectrum are only possible for $\beta = -N/2$

Numerical Analysis (ctd.)

The final result: $\sigma(\mathcal{P})$



Summary and Outlook

- Self-adjointness of the Farey operator $\mathcal{P}$ on a suitable Hilbert space
- Spectral properties: $\sigma = \sigma_c \cup \sigma_p$ where
  - $\sigma_c = [0, 1]$
  - $\sigma_p \backslash \{0\}$ isolated eigenvalues of finite multiplicity
- Numerical analysis of eigenvalues via induced operator
- Conjecture: eigenvalues embedded in σ_c only at $\beta = -n/2$
- Open problem: proof of convergence of numerical method