On the Number of Walks in a Triangular Domain

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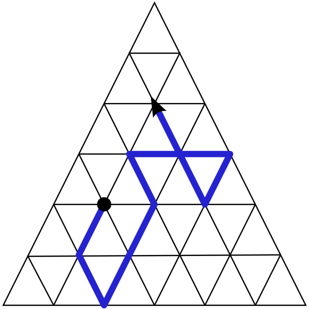
Topic Outline

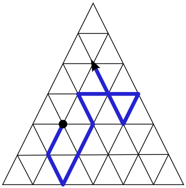
Outline

Walks in a Triangle



Walk with 10 steps inside a triangle of side-length 6

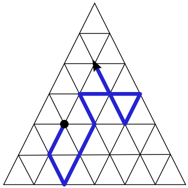
Counting Walks



Parameters

- Side-length L
- Number of steps n
- Starting point $\mathbf{a}$
- End point $\mathbf{b}$

Counting Walks



Parameters

- Side-length L
- Number of steps n
- Starting point $\mathbf{a}$
- End point $\mathbf{b}$

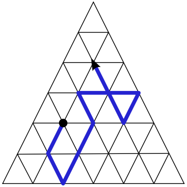
Number of n-step walks from **a** to **b** within triangle of side-length L

$$c_{n,L}^{\mathbf{a},\mathbf{b}}$$

Generating function

$$Z_L^{\mathbf{a},\mathbf{b}}(t) = \sum c_{n,L}^{\mathbf{a},\mathbf{b}} t^n$$

Counting Walks



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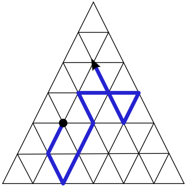

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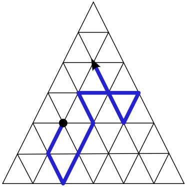
$$Z_L^{\mathbf{a},\mathbf{b}}(t) = \sum c_{n,L}^{\mathbf{a},\mathbf{b}} t^n$$

Finite transition matrix ⇒ Rational generating function

$\binom{L+2}{2}$ vertices ⇒ Degree of polynomials grows quadratically in L

Counting Walks



Generating function

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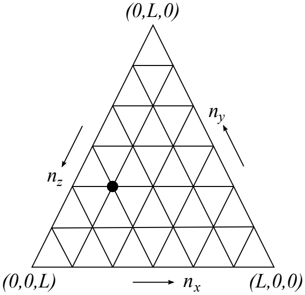
Finite transition matrix ⇒ Rational generating function

$\binom{L+2}{2}$ vertices ⇒ Degree of polynomials grows quadratically in L

It is surprisingly difficult to give a closed-form expression for $Z_L^{\mathbf{a},\mathbf{b}}(t)$

Some Notation

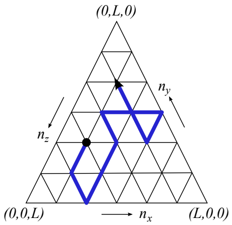
Associate to each vertex three coordinates (n_x, n_y, n_z)



$$n_x + n_y + n_z = L$$

For example, the point in the triangle above is given by $\mathbf{p} = (1, 2, 3)$

Some Notation



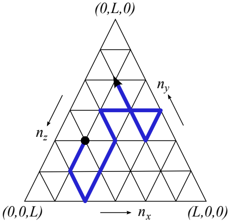
Generating function

$$Z_L^{\mathbf{a},\mathbf{b}}(t) = \sum_n c_{n,L}^{\mathbf{a},\mathbf{b}} t^n$$

Consider new generating function by summing over end-point positions

$$G_L^{\mathbf{a}}(x,y,z;t) = \sum_{n_x,n_y,n_z} Z_L^{\mathbf{a},(n_x,n_y,n_z)}(t) x^{n_x} y^{n_y} z^{n_z}$$

Some Notation



Fix L and **a**, drop t and write

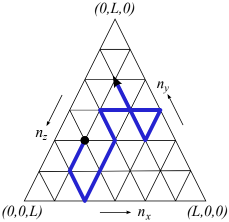
$$G(x, y, z) \equiv G_L^{\mathbf{a}}(x, y, z; t)$$

As $n_x + n_y + n_z = L$, G is homogeneous of degree L

$$G(\gamma x, \gamma y, \gamma z) = \gamma^L G(x, y, z)$$

Changing the Dimension

Triangle of side-length L, starting point $\mathbf{a} = (u, v, w)$ with $u + v + w = L$:



$$G(x, y, z) = \sum_{n_x, n_y, n_z, t} c_{n,L}^{(u,v,w),(n_x,n_y,n_z)} x^{n_x} y^{n_y} z^{n_z} t^n$$

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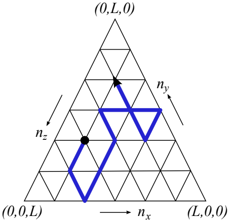
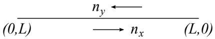


$$G(x,y,z) = \sum_{n_x,n_y,n_z,t} c_{n,L}^{(u,v,w),(n_x,n_y,n_z)} x^{n_x} y^{n_y} z^{n_z} t^n$$

Line of length L, starting point $\mathbf{a} = (u, v)$ with $u + v = L$:



$$G(x,y) = \sum_{n_x,n_y,t} c_{n,L}^{(u,v),(n_x,n_y)} x^{n_x} y^{n_y} t^n$$

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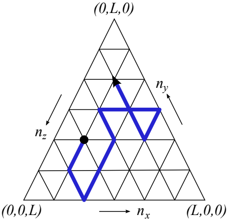

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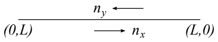


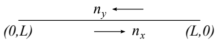
$$G(x, y) = \sum_{n_x, n_y, t} c_{n,L}^{(u,v),(n_x,n_y)} x^{n_x} y^{n_y} t^n$$

Obvious generalisation to tetrahedron and higher dimensions

Outline

The Line

Line of length L, starting point $\mathbf{a} = (u, v)$ with $u + v = L$:



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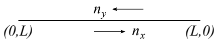

$$G(x,y) = \sum_{n_x, n_y, t} c_{n,L}^{(u,v),(n_x,n_y)} x^{n_x} y^{n_y} t^n$$

$G(x,y) =$ $x^u y^v$ zero-length walk

The Line

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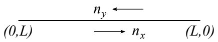


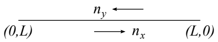
$$G(x,y) = \sum_{n_x, n_y, t} c_{n,L}^{(u,v),(n_x,n_y)} x^{n_x} y^{n_y} t^n$$

$$G(x,y) = x^u y^v \qquad \text{zero-length walk}$$

$$+G(x,y) t \frac{x}{y} \qquad \text{take walk and add a step to the right}$$

The Line

Line of length L, starting point $\mathbf{a} = (u, v)$ with $u + v = L$:



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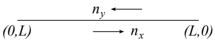
$$\begin{aligned} G(x,y) = \quad & x^u y^v && \text{zero-length walk} \\ & + G(x,y) t \frac{x}{y} && \text{take walk and add a step to the right} \\ & + G(x,y) t \frac{y}{x} && \text{take walk and add a step to the left} \end{aligned}$$

The Line

Line of length L, starting point $\mathbf{a} = (u, v)$ with $u + v = L$:



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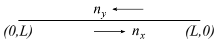
& + G(x, y) t \frac{y}{x} && \text{take walk and add a step to the left} \\
& - G(x, 0) t \frac{x}{y} && \text{forbid stepping past the right boundary}
\end{aligned}$$

The Line

Line of length L, starting point $\mathbf{a} = (u, v)$ with $u + v = L$:



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\end{aligned}$$

The Line

Line of length L, starting point $\mathbf{a} = (u, v)$ with $u + v = L$:

Functional equation

$$\underbrace{\left[1 - t\left(\frac{x}{y} + \frac{y}{x}\right)\right]}_{\text{Kernel}} G(x, y) = x^u y^v - t\frac{x}{y}G(x, 0) - t\frac{y}{x}G(0, y)$$

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Functional equation

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Note that the length L only enters the functional equation through $x^u y^v$

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Note that the length L only enters the functional equation through $x^u y^v$

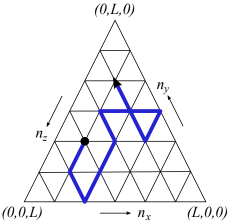
The *Kernel*

$$K(x, y) = 1 - t\left(\frac{x}{y} + \frac{y}{x}\right)$$

will be central to finding a solution of this functional equation

The Triangle

Triangle of side-length L, starting point $\mathbf{a} = (u, v, w)$ with $u + v + w = L$:



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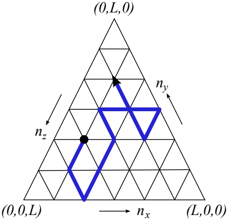

$$K(x,y,z)G(x,y,z) = x^u y^v z^w - t\left(\frac{y}{x} + \frac{z}{x}\right) G(0,y,z) - t\left(\frac{z}{y} + \frac{x}{y}\right) G(x,0,z) - t\left(\frac{x}{z} + \frac{y}{z}\right) G(x,y,0)$$

with the *Kernel*

$$K(x,y,z) = 1 - t\left(\frac{y}{x} + \frac{z}{x} + \frac{z}{y} + \frac{x}{y} + \frac{x}{z} + \frac{y}{x}\right)$$

Outline

The Kernel Method

Loosely speaking, the Kernel Method consists of

- setting the Kernel equal to zero and manipulating the RHS, or
- using variable transformations that leave the Kernel invariant and cancelling terms on the RHS, or
- staring at the stuff until you get a good idea ...

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- setting the Kernel equal to zero and manipulating the RHS, or
- using variable transformations that leave the Kernel invariant and cancelling terms on the RHS, or
- staring at the stuff until you get a good idea ...

This gives rise to various variations, known as the elementary Kernel method, the algebraic Kernel method, the obstinate Kernel method, the iterative Kernel method, and so forth

The experts will recognize in what follows yet another variation of the theme

The Line

Recall the functional equation

$$K(x,y)G(x,y) = x^u y^v - t\frac{x}{y}G(x,0) - t\frac{y}{x}G(0,y)$$

with the Kernel

$$K(x,y) = 1 - t\left(\frac{x}{y} + \frac{y}{x}\right)$$

The Line

Recall the functional equation

$$K(x,y)G(x,y) = x^u y^v - t\frac{x}{y}G(x,0) - t\frac{y}{x}G(0,y)$$

with the Kernel

$$K(x,y) = 1 - t\left(\frac{x}{y} + \frac{y}{x}\right)$$

The Kernel is invariant under

$$K(x,y) = K(y,x) = K(\lambda x, \lambda y)$$

In particular,

$$K(p,1) = K(1,p) = 1 - t(p + 1/p)$$

The Line

Now choose p such that

$$K(p,1) = K(1,p) = 1 - t(p + 1/p) = 0$$

The Line

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- Specialize $(x, y) = (1, p)$ and $(x, y) = (p, 1)$

$$0 = p^u - tpG(p,0) - \frac{t}{p}G(0,1)$$

$$0 = p^v - \frac{t}{p}G(1,0) - tpG(0,p)$$

The Line

Now choose p such that

$$K(p,1) = K(1,p) = 1 - t(p + 1/p) = 0$$

- Specialize $(x,y) = (1,p)$ and $(x,y) = (p,1)$

$$0 = p^u - tpG(p,0) - \frac{t}{p}G(0,1)$$
$$0 = p^v - \frac{t}{p}G(1,0) - tpG(0,p)$$

- Use homogeneity $G(p,0) = p^L G(1,0)$ and $G(0,p) = p^L G(0,1)$

$$0 = p^u - tp^{1+L}G(1,0) - \frac{t}{p}G(0,1)$$
$$0 = p^v - \frac{t}{p}G(1,0) - tp^{1+L}G(0,1)$$

The Line

This leads to a complete solution

Proposition

The generating function $G(x, y)$ counting n-step walks on a line of length L starting at (u, v) is given by

$$G(x,y) = \frac{1}{1 - \frac{\frac{x}{y} + \frac{y}{x}}{p + \frac{1}{p}}} \left(x^u y^v - \frac{x^{u+v+1} p^{v+1} (1 - p^{2u+2})}{y(1 - p^{2u+2v+4})} - \frac{y^{u+v+1} p^{u+1} (1 - p^{2v+2})}{x(1 - p^{2u+2v+4})} \right),$$

where

$$p = \frac{1 - \sqrt{1 - 4t^2}}{2t}$$

The Line

Specializing to $x = y = 1$, the result looks quite pleasant

Corollary

The generating function $G(1,1)$ counting n-step walks on a line of length L starting at (u, v) with no restrictions on the endpoint is given by

$$G(1,1) = \frac{(1+p^2)(1-p^{u+1})(1-p^{v+1})}{(1-p)^2(1+p^{u+v+2})},$$

where

$$p = \frac{1-\sqrt{1-4t^2}}{2t}.$$

The Triangle

The Kernel

$$K(x,y,z) = 1 - t\left(\frac{y}{x} + \frac{z}{x} + \frac{z}{y} + \frac{x}{y} + \frac{x}{z} + \frac{y}{x}\right)$$

is invariant under

$$K(1,1,p) = K(1,p,1) = K(p,1,1) = K(1,p,p) = K(p,1,p) = K(p,p,1)$$

The Triangle

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is invariant under

$$K(1,1,p) = K(1,p,1) = K(p,1,1) = K(1,p,p) = K(p,1,p) = K(p,p,1)$$

Choose p such that

$$K(1,1,p) = 1 - 2t(p + 1 + 1/p) = 0$$

The Triangle

We find

$$\frac{2t}{p}G(0,1,1)+t(1+p)G(p,0,1)+t(p+1)G(p,1,0)=p^u$$

$$\frac{2t}{p}G(1,0,1)+t(p+1)G(0,p,1)+t(1+p)G(1,p,0)=p^v$$

$$\frac{2t}{p}G(1,1,0)+t(1+p)G(0,1,p)+t(p+1)G(1,0,p)=p^w$$

$$2tpG(0,p,p)+t\left(1+\frac{1}{p}\right)G(1,0,p)+t\left(\frac{1}{p}+1\right)G(1,p,0)=p^vp^w$$

$$2tpG(p,0,p)+t\left(\frac{1}{p}+1\right)G(0,1,p)+t\left(1+\frac{1}{p}\right)G(p,1,0)=p^up^w$$

$$2tpG(p,p,0)+t\left(1+\frac{1}{p}\right)G(0,p,1)+t\left(\frac{1}{p}+1\right)G(p,0,1)=p^up^v$$

The Triangle

While this is insufficient to find a full solution, we are able to show

Theorem (Mortimer, Prellberg)

The generating function $G(1,1,1)$ which counts n-step walks in a triangle of side-length L starting at (u,v,w) with no restrictions on the endpoint is given by

$$G(1,1,1) = \frac{(1-p^3)(1-p^{u+1})(1-p^{v+1})(1-p^{w+1})}{(1-p)^3(1-p^{u+v+w+3})} \;,$$

where

$$p = \frac{1-2t-\sqrt{(1+2t)(1-6t)}}{4t}$$

The Triangle

Starting the walks in a corner of the triangle, we find

Corollary (Mortimer, Prellberg)

The generating function which counts n-step walks in a triangle of side-length L starting at a chosen corner with no restrictions on the endpoint is given by

$$\frac{(1-p^3)(1-p^{1+L})}{(1-p)(1-p^{3+L})}$$

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where

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It is not obvious that this is a rational function in t, but it is indeed one, and it has a very special structure ...

Outline

Continued Fractions

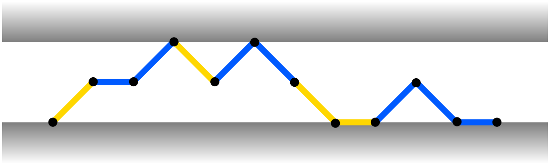
For $L = 2H$ even, the GF expands as a continued fraction of length H,

$$\frac{(1-p^3)(1-p^{1+2H})}{(1-p)(1-p^{3+2H})} = \underbrace{\cfrac{1}{1-2t-\cfrac{4t^2}{1-2t-\cfrac{4t^2}{\ddots-\cfrac{4t^2}{1-2t-4t^2}}}}}_{\text{length } H}$$

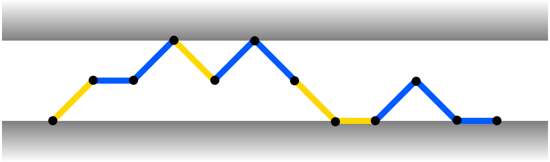
and for $L = 2H+1$ odd, a continued fraction of length $H+1$,

$$\frac{(1-p^3)(1-p^{2+2H})}{(1-p)(1-p^{4+2H})} = \underbrace{\cfrac{1}{1-2t-\cfrac{4t^2}{1-2t-\cfrac{4t^2}{\ddots-\cfrac{4t^2}{1-2t}}}}}_{\text{length } H+1}$$

Bi-Colored Motzkin Paths



Bi-Colored Motzkin Paths



Corollary (Mortimer, Prellberg)

(a) n-step walks starting in a corner of a triangle of side-length $L = 2H + 1$ with arbitrary endpoint are in bijection with bi-colored n-step Motzkin paths in a strip of height H.

(b) n-step walks starting at a corner of a triangle of side-length $L = 2H$ with arbitrary endpoint are in bijection with bi-colored n-step Motzkin paths in a strip of height H, such that horizontal steps at height H are forbidden.

Outline

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Open Problem

Find a bijection for triangles!