On the Asymptotics of a partial theta function

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Abstract

The MSci project is based on V.P Kostov, Rev. Mat. Complut. "Asymptotics of the spectrum of partial theta function", from which we call the spectrum of $\theta(q, x) :=$ $\sum_{j=0}^{\infty} q^{\binom{j+1}{2}} x^j$, the set Γ of values of $q \in (0, 1)$ for which $\theta(q, .)$ has a multiple real zero. We refer to its elements as spectral numbers, denoted by

 $0 < \tilde{q} = \tilde{q_1} < \tilde{q_2} < \cdots < \tilde{q_j} < \cdots < 1$, $\lim_{j \to +\infty} \tilde{q_j} = 1$. As q increases from 0 to 1 and when it passes through a value $\tilde{q_j}$ of the spectrum, the rightmost two of the real zeros coalesce and then form a complex pair. Hence geometrically speaking we think of spectral numbers as being bifurcation points of θ . At the value of these bifurcation points $\tilde{q_j}$, $\theta(\tilde{q_j}, x)$ has a corresponding local minimum for $x = y_j$.

Essentially, Kostov's paper extends the approximations of the spectral numbers, \tilde{q}_j 's made in an earlier paper of Kostov's [2], whilst giving rise to a truly remarkable and ingenious result, of which absolutely astounds me, by providing us with an asymptotic approximation for their associative y_j 's, namely

$$\lim_{j \to \infty} y_j = -e^{\pi}.$$

I will be explaining the proof of this, as well as suggesting ideas of further work to possibly generalise this theorem for the β -family of functions $\theta_{\beta}(q, x) := \sum_{j=0}^{\infty} q^{\binom{j+1+\beta}{2+\beta}} x^j$, where $\beta = 0$ is the case considered by Kostov himself.

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1 Introduction

1.1 Some facts about the function θ

Consider the bivariate series $\theta(q, x) := \sum_{j=0}^{\infty} q^{j(j+1)/2} x^j$. For each fixed $q \in [0, 1)$ it defines an entire function in x. In what follows we are interested in the case $x \in \mathbb{R}$. We say that the series defines a *partial theta function* and we regard x as a variable and q as a parameter. The terminology 'partial theta function' is explained by the fact that the Jacobi theta function is defined by the series $\sum_{j=-\infty}^{\infty} q^{j^2} x^j$ and one has $\theta(q^2, \frac{x}{q}) = \sum_{j=0}^{\infty} q^{j^2} x^j$ (i.e. in the definition of θ only a partial summation is performed). The function θ satisfies the following functional equation:

$$\theta(q, x) = 1 + qx\theta(q, qx) \tag{1}$$

This can be shown true since

$$\sum_{j=0}^{\infty} q^{j(j+1)/2} x^j = 1 + \sum_{j=1}^{\infty} q^{j(j+1)/2} x^j$$
$$= 1 + \sum_{j=0}^{\infty} q^{(j+1)(j+2)/2} x^{j+1}$$
$$= 1 + \sum_{j=0}^{\infty} q^{j(j+1)/2} (qx)^{j+1}$$
$$= 1 + qx \sum_{j=0}^{\infty} q^{j(j+1)/2} x^j$$

The function θ also satisfies the following differential equation, which is merely stated in Kostov's paper

$$2q\partial\theta/\partial q = 2x\partial\theta/\partial x + x^2\partial^2\theta/\partial x^2 = x\partial^2(x\theta)/\partial x^2 \tag{2}$$

To derive the first equality

$$\partial \theta / \partial q = \sum_{j=0}^{\infty} [j(j+1)/2] q^{\frac{j(j+1)}{2} - 1} x^j$$

$$\Rightarrow 2q\partial\theta/\partial q = 2\sum_{j=0}^{\infty} [j(j+1)/2] q^{\frac{j(j+1)}{2}} x^j = \sum_{j=0}^{\infty} [j(j+1)] q^{\frac{j(j+1)}{2}} x^j \text{ and we also have } \partial\theta/\partial x = \sum_{j=0}^{\infty} j q^{\frac{j(j+1)}{2}} x^{j-1} \Rightarrow 2x\partial\theta/\partial x = 2\sum_{j=0}^{\infty} j q^{\frac{j(j+1)}{2}} x^j$$
which then means we have $\partial^2 \theta/\partial x^2 = \sum_{j=0}^{\infty} j(j-1)q^{\frac{j(j+1)}{2}} x^{j-2}$

$$\Rightarrow x^2 \partial^2 \theta/\partial x^2 = \sum_{j=0}^{\infty} j(j-1)q^{\frac{j(j+1)}{2}} x^j$$

$$\Rightarrow 2x\partial\theta/\partial x + x^2 \partial^2 \theta/\partial x^2 = \sum_{j=0}^{\infty} 2jq^{\frac{j(j+1)}{2}} x^j + \sum_{j=0}^{\infty} j(j-1)q^{\frac{j(j+1)}{2}} x^j = \sum_{j=0}^{\infty} [2j+j(j-1)]q^{\frac{j(j+1)}{2}} x^j = \sum_{j=0}^{\infty} [2j+j^2-j)]q^{\frac{j(j+1)}{2}} x^j = \sum_{j=0}^{\infty} [j(j-1)]q^{\frac{j(j+1)}{2}} x^j = 2q\partial\theta/\partial q$$

Similarly, to derive the second equality

$$\begin{split} x\theta &= \sum_{j=0}^{\infty} q^{\frac{j(j+1)}{2}} x^{j+1} \\ \Rightarrow \partial(x\theta)/\partial x &= \sum_{j=0}^{\infty} (j+1)q^{\frac{j(j+1)}{2}} x^j \Rightarrow \partial^2(x\theta)/\partial x^2 = \sum_{j=0}^{\infty} j(j+1)q^{\frac{j(j+1)}{2}} x^{j-1} \Rightarrow x\partial^2(x\theta)/\partial x^2 = \\ \sum_{j=0}^{\infty} j(j+1)q^{\frac{j(j+1)}{2}} x^j &= 2q\partial\theta/\partial q = 2x\partial\theta/\partial x + x^2\partial^2\theta/\partial x^2 \end{split}$$

For any fixed $q \in (0,1)$, the function $\theta(q,.)$ has infinitely many real zeros all of which are negative. It is shown in [1] that \exists constant $\tilde{q} \in (0,1)$ with the following property: for $q \in (0, \tilde{q})$ (where the more accurate value 0.3092493386...of $\tilde{q} = (\tilde{q}_1)$ is given in [2]) the function $\theta(q,.)$ has only real negative simple zeros and that $\theta(\tilde{q},.)$ has a double negative zero, the rest of the zeros being negative and simple.

In [2] one can find the value $-7.5032559833...(=y_1)$ of the double zero of $\theta(\tilde{q}, .)$.

Remark 1.1. This value of \tilde{q} (= $\tilde{q_1}$) above is called the first spectral number of $\theta(q,.)$ and from [2] we find its corresponding local minimum (to 10 decimal places), which is the value y_1 of the negative double root of $\theta(\tilde{q_1},.)$ to be -7.5032559833.

I will give some formal definitions regarding spectral numbers and their corresponding local minimums below and in the next section I will be explaining in detail (and showing visually) what these values represent.

Remark 1.2. In equation (3) the term $2x\partial\theta/\partial x = 0$ at a local minimum while the term $x^2\partial^2\theta/\partial x^2 > 0$ (due to definition of local minimum). Hence $\partial\theta/\partial q > 0$, i.e. the values of θ at

its local minima (when considered as functions of q) increase. In the same way, the values at its local maxima decrease.

Remark 1.3. It is shown in [3] that it is indeed at a local minimum of $\theta(q, .)$ that a double zero and then a complex conjugate pair is born.

1.2 Formal definition & related remarks

Now, for a definition from Kostov's paper and vital related remarks, needed to understand the main theorem in Kostov's paper:

Definition 1.4. We call the **spectrum of** θ , the set Γ of values of $q \in (0, 1)$ for which $\theta(q, .)$ has a multiple real zero. These such values of q are referred to as **spectral numbers**.

Remarks:

 It is proved in [3] that Γ consists of countably-many real numbers (referred to as spectral numbers) denoted by

$$0 < \tilde{q} = \tilde{q_1} < \tilde{q_2} < \cdots < \tilde{q_j} < \cdots < 1, \lim_{j \to +\infty} \tilde{q_j} = 1.$$

- For q̃_j ∈ Γ, θ(q̃_j,.) has exactly one multiple real zero which is negative, multiplicity 2 and is the rightmost of its real zeros (i.e. largest real zero).
 We denote it by y_j.
- 3. The function $\theta(\tilde{q}_j, .)$ has a local minimum at y_j . (i.e. the local minima of $\theta(\tilde{q}_j, .)$ give rise to double zeros as q increases)
- 4. The function $\theta(q_j \epsilon, .)$ [resp. $\theta(q_j + \epsilon, .)$] has two real zeros close to y_j , namely ϵ_{2j} and ϵ_{2j-1} .
- 5. For $q \in (\tilde{q}_j, q_{j+1})$ the function θ has exactly j complex conjugate pairs of zeros (counted with multiplicity).

6. Denote by $\cdots < \xi_2 < \xi_1 < 0$ the zeros of $\theta(q, .)$ for $q < \tilde{q}$. When q increases from 0 to 1 and when it passes through a value \tilde{q}_j of the spectrum, the rightmost two of the real zeros coalesce and then form a complex pair (think of this as a bifurcation of real roots into complex ones). The other real zeros remain real negative and distinct.

In the next section, I shall be giving a clear and visual explanation of Remarks (3) – (6), but first, allow me to introduce Kostov's Theorem from his paper, in which he gives an asymptotic estimate of the spectral numbers \tilde{q}_j and an asymptotic approximation of their corresponding local minima y_j 's:

Theorem 1.5. (1) One has $\tilde{q_j} \approx 1 - (\frac{\pi}{2j}) + o(\frac{1}{j})$

(2) One has $\lim_{j \to \infty} y_j = -e^{\pi} = -23.1407 \cdots$

Note: In [2] the following approximation of \tilde{q}_j is suggested:

$$\tilde{q}_j \approx 1 - \frac{1}{0.6388j + 0.8022} \tag{3}$$

The first 40 spectral numbers \tilde{q}_j are well approximated by this formula. This is due to the fact that

 $1 - \frac{1}{0.6388j + 0.8022} = 1 - \frac{1}{0.6388j} + O(\frac{1}{j^2}) \text{ and we have } 0.6388 \text{ is quite close to } \frac{2}{\pi} = 0.6366197723.$ Remark 1.6. Essentially, Kostov's paper extends the approximations of the spectral numbers, $\tilde{q}_j's$ made in [2]. Not only this, but provides an asymptotic approximation for their associative y'_js with $(\lim_{j\to\infty} \tilde{q}_j = 1) \land (\lim_{j\to\infty} y_j = -e^{\pi})$

Remarks 2 (1) [4] announces the following result: The sum of the series $\theta(q, x) := \sum_{j=0}^{\infty} q^{j(j+1)/2} x^j$ (considered for $q \in (0, 1)$ and x complex) tends to 1/(1-x) (for x fixed and as $q \to 1^-$) exactly when x belongs to the interior of the closed Jordan curve $e^{|s|+is}, s \in [-\pi, \pi]$.

(2) When both x and q are real this theorem implies that for $q \in (0,1)$ and as $q \to 1^-$, the

function $\theta(q, .)$ tends to 1/(1 - x) exactly if $x \in (-e^{\pi}, 1)$. This result is in line with part (2) of Kostov's Theorem.

1.3 Further directions and miscellanea

Definition 1.7. The Laguerre-P \acute{o} **lya class** (*L-PI*) is the class of entire functions consisting of those functions which are locally the limit of a series of polynomials whose roots are all real.

One can easily see that $\theta(q, x)$ has a positive radius of convergence as a function of x if and only if $|q| \leq 1$. If |q| = 1, then $\theta(q, x)$ has a radius of convergence equal to 1; while $\forall q$ with |q|<1, the function $\theta(q, x)$ is entire. Moreover, for small positive q, the series $\theta(q, x)$, considered as a function of x belongs to the Laguerre-Pólya class (*L-PI*), i.e., it has all roots negative.

In fact, we have the following interesting statement:

$$\theta(q, x) \in L\text{-}PI \Leftrightarrow q \in (0, \tilde{q_1}) \tag{4}$$

Remark 1.8. A consequence of $\theta(q, x) \in L$ -PI (for $q \in (0, \tilde{q}_1)$) gives the following three properties of $\theta(q, x)$ for $q \in (0, \tilde{q}_1)$:

- The roots of $\theta(q, x)$ are all real.
- The nonzero zeros x_n satisfy $\sum_n \frac{1}{|x_n|^2}$ converges, with zeros counted according to their multiplicity.
- The function $\theta(q, x)$ can be expressed in the form of a Hadamard product

$$x^m e^{a+bx+cx^2} \prod_n (1-\frac{x}{x_n}) exp(\frac{x}{x_n})$$
(5)

with $b, c \in \mathbb{R}$, $c \leq 0$. (The non-negative integer m will be positive if $\theta(q, 0) = 0$. Note that if the number of zeros is infinite (which is true in this case), one may have to define how to take the infinite product.)

2 Visual explanation of Kostov's Theorem & previous related remarks

In this chapter, I will be giving visual explanations of remarks (3)-(6) of the first chapter and of Kostov's main Theorem. I start by using maple to plot $\theta(q, x)$, truncated to order 20, to first give an idea of the type of function we are dealing with:

> theta:=(x,q,N) -> sum(x^n*q^binomial(n+1,2),n=0..N);

 $\theta:=(x,q,N) \rightarrow \sum_{n=0}^N x^n q^{binomial(n+1,2)}$

> series(theta(x,y,10),x);

 $1 + yx + y^3x^2 + y^6x^3 + y^{10}x^4 + y^{15}x^5 + O(x^6)$

> with(plots): implicitplot(theta(x,q,20),q=0..1,x=-30..30, gridrefine=5,view=[0..1,-30..0]);



Note: I will denote the truncated partial theta function by $\theta_N(q, x)$.

From [2] we have the following approximations of the first three spectral numbers:

 $\tilde{q_1} \approx 0.3092, \, \tilde{q_2} \approx 0.5169, \, \tilde{q_3} \approx 0.6306$, from which, using the above graph one can read off approximations of their corresponding y_j values.

Now, increasing the order of $\theta_N(q, x)$ to say N=30, gives a more approximate graph of $\theta(q, x)$ and hence a clearer approximation of the spectral numbers \tilde{q}_j 's and their associative y_j 's:

>implicitplot(theta(x,q,30),q=0..1,x=-30..30,gridrefine=5, view=[0..1,-30..0]);



The following graph includes the limit point of both the y_j $(\lim_{j\to\infty} y_j = -e^{\pi})$ and the \tilde{q}_j $(\lim_{j\to+\infty} \tilde{q}_j = 1)$ values as well as the graph of $\theta(q, x)$ truncated to Order 40:

```
> with(plots):p1:=implicitplot(theta(x,q,40),q=0..1,x=-30..30,gridrefine=5,
view=[0..1,-30..0]):
> p2:=plot([[1,-exp(Pi)]],style=point,symbolsize=30,color=blue):
> display(p1,p2);
```



Now, for the visual explanations on Remarks (3)-(6) (page 5-6):

Regarding Remark (6), the following slides from an animation of the roots of $\theta_{30}(q, x)$ as q increases from 0 to 1 shows this bifurcation of real roots into complex ones very clearly. One sees that (as already stated generally in Remark (6)) when q increases from 0 to 1 and passes through the first value \tilde{q}_1 of the spectrum, the rightmost two of the real zeros (ϵ_1, ϵ_2) coalesce and then form a complex conjugate pair.

animate(complexplot,[['fsolve(theta(x,q,30),complex)'],-10..10,-10..10,style=point],
q=[seq(k/1000,k=306..311)]);

Instead of the actual animation, on the next page I show specific slides showing how the rightmost two real roots of $\theta(x, q, 30) = 0$ for q increasing from 0.306 (top) via 0.308 (middle) to 0.311 (bottom) collide and form a complex conjugate pair.



Remark 2.1. In slides q=0.306, q=0.308 above, ϵ_2 is seen as the leftmost real simple zero whereas ϵ_1 is the rightmost.

Remark 2.2. In between q=0.308 and q=0.311, the two real roots (ϵ_1, ϵ_2) collide and a bifurcation takes place (at $x = y_1$:=min turning point), with a complex conjugate pair being created.

Regarding Remark (5): For the purpose of simply explaining what this statement tries to show, allow me to truncate $\theta(q, .)$ to say Order 30, disregarding error terms produced by truncation as irrelevent for this purpose.

Using Maple again, let:

> theta:=(x,q,N) -> sum(x^n*q^binomial(n+1,2),n=0..N);

 $\theta := (x,q,N) \to \sum_{n=0}^{N} x^n q^{binomial(n+1,2)}$

Let N=30 and solving the roots for q=0.306, 0.308, 0.311 we get:

fsolve(theta(x,0.306,30),complex);

 $-1.278896245 \cdot 10^{15}, -1.041614412 \cdot 10^{15}, -2.501560001 \cdot 10^{14}, -7.683085837 \cdot 10^{13}, -2.350947190 \cdot 10^{14}, -2.3$

 $10^{13}, -7.193899030 \cdot 10^{12}, -2.201333103 \cdot 10^{12}, \dots, -114.0512615, -35.02876733, \epsilon_2 = -8.412572144, \epsilon_1 = -6.851733526.$

fsolve(theta(x,0.308,30),complex);

 $-1.012363550 \cdot 10^{15}, -8.914705317 \cdot 10^{14}, -2.084025643 \cdot 10^{14}, -6.443859282 \cdot 10^{13}, -1.984638117 \cdot 10^{13}, -6.112686009 \cdot 10^{12}, -1.882707287 \cdot 10^{12}, \dots, -111.1173155, -34.35775116, \epsilon_2 = -8.031946335, \epsilon_1 = -7.072798526.$

fsolve(theta(x,0.311,30),complex);

$$\begin{split} \beta_1 &= -7.090368538 \cdot 10^{14} - 5.342026391 \cdot 10^{13}I, \\ \beta_2 &= -7.090368538 \cdot 10^{14} + 5.342026391 \cdot 10^{13}I, \\ -1.588152030 \cdot 10^{14}, -4.960059198 \cdot 10^{13}, \\ \dots, -343.7146990, -106.8909855, -33.38380732, \\ \alpha_1 &= -7.435340550 - .5601934500I, \\ \alpha_2 &= -7.435340550 + .5601934500I. \end{split}$$

Notice how for $q = 0.311 \in (\tilde{q}_1, \tilde{q}_2) \theta_N(q, .)$ has **2** complex conjugate pair of zeros. However, only the smaller conjugate pair (α_1, α_2) is a legitimate pair of zeros of θ . The extremely large conjugate pair (β_1, β_2) is a consequence of truncating θ (only small values of x close to the origin are considered actual zeros of θ and we disregard the extreme values due to truncation errors). Hence we only really have 1 conjugate pair when considering roots of θ .

As stated previously, we have in general for $q \in (\tilde{q}_j, q_{j+1})$ the function θ has exactly j complex conjugate pairs of zeros (counted with multiplicity).

Regarding Remarks (3) & (4): The following graphs of $\theta(q, .)$ show how the local minima of $\theta(q, .)$ give rise to double zeros as q increases:

- > theta2:=(x,q)->sum(x^n*q^(n*(n+1)/2),n=0..20);
- $\theta 2 := (x,q) \to \sum_{n=0}^{20} x^n q^{\frac{1}{2}n(n+1)}$
- > plot([theta2(x,0.306),theta2(x,0.308),theta2(x,0.311)],x=-10..0, view=[-9..-6,-0.05..0.05]);



The graph above shows curves for $\theta(0.306, x)$ (bottom), $\theta(0.308, x)$ (middle), $\theta(0.311, x)$ (top) As you can see, $\exists q \in (0.306, 0.311)$ s.t. $\theta(q, .)$ has a double zero. This value of q is the first spectral number $\tilde{q_1}$ and note that the local minimum is $x = y_1 \approx -7.5$, which gives rise to a double zero.

Regarding Remark (4); In the above example, we let $\epsilon > 0$ s.t. $\theta(\tilde{q}_1 - \epsilon, .) = \theta(0.306, .)$. It's clearly visible that \exists two real zeros close to $x = y_1$, namely ϵ_1 and ϵ_2 in this case (where in the above graph, again we have for q=0.306, q=0.308, ϵ_2 is the leftmost real simple zero, whereas ϵ_1 is the rightmost).

Graphing the same plots for the second spectral number \tilde{q}_2 gives:

> plot([theta2(x,0.50),theta2(x,0.51),theta2(x,0.52)],x=-20..0);



The graph above shows curves for $\theta(0.50, x)$ (bottom), $\theta(0.51, x)$ (middle), $\theta(0.52, x)$ (top). Now $\exists q \in (0.51, 0.52)$ s.t. $\theta(q, .)$ has a double zero and this value of q is the second spectral number \tilde{q}_2 .

Note that the local minimum of $\theta(\tilde{q}_2, .)$ is $x = y_2 \approx -11.2$.

Note also that here, regarding Remark (4), in this example (for $\theta(0.51, .)$), we let $\epsilon > 0$ s.t. $\theta(\tilde{q}_2 - \epsilon, .) = \theta(0.51, .)$. It's clearly visible that \exists two real zeros close to $x = y_2$, namely ϵ_3 and ϵ_4 in this case (where in the above graph, we have for q=0.5, q=0.51, ϵ_4 is the leftmost real simple zero, whereas ϵ_3 is the rightmost).

In general, $\theta(\tilde{q}_j - \epsilon, .)$ and $\theta(\tilde{q}_j + \epsilon, .)$ has two real zeros close to y_j , namely ϵ_{2j} and ϵ_{2j-1} (which is exactly what Remark (4) stated).

3 Proof of Kostov's Theorem

In this section, I am going to be explaining the proof of Kostov's Theorem (seen in Introduction 1.2) in detail.

3.1 Beginning of the proof of Kostov's Theorem

Facts:

- 1. We have already seen and explained previously that the local minima of θ give rise to double zeros as q increases.
- 2. From 'definition 5' of Kostov's paper we have; for k even (resp. odd) the number $-q^{-k-\frac{1}{2}}$ corresponds to a local maximum (resp. minimum) of $\theta(q, .)$.

Idea:

We want to first prove the asymptotic behaviour indicated in the Theorem correct not for the values of the spectrum of θ , but for the values for which one has $\theta(q, -q^{-k-\frac{1}{2}}) = 0$. In what follows we are interested in the odd values of k. For these values (seen in fact (2) above) the number $-q^{-k-\frac{1}{2}}$ corresponds to a local minimum of θ , where the double zeros arise. Set k := 2s - 1, where s = 1, 2, ... Denote by $\tilde{r_s}$ the solution to the equation $\theta(q, -q^{-2s+\frac{1}{2}}) = 0$ and set $z_s := -(\tilde{r_s})^{-2s+\frac{1}{2}}$ (It's clear why we have defined z_s to be this- \forall s, we have r_s is the s - th spectral number and as $-q^{-k-\frac{1}{2}}$ corresponds to each successive local minimum, we have $z_s := -(\tilde{r_s})^{-2s+\frac{1}{2}}$).

We first prove the following propostion about quantities \tilde{r}_s and z_s . Later we compare these numbers (\tilde{r}_s, z_s) with \tilde{q}_s and y_s respectively and we show that the Theorem holds true (see the completion of the proof of Kostov's Theorem).

Proposition 3.1. 1. One has $\tilde{r_s} = 1 - (\frac{\pi}{2s}) + o(\frac{1}{s})$

2. One has $\lim_{s \to \infty} z_s = -e^{\pi} = -23.1407 \cdots$

The following equalities about $\theta(q, .)$ hold true:

$$\theta(q, -q^{-k-\frac{1}{2}}) = \sum_{j=0}^{\infty} (-1)^j q^{\frac{j(j-2k)}{2}} = \sum_{j=-k}^{\infty} (-1)^{j+k} q^{\frac{j^2-k^2}{2}} = \sum_{j=-\infty}^{\infty} (-1)^{j+k} q^{\frac{j^2-k^2}{2}} - \sum_{j=k+1}^{\infty} (-1)^{j+k} q^{\frac{j^2-k^2}{2}} (**)$$

We have that
$$\sum_{j=-\infty}^{\infty} (-1)^{j+k} q^{\frac{j^2-k^2}{2}} = (-1)^k q^{\frac{-k^2}{2}} \sum_{j=-\infty}^{\infty} (-1)^j q^{\frac{-k^2}{2}} (\sum_{j=-\infty}^1 (-1)^j q^{\frac{j^2}{2}} + 1 + \sum_{j=1}^\infty (-1)^{j+k} q^{\frac{j^2}{2}}) = (-1)^k q^{\frac{-k^2}{2}} (1 + 2\sum_{j=1}^\infty (-1)^{j+k} q^{\frac{j^2}{2}})$$
$$= (-1)^k q^{\frac{-k^2}{2}} \psi(q^{\frac{1}{2}}) \text{ (where } \psi(q) = 1 + 2\sum_{j=1}^\infty (-1)^j q^{j^2}).$$
From (**) it follows that $(-1)^k q^{\frac{-k^2}{2}} \psi(q^{\frac{1}{2}}) - \sum_{j=k+1}^\infty (-1)^{j+k} q^{\frac{j^2-k^2}{2}} = 0.$ This implies that $\psi(q^{\frac{1}{2}}) = \sum_{j=k+1}^\infty (-1)^{j+k} q^{\frac{j^2-k^2}{2}}.$

For k := 2s - 1, this becomes:

$$\psi(q^{\frac{1}{2}}) = \sum_{j=2s}^{\infty} (-1)^j q^{\frac{j^2}{2}} =: \lambda_s(q) = q^{2s^2} \chi_s(q)$$
(6)

The important thing to notice is that we have turned solving

 $\theta(q, -q^{-2s+\frac{1}{2}}) = 0$ into solving $\psi(q^{\frac{1}{2}}) = \sum_{j=2s}^{\infty} (-1)^j q^{\frac{j^2}{2}}$, where the function $\psi(q^{\frac{1}{2}}) := 1 + 2\sum_{j=1}^{\infty} (-1)^j q^{\frac{j^2}{2}} = -1 + 2\theta(q^2, -\frac{1}{q})$ has many important properties (Theorem 6 of Kostov's paper, proved in [5]), which allow us to prove the following Lemma, which in turn we deduce Proposition 3.1 from it.

Lemma 3.2. 1. One has $\lim_{q \to 1^-} \lambda_s(q) = \lim_{q \to 1^-} \chi_s(q) = \frac{1}{2}$

- 2. For $s \in \mathbb{N}$ sufficiently large the graphs of the functions $\psi(q^{\frac{1}{2}})$ and $\lambda_s(q)$ (considered for $q \in [0,1]$) intersect at exactly one point belonging to (0,1) and at 1.
- 3. For $q \in [0,1]$, the inequality $\lambda_s(q) \ge \lambda_{s+1}(q)$ holds true with equality for q = 0 and q=1.
- 4. For $q \in [0, 1]$ one has $\frac{1}{2} \le \chi_s(q) \le 1$.

Before proving Lemma 3.2, we deduce Proposition 3.1 from it. Part 2 of Lemma 3.2 implies that for each s sufficiently large the number \tilde{r}_s is correctly defined (since $\psi(q^{\frac{1}{2}})$ and $\lambda_s(q)$ intersect at exactly one point). Part 3 implies that the numbers $\tilde{r_s}$ form an increasing sequence. Indeed this follows from $\psi(q^{\frac{1}{2}})$ being a decreasing function, see part 1 of Theorem 6 of Kostov's paper.

Set $\tilde{r_s} := 1 - \frac{h_s}{s}$. Consider the equalities (6). It is stated in Kostov's paper that the left-hand side is representable in the form $e^{-((\pi^2 - \epsilon_s)/4)(2s/h_s)}$ (where $\epsilon_s \ge 0$ and $\lim_{s \to \infty} \epsilon_s = 0$, see part 5 of Theorem 6 of Kostov's paper).

My proof that the LHS of (6) is representable in the form $e^{-((\pi^2 - \epsilon_s)/4)(2s/h_s)}$.

By Theorem 6 of Kostov's Theorem (part 6) we have $\forall \epsilon > 0, \exists \delta > 0$ s.t.

$$e^{\frac{\pi^2}{4(q-1)}} < \psi(q) < e^{\frac{\pi^2 - \epsilon}{4(q-1)}}.$$

Now, by the IVT, $\exists \epsilon_s \text{ s.t. } \psi(q^{\frac{1}{2}}) = e^{\frac{\pi^2 - \epsilon_s}{1}} \text{ where } \epsilon_s \ge 0, \lim_{s \to \infty} \epsilon_s = 0.$ Let $q = \tilde{r_s} = 1 - \frac{h_s}{s} \Rightarrow \psi(q^{\frac{1}{2}}) = e^{-((\pi^2 - \epsilon_s)/4)(2s/h_s)}.$

The right-hand side of (6) equals $(1 - \frac{h_s}{s})^{2s^2} \chi_s (1 - \frac{h_s}{s})$ (simply by substituting $q = 1 - \frac{h_s}{s}$). It is further stated that

$$\left(1 - \frac{h_s}{s}\right)^{2s^2} = \left(\left(1 - \frac{h_s}{s}\right)^{\frac{s}{h_s}}\right)^{2h_s s} = \left(e^{-1} + \eta_s\right)^{2h_s s} \tag{7}$$

where $\lim_{s \to \infty} \eta_s = 0.$

My proof of (7). Firstly, It's clearly visible that $(1 - \frac{h_s}{s})^{2s^2} = ((1 - \frac{h_s}{s})^{\frac{s}{h_s}})^{2h_s s}$

For the second equality, we simply use the fact that

$$\lim_{n \to \infty} (1 + \frac{k}{n})^n = e^k \text{ by letting } n = \frac{s}{h_s} \Rightarrow \frac{1}{n} = \frac{h_s}{s}$$
$$\Rightarrow \lim_{s \to \infty} ((1 - \frac{h_s}{s})^{\frac{s}{h_s}}) = \lim_{n \to \infty} (1 + -\frac{1}{n})^n = e^{-1} \text{ (with } n = \frac{s}{h_s} \text{ as above)}.$$

Hence, as $n \to \infty$ (equivalently as $s \to \infty$), we have $\left(\left(1 - \frac{h_s}{s}\right)^{\frac{s}{h_s}}\right) \to e^{-1}$ but as s is simply fixed and not specified as tending to infinity we have

$$((1 - \frac{h_s}{s})^{\frac{s}{h_s}}) = e^{-1} + \eta_s$$

With $\eta_s \to 0$ as $s \to \infty$

Taking the logarithms of both sides of (7) yields:

$$-\left((\pi^2 - \epsilon_s)/4\right)\left(\frac{2s}{h_s}\right) = -(2h_s s)(1 + o(1)) + \log(\chi_s(1 - \frac{h_s}{s}))$$
(8)

$$\begin{split} &My \ proof \ of \ (8). \ \log(e^{-((\pi^2 - \epsilon_s)/4)(\frac{2s}{h_s})}) = \log((1 - \frac{h_s}{s})^{2h_s s} \chi_s(1 - \frac{h_s}{s})) \ \text{since} \\ &(1 - \frac{h_s}{s})^{2s^2} = (e^{-1} + \eta_s)^{2h_s s}. \\ &\text{This implies } -((\pi^2 - \epsilon_s)/4)(\frac{2s}{h_s}) = \log((e^{-1} + \eta_s)^2 h_s s) + \log(\chi_s(1 - \frac{h_s}{s})) \\ &\text{We have } \log((e^{-1} + \eta_s)^{2h_s s}) = 2h_s s[\log(e^{-1} + \eta_s)] = 2h_s s[\log(e^{-1}(1 + \frac{\eta_s}{e^{-1}}))] = 2h_s s[\log(e^{-1} + \eta_s)] \\ &\log(1 + \frac{\eta_s}{e^{-1}})] \end{split}$$

Where $\log(1 + \frac{\eta_s}{e^{-1}}) = o(1)$ since $\lim_{s \to \infty} \eta_s = 0$

Hence $\log(1 + \frac{\eta_s}{e^{-1}})$ is bounded above by 1 as $s \to \infty$ so we get $2h_s s[\log(e^{-1} + \log(1 + \frac{\eta_s}{e^{-1}})] = 2h_s s(-1 + o(1)) = -2h_s s(1 + o(1))$

As $\chi_s(q) \in [\frac{1}{2}, 1]$ (see part 4 of the Lemma), $\log(\chi_s(1 - \frac{h_s}{s})) \in [\log(\frac{1}{2}), 0]$.

Dividing both sides of Eq. (8) by s and comparing expressions afting taking the large s limits one gets $h_s^2 = \frac{\pi^2}{4} + o(1)$, i.e. $h_s = \frac{\pi}{2} + o(1)$.

My proof. Dividing both sides of Eq. (8) by s we get:

$$\frac{-(\pi^2 - \epsilon_s)2s}{4h_s s} = -2h_s(1 + o(1)) + \frac{\log(\chi_s(1 - \frac{h_s}{s}))}{s}$$

Now, since we have $\lim_{s \to \infty} \epsilon_s = 0$ and $\frac{\log(\chi_s(1-\frac{h_s}{s}))}{s} \to 0$ as $s \to \infty$, we then get

$$\frac{\pi^2}{2h_s} = 2h_s(1+o(1)) \Rightarrow \pi^2 = 4h_s^2(1+o(1)) \Rightarrow \frac{\pi^2}{4} = h_s^2(1+o(1)) \Rightarrow h_s^2 = \frac{\pi^2}{4} + o(1) \Rightarrow h_s = \frac{\pi}{2} + o(1).$$

This proves part 1 of Proposition 3.1.

Remark 3.3. I noted an error in Kostov's paper. See the Proof of parts 1 and 4 below for details regarding this.

Part 2:

Proof.
$$z_s = -(\tilde{r_s})^{-2s+\frac{1}{2}} = -(1 - \frac{\pi}{2s} + o(\frac{1}{s}))^{-2s+\frac{1}{2}}$$
.

Taking the limit as s tends to ∞ we get:

$$\lim_{s \to \infty} \left[-\left(1 - \frac{\pi}{2s}\right) + o\left(\frac{1}{s}\right) \right)^{-2s + \frac{1}{2}} \right] \cdot s \to \infty \Rightarrow \frac{1}{s} \to 0 \text{ in which we now have}$$
$$-\lim_{s \to \infty} \left(1 - \left(\frac{\pi}{2s}\right)\right)^{-2s + \frac{1}{2}} = -\lim_{s \to \infty} \left(1 + \left(\frac{\pi}{-2s}\right)\right)^{-2s} = -e^{\pi}.$$
Since
$$\lim_{n \to \infty} \left(1 + \frac{k}{n}\right)^n = e^k.$$

Thus, we have have proved that we deduce Proposition 3.1 from Lemma 3.2. Now, onto the proof of Lemma 3.2.

Proof of parts 1 and 4:

For $\nu > 0$ (where ν need not be integer) we consider the function $\varphi_v(q) := \sum_{j=0}^{\infty} (-1)^j q^{\nu j + j(j-1)/2}$. This function is analytic on [0,1) and $\lim_{q \to 1^-} \varphi_v(q) = \frac{1}{2}$ (see Propostion 14 in [3]).

Kostov's paper suggests that $\lambda_k = q^{(2k+1)/2} \varphi_{(2k+3)/2}(q)$ and $\chi_s(q) = \varphi_{(4s+1)/2}(q)$ (*).

This second equation is correct. However, the first is *not*:

$$\begin{split} \lambda_s(q) &= \sum_{j=2s}^{\infty} (-1)^j q^{\frac{j^2}{2}}, \, \varphi_v(q) := \sum_{j=0}^{\infty} (-1)^j q^{\nu j + j(j-1)/2} \\ \Rightarrow \lambda_s(q) &= \sum_{j=0}^{\infty} (-1)^{j+2s} q^{\frac{(j+2s)^2}{2}} = q^{2s^2} \sum_{j=0}^{\infty} (-1)^j q^{\frac{j^2}{2} + 2js} = q^{2s^2} \varphi_{2s+\frac{1}{2}}(q) = q^{2s^2} \chi_s(q). \end{split}$$
Now, since $h := 2s - 1$, we get this is equal to $q^{(k+1)^2/2} = q^{2s^2} \varphi_{2s+\frac{1}{2}}(q) = q^{2s^2} \chi_s(q).$

Now, since k := 2s - 1, we get this is equal to $q^{(k+1)^2/2}\varphi_{k+\frac{3}{2}}(q)$ by substituting $s = \frac{k+1}{2}$. Not what Kostov's paper suggests $q^{(2k+1)/2}\varphi_{k+\frac{3}{2}}(q)$, which is presumeably a typographical error. Consequently however, this does not change the result of part 1 or part 4 of Lemma 3.2. Since we have $\lim_{q \to 1^-} \varphi_v(q) = \frac{1}{2}$ by Proposition 14 in [3], we hence have:

$$\lim_{q \to 1^{-}} \varphi_k(q) = \lim_{q \to 1^{-}} q^{(k+1)^2/2} \varphi_{k+\frac{3}{2}}(q) = 1 \cdot \frac{1}{2} = \frac{1}{2} = \frac{1}{2} = \lim_{q \to 1^{-}} \chi_s(q) = \frac{1}{2}.$$

This proves part 1.

Equality (*) implies part 4 of Lemma 3.2 for $s \ge 1$ because $\varphi_{\nu}(q) = \theta(q, -q^{\nu-1})$ (see below).

$$\begin{aligned} \theta(q,x) &= \sum_{j=0}^{\infty} q^{j(j+1)/2} x^j \Rightarrow \theta(q,-q^{\nu-1}) = \sum_{j=0}^{\infty} q^{j(j+1)/2} (-q^{\nu-1})^j = \sum_{j=0}^{\infty} (-1)^j q^{\frac{j(j+1)}{2} + (\nu-1)j} = \\ \sum_{j=0}^{\infty} (-1)^j q^{\nu j + \frac{j(j-1)}{2}} = \varphi_{\nu}(q). \end{aligned}$$

For $s \ge 1$ one has $\nu > 1$ hence $\theta(q, -q^{\nu-1}) \in [\frac{1}{2}, 1]$, see part 6 of Theorem 4 of Kostov's paper.

For s = 0 (i.e. $\nu = \frac{1}{2}$) we observe that

$$\chi_0(q) = \varphi_{\frac{1}{2}}(q) := \sum_{j=0}^{\infty} (-1)^j q^{\frac{j^2}{2}} = \frac{1}{2} + \frac{\psi(q^{\frac{1}{2}})}{2}$$
(9)

Since $\psi(q^{\frac{1}{2}}) = 1 + 2\sum_{j=1}^{\infty} (-1)^j q^{\frac{j^2}{2}}$ we have $\frac{1}{2}\psi(q^{\frac{1}{2}}) = \frac{1}{2} + \sum_{j=1}^{\infty} (-1)^j q^{\frac{j^2}{2}}$ $\Rightarrow \frac{1}{2} + \frac{\psi(q^{\frac{1}{2}})}{2} = 1 + \sum_{j=1}^{\infty} (-1)^j q^{\frac{j^2}{2}} = \sum_{j=0}^{\infty} (-1)^j q^{\frac{j^2}{2}} = \varphi_{\frac{1}{2}}(q).$

From Theorem 6, parts 1 and 2 we hence have $\chi_0(q) (= \varphi_{\frac{1}{2}}(q)) \in [\frac{1}{2}, 1]$. This proves part 4.

Proof of part 2:

We represent the function $\lambda_s(q)$ in the form $\rho_1(q) + \rho_2(q) + \cdots$, where $\rho_i(q) = q^{\frac{(2s+2i-2)^2}{2}} - q^{\frac{(2s+2i-1)^2}{2}}$, $i = 1, 2, \cdots$. One checks directly that the following property of the functions $\rho_i(q)$ holds:

(A) Let $A_i := (2s + 2i - 2)^2/2$ and $B_i := (2s + 2i - 1)^2/2$. The function ρ_i is positive valued on (0,1) (obviously $\rho_i(0) = \rho_i(1) = 0$). It is increasing on $[0, \tau_i]$, decreasing on $[\tau_i, 1]$, convex on $[0, \sigma_i]$ and concave on $[\sigma_i, 0]$, where $\tau_i = (\frac{A_i}{B_i})^{\frac{1}{B_i - A_i}}$, $\sigma_i = (\frac{A_i(A_i - 1)}{B_i(B_i - 1)})^{\frac{1}{B_i - A_i}}$. One has $\sigma_i < \tau_i < \tau_{i+1}$.

(The inequality $\tau_i < \tau_{i+1}$ is equivalent to

$$\left(1 + \frac{1}{2s+2i}\right)^{1/(4s+4i+1)} < \left(1 + \frac{1}{2s+2i-2}\right)^{1/(4s+4i-3)} \tag{10}$$

and can be deduced from the fact that for x > 0 the function $(1+1/x)^{1/(2x+1)}$ (above inequality for x = 2s + 2i) is decreasing). We prove below that (B) For s large enough and $q \in (0,1)$ the graphs of $\psi(q^{1/2})$ and $\rho_1(q)$ intersect at only one point $\mu \in (0, \tau_1)$.

(C) If s is large enough, then $\psi(q^{1/2}) < \rho_1(\tau_1)$.

My proof of the above inequality (10).

$$\begin{split} A_{i} &:= (2s+2i-2)^{2}/2, B_{i} := (2s+2i-1)^{2}/2 \text{ and } \tau_{i} = (\frac{A_{i}}{B_{i}})^{\frac{1}{B_{i}-A_{i}}} \Rightarrow \tau_{i} < \tau_{i+1} \Leftrightarrow (\frac{A_{i}}{B_{i}})^{\frac{1}{B_{i}-A_{i}}} < (\frac{A_{i+1}}{B_{i+1}})^{\frac{1}{B_{i+1}}-A_{i+1}} \\ (\frac{A_{i}}{B_{i}})^{\frac{1}{B_{i}}-A_{i}} &= ? \\ \text{Let } A_{i} &:= \frac{u_{i}^{2}}{2}, B_{i} := \frac{(u_{i}+1)^{2}}{2} \Rightarrow B_{i} - A_{i} = \frac{2u_{i}+1}{2} \\ \Rightarrow \tau_{i} &= (\frac{u_{i}^{2}}{u_{i+1}^{2}})^{2/(2u_{i}+1} = (\frac{u_{i}}{u_{i+1}})^{4/(2u_{i}+1)} \\ \text{with } u_{i} &= 2s + 2i - 2, \ u_{i+1} &= u_{i} + 2. \\ \text{So, } \tau_{i} < \tau_{i+1} \Leftrightarrow (\frac{u_{i}}{u_{i+1}})^{4/(2u_{i}+1} < (\frac{u_{i+2}}{u_{i+2}})^{4/(2(u_{i+1}))+1} \Leftrightarrow (\frac{u_{i}}{u_{i+1}})^{4/(2u_{i}+1} < (\frac{u_{i}+2}{u_{i}+4})^{4/(2(u_{i}+2))+1} \\ \Leftrightarrow (\frac{u_{i}}{u_{i+1}})^{4/(2u_{i}+1} < (\frac{u_{i+2}}{u_{i}+4})^{4/(2u_{i}+5)} \\ \Leftrightarrow (\frac{u_{i}}{u_{i+1}})^{1/(2u_{i}+1} < (\frac{u_{i}+4}{u_{i+2}})^{1/(2u_{i}+5)} \\ \Leftrightarrow (1 + \frac{1}{u_{i}})^{1/(2u_{i}+1} > (1 + \frac{1}{u_{i+2}})^{1/(2u_{i}+5)} \\ u_{i} &= 2s + 2i - 2 \text{ hence gives our required inequality:} \end{split}$$

$$(1 + \frac{1}{2s+2i})^{1/(4s+4i+1)} < (1 + \frac{1}{2s+2i-2})^{1/(4s+4i-3)}$$

From the above properties (A), (B) and (C) of the functions p_i , we have that part 2 of Lemma 3.2 follows. Indeed, the properties (A) implies not only that p_i , but also λ_s is increasing on $[0, \tau_1]$ (since p_i all increasing on $[0, \tau_i]$ implies $\lambda_s(q) := \sum p_i$ also increasing on $[0, \tau_1]$; sum of increasing functions is also increasing).

One has
$$\lambda_s(0) = p_1(0) + p_2(0) + p_3(0) + \dots = 0$$
 (since $p_i(0) = 0 \ \forall i$).

 $\lambda_{s}(\tau_{1}) = p_{1}(\tau_{1}) + p_{2}(\tau_{1}) + \dots > p_{1}(\tau_{1}) \text{ (since } p_{2}(\tau_{1}), p_{3}(\tau_{1}), \dots \text{ are all positive valued on } (0, 1)).$ $\psi(\tau_{1}^{\frac{1}{2}}) = 1 + 2 \sum_{j=1}^{\infty} (-1)^{j} \tau_{1}^{\frac{j^{2}}{2}} \text{ is clearly less than } \lambda_{s}(\tau_{1}) = \sum_{j=2s}^{\infty} (-1)^{j} \tau_{1}^{\frac{j^{2}}{2}}.$ So, $\lambda_{s}(\tau_{1}) > p_{1}(\tau_{1}) > \psi(\tau_{1}^{\frac{1}{2}}).$

Now, before we make use of Bolzano's Theoem, let me first define what this is:

Bolzano's Theorem: Bolzano's Theorem is a specialization of the Intermediate Value Theorem and states that if a continuous function has values of opposite sign inside an interval, then it has a root inside that interval.

We have the following facts:

(i) ψ is decreasing $\forall q \in (-1, 1)$ (Theorem 6 part 1).

(ii)
$$\psi(q) := 1 + 2 \sum_{j=1}^{\infty} (-1)^j q^{j^2} \Rightarrow \psi(0) = 1$$

(iii) $\lambda_s(0) = 0$, λ_s increasing on $[0, \tau_1]$ and $\lambda_s(\tau_1) > p_1(\tau_1) > \psi(\tau_1^{\frac{1}{2}})$.

Then, using Bolzano's Theorem, it's clear to see that the graphs of $\psi(q^{\frac{1}{2}})$ and $\lambda_s(q)$ intersect at a point $\in (0, \tau_1)$. This point is unique because $p_1(\tau_1)$ is sandwiched between two strictly monotonic functions.

The rest of the proof of (A) is very clear so I won't be ellaborating on what has already been written. I will however prove property (C). Obviously (B) follows from (C) and Bolzano's theorem $(\psi(\tau_1^{\frac{1}{2}})$ is decreasing, $\psi(0) = 1$, p_1 is increasing, $p_1(0) = 0$ and $p_1(\tau_1) > \psi(\tau_1^{\frac{1}{2}})$).

Now we prove property (C). Consider the straight line L through the points $(\tau_1, p_1(\tau_1))$ and (1, 0). The slope of this line L equals

$$-\frac{p_1(\tau_1) - 0}{1 - \tau_1} = -\frac{(A_1/B_1)^{A_1/(B_1 - A_1)}(B_1 - A_1)/B_1}{1 - (A_1/B_1)^{1/(B_1 - A_1)}}.$$
(11)

Since $\tau_1 = (A_1/B_1)^{1/(B_1-A_1)}$ we therefore have $p_1(\tau_1) = p_1((A_1/B_1)^{1/(B_1-A_1)}) = \tau_1^{A_1} - \tau_1^{B_1}$ (where $A_1 = (2s+2-2)^2/2$, $B_1 = (2s+2-1)^2/2$) $= \tau_1^{A_1}(1-\tau_1^{(B_1-A_1)}) = (A_1/B_1)^{A_1/(B_1-A_1)}(1-\tau_1^{(B_1-A_1)})$

$$(A_1/B_1)) = (A_1/B_1)^{A_1/(B_1 - A_1)} (\frac{B_1 - A_1}{B_1}) = p_1(\tau_1).$$

 \Rightarrow the slope of line L equals

$$-\frac{(A_1/B_1)^{A_1/(B_1-A_1)}(B_1-A_1)/B_1}{1-(A_1/B_1)^{1/(B_1-A_1)}}$$

It is merely stated in Kostov's paper that the factor $(A_1/B_1)^{A_1/(B_1-A_1)}$ tends to e^{-1} as $s \to \infty$:

$$My \text{ proof. } A_1 = (2s)^2/2 = 2s^2, \ B_1 = (2s+1)^2/2 \Rightarrow \frac{A_1}{B_1} = \frac{(2s)^2}{(2s+1)^2} = (\frac{2s}{2s+1})^2 \text{ and } B_1 - A_1 = \frac{(2s+1)^2 - (2s)^2}{2} = \frac{4s^2 + 4s + 1 - 4s^2}{2} = (\frac{4s^2}{4s+1})$$

$$\Rightarrow A_1/(B_1 - A_1) = (2s^2)/((4s+1)/2) = (\frac{4s^2}{4s+1})$$

$$\Rightarrow (A_1/B_1)^{A_1/(B_1 - A_1)} = (\frac{4s^2}{4s^2 + 4s+1})^{4s^2/(4s+1)} = (\frac{1}{(\frac{4s^2 + 4s+1}{4s^2})})^{(4s^2/(4s+1))} = (\frac{1}{1 + \frac{4s+1}{4s^2}})^{(4s^2/(4s+1))} = \frac{1}{[(1 + \frac{4s+1}{4s^2})^{(4s^2/(4s+1))}]}$$

let
$$n = \frac{4s^2}{4s+1}$$
, then letting $s \to \infty \Rightarrow n \to \infty$ also. The above is then equal to

$$\frac{1}{\left[(1+\frac{1}{n})^n\right]}$$

Now, taking the limit as $n \to \infty$ we get $\lim_{n \to \infty} \frac{1}{[(1+\frac{1}{n})^n]} = \frac{1}{e^1} = e^{-1}$.

It is also stated that the factor $(B_1 - A_1)/B_1 = O(\frac{1}{s})$.

My proof.
$$(B_1 - A_1)/B_1 = (\frac{4s+1}{2})/\frac{(2s+1)^2}{2} = \frac{(4s+1)}{(2s+1)^2} = \frac{(4s+1)}{4s^2+4s+1} = O(\frac{1}{s}).$$

It is further stated that the factor $1 - (A_1/B_1)^{1/(B_1 - A_1)} = O(\frac{1}{s^2}).$

My proof. $B_1 - A_1 = \frac{4s+1}{2} \Rightarrow \frac{1}{B_1 - A_1} = \frac{2}{4s+1}$

and $(A_1/B_1) = (\frac{2s}{2s+1})^2 \Rightarrow (A_1/B_1)^{1/(B_1-A_1)} = (\frac{4s^2+4s+1-4s-1}{(2s+1)^2})^{\frac{2}{4s+1}} = (1 - \frac{4s+1}{(2s+1)^2})^{2/(4s+1)} = 1 - \frac{(4s+1)}{(2s+1)^2} \frac{2}{(4s+1)} + \text{higher order terms.}$ $\Rightarrow 1 - (A_1/B_1)^{1/(B_1-A_1)} = O(\frac{1}{s^2}).$

Hence we have that the slope of L is O(s), i.e. tends to $-\infty$ as $s \to \infty$. On the other hand $\psi(q^{\frac{1}{2}})$ is flat at 1 (We use "flat" here as defined in Theorem 6 part 3 of Kostov's paper; for any $l \in \mathbb{N}$, $\psi(q) = o((q-1)^l)$ as $q \to 1^-$) from which property (C) follows.

Proof of part 3:

From the proof of part 2, we have $p_1(q) = q^{\frac{(2s+2-2)^2}{2}} - q^{\frac{(2s+2-1)^2}{2}} = q^{2s^2} - q^{\frac{(2s+1)^2}{2}}$.

We then have:

$$\lambda_{s}(q) - \lambda_{s+1}(q) = \sum_{j=2s}^{\infty} (-1)^{j} q^{\frac{j^{2}}{2}} - \sum_{j=2s+2}^{\infty} (-1)^{j} q^{\frac{j^{2}}{2}} = \sum_{j=2s}^{2s+1} (-1)^{j} q^{\frac{j^{2}}{2}} = (-1)^{2s} q^{2s^{2}} + (-1)^{2s+1} q^{\frac{(2s+1)^{2}}{2}} = q^{2s^{2}} - q^{\frac{(2s+1)^{2}}{2}}$$
(for any $s \in \mathbb{N}$) = $p_{1}(q)$.

So we have that $\lambda_s(q) - \lambda_{s+1}(q) = p_1(q) \ge 0$ (from (A)).

Clearly $p_1(0) = p_1(1) = 0$. So we hence have that $\lambda_s(q) \ge \lambda_{s+1}(q)$ for $q \in [0,1]$ and $\lambda_s(q) = \lambda_{s+1}$ (q) only for q = 0 or q = 1.

Thus we have proved the Lemma and consequently the Proposition too. In doing so, we have proved Kostov's Theorem for the values $q \in (0, 1)$ s.t. $\theta(q, -q^{-2s+\frac{1}{2}}) = 0$. We denoted these q by $\tilde{r_s}$ and have $\tilde{r_s} = 1 - (\frac{\pi}{2s}) + o(\frac{1}{s})$ with their corresponding local minimum turning points z_s .

Now, for completion of the Theorem, we need to prove the Theorem for the values of the spectrum of θ . We do this by comparing \tilde{r}_s and z_s with \tilde{q}_s and y_s respectively and we show that the Theorem must hold true for \tilde{q}_s and y_s also.

3.2 Completion of the proof of Kostov's Theorem

This next section is mostly very clearly explained in Kostov's paper so I will not be greatly elaborating on what has already been written. I will however be making some parts more explicitly understandable where I believe it is needed.

It is clear that $\tilde{q_s} \geq \tilde{r_s}$. Indeed, for $q = \tilde{r_s}$ one has $\theta(q, -q^{-2s+\frac{1}{2}}) = 0$, but at $-q^{-2s+\frac{1}{2}}$ the function $\theta(\tilde{r_s}, .)$ does not necessarily have a double zero. That is, it might have two simple zeros on the interval $(-\tilde{r_s}^{-2s}, -\tilde{r_s}^{-2s+1})$. This follows directly from part 7 of Theorem 4 of Kostov's paper where we have $-q^{-2k} < \epsilon_{2k} < -q^{-2k+\frac{1}{2}} < \epsilon_{2k-1} < -q^{-2k+1}$ for any real participating zeros of $\theta(q, .), \epsilon_{2k}, \epsilon_{2k-1}$.

So a double zero on the interval $(-q^{-2s}, -q^{-2s+1})$ occurs for some value of q greater than(or at most equal to) $\tilde{r_s}$, see Remark 1, i.e. $\tilde{q_s} \geq \tilde{r_s}$. Now, suppose that the sequence $\{\tilde{q_s}\}$ tends to 1 faster than $1 - (\frac{2}{\Pi s})$. (To be more precise. One should speak about a subsequence, but given that $\{\tilde{q_s}\}$ is monotone, the existence of a subsequence tending to 1 faster than $1 - (\frac{2}{\pi s})$ implies that the whole sequence tends to it faster). Then the sequence of double zeros of $\theta(\tilde{q_s}, .)$ cannot tend to $-e^{\pi}$.

More precisely, $\exists \delta > 0, N \in \mathbb{N}$ such that these zeros are less than or equal to $-e^{\pi} - \delta$ for $s \ge N$. Indeed, the double zero y_s belongs to the interval $J := (-\tilde{q_s}^{-2s}, -\tilde{q_s}^{-2s+1})$, i.e. we have

$$-\tilde{q_s}^{-2s} < y_s < -\tilde{q_s}^{-2s+1} \tag{12}$$

(It is the result of the confluence of the simple zeros ϵ_{2s} and ϵ_{2s-1} and one has

$$-\tilde{q_s}^{-2s} < \epsilon_{2s} = \epsilon_{2s-1} = y_s < -\tilde{q_s}^{-2s+1},$$
(13)

see part 7 of theorem 4 of Kostov's paper).

If $\{\tilde{q}_s\}$ tends to 1 faster than $1 - (\frac{2}{\pi s})$, then both extremities of the interval J are $\leq -e^{\pi} - \delta$ for s large enough, i.e. we have

$$-\tilde{q_s}^{-2s+1} \le -e^{\pi} - \delta \tag{14}$$

(We make use of $\lim_{x\to 0} (1+x)^{\frac{1}{x}} = e^1$). This, however, is impossible. Indeed, consider any sequence of numbers $q_s \in I_s$, $I_s := [\tilde{r_s}, \tilde{r_{s+1}}]$. Hence one has $q_s = 1 - (\frac{2}{\pi s}) + o(\frac{1}{s})$, the numbers $-q_s^{-2s-\frac{1}{2}}$ tend to $-e^{\pi}$ as $s \to \infty$ and the convergence can be estimated independently of the choices of q_s in the intervals I_s . Thus for s large enough all such numbers

$$-q_s^{-2s-\frac{1}{2}} \ge -e^{\pi} - \frac{\delta}{2}$$
(15)

On the other hand

 $-\tilde{q_s}^{-2s-\frac{1}{2}} < -\tilde{q_s}^{-2s} < y_s$, see (13). For each index s there exists $\tau(s) \in \mathbb{N} \cup \{0\}$ such that $\tilde{q_s} \in I_{s+\tau(s)}$ (because $\tilde{q_s} \geq \tilde{r_s}$). Hence

$$-\tilde{r}_{s+\tau(s)}^{-2(s+\tau(s))-\frac{1}{2}} = -\tilde{r}_{s+\tau(s)}^{-2s-2\tau(s)-\frac{1}{2}} \le -\tilde{q_s}^{-2s-2\tau(s)-\frac{1}{2}} \le -\tilde{q_s}^{-2s-\frac{1}{2}}$$
(16)

Where $-\tilde{r}_{s+\tau(s)}^{-2(s+\tau(s))-\frac{1}{2}} \leq -\tilde{q_s}^{-2s-2\tau(s)-\frac{1}{2}}$ holds true since $\tilde{q_s} \geq \tilde{r_s}$.

Now, using some of the above information (I will specify which ones after the following inequality), we have that:

$$-e^{\pi} - \frac{\delta}{2} \le -\tilde{r}_{s+\tau(s)}^{-2s-2\tau(s)-\frac{1}{2}} \le -\tilde{q}_s^{-2s-2\tau(s)-\frac{1}{2}} \le -\tilde{q}_s^{-2s-\frac{1}{2}} < -\tilde{q}_s^{-2s} < y_s < -\tilde{q}_s^{-2s+1} \le -e^{\pi} - \delta$$
(17)

(Where the first inequality holds due to (15) and because $q_s \ge \tilde{r_s}$, the second since $\tilde{q_s} \ge \tilde{r_s}$,

the third since \tilde{q}_s is a monotonically increasing sequence, the fourth comes from (13), the fifth and sixth both come from (12) and the last inequality follows from (14)).

Which is a contradiction since $-e^{\pi} - \frac{\delta}{2} > -e^{\pi} - \delta$.

Thus, we have that the sequence of double zeros of $\theta(\tilde{q}_s, .), y_s, \to -e^{\pi}$ as $s \to \infty$, i.e. $\lim_{s \to \infty} y_s = -e^{\pi}.$

This now completes the proof of Kostov's Theorem. In the next section, I shall summarise the main points of Kostov's Theorem, as well as suggesting possible further work to generalise this theorem.

4 Conclusion and possible further work

4.1 Summary of Kostov's paper and of the proof of his Theorem

Regarding Kostov's paper "Asymptotics of the spectrum of partial theta function", we call the spectrum of $\theta(q, x) := \sum_{j=0}^{\infty} q^{\binom{j+1}{2}} x^j$, the set Γ of values of $q \in (0, 1)$ for which $\theta(q, .)$ has a multiple real zero. We refer to its elements as spectral numbers, denoted by $0 < \tilde{q} = \tilde{q_1} < \tilde{q_2} < \cdots < \tilde{q_j} < \cdots < 1$, $\lim_{j \to +\infty} \tilde{q_j} = 1$. As q increases from 0 to 1 and when it passes through a value $\tilde{q_j}$ of the spectrum, the rightmost two of the real zeros coalesce and then form a complex pair. Hence geometrically speaking we think of spectral numbers as being bifurcation points of θ . At the value of these bifurcation points $\tilde{q_j}$, $\theta(\tilde{q_j}, x)$ has a corresponding local minimum for $x = y_j$.

Essentially, Kostov's paper extends the approximations of the spectral numbers, \tilde{q}_j 's made in an earlier paper of Kostov's [2], whilst giving rise to a truly remarkable and ingenious result, of which absolutely astounds me, by providing us with an asymptotic approximation for their associative y_j 's, namely

$$\lim_{j \to \infty} y_j = -e^{\pi}.$$

As you already know by my detailed explanation of the proof of Kostov's Theorem, this proof contains a substantial amount of mathematically challenging and dense material, however, it is quite concise in the way it has been constructed.

I am now going to summarise the main points of the proof of Kostov's Theorem.

Firstly, we prove the asymptotic behaviour indicated in Kostov's Theorem for the values for which one has $\theta(q, -q^{-k-\frac{1}{2}}) = 0$ (k odd, k:=2s-1, s=1,2,...), for these values the number $-q^{-k-\frac{1}{2}}$ corresponds to a local minimum of θ , where the double zeros arise. Denote by \tilde{r}_s the solution to $\theta(q, -q^{-2s+\frac{1}{2}}) = 0$ and set $z_s := -(\tilde{r}_s)^{-2s+\frac{1}{2}}$. After proving this asymptotic behaviour for \tilde{r}_s and z_s , later we compare these numbers (\tilde{r}_s, z_s) with \tilde{q}_s and y_s respectively and we show that the theorem holds true. To prove this behaviour for \tilde{r}_s and z_s , we use a shift of limits to split $\theta(q, -q^{-k-\frac{1}{2}}) = \sum_{j=0}^{\infty} (-1)^j q^{\frac{j(j-2k)}{2}}$ into $\sum_{j=-\infty}^{\infty} (-1)^{j+k} q^{\frac{j^2-k^2}{2}} - \sum_{j=k+1}^{\infty} (-1)^{j+k} q^{\frac{j^2-k^2}{2}}$. After some rearrangement, we conclude that solving $\theta(q, -q^{-k-\frac{1}{2}}) = 0$ (k:=2s-1) is equivalent to solving $\psi(q^{\frac{1}{2}}) = \sum_{j=2s}^{\infty} (-1)^j q^{\frac{j^2}{2}}$, where the function $\psi(q^{\frac{1}{2}}) := 1 + 2 \sum_{j=1}^{\infty} (-1)^j q^{\frac{j^2}{2}} = -1 + 2\theta(q^2, -\frac{1}{q})$ has many important properties (Theorem 6 of Kostov's paper, proved in [5]), which allow us to prove Lemma 3.2, which in turn we deduce Proposition 3.1 from it. Now, for completion of the Theorem, we need to prove the Theorem for the values of the spectrum of θ . We do this by comparing \tilde{r}_s and z_s with \tilde{q}_s and y_s respectively and we show that the Theorem must hold true for \tilde{q}_s and y_s also.

This comparison of these two pairs of numbers uses the fact that $\tilde{q}_s \geq \tilde{r}_s$. We suppose that the sequence $\{\tilde{q}_s\}$ tends to 1 faster than $1 - (\frac{2}{\Pi s})$, which in turn implies that the sequence of double zeros of $\theta(\tilde{q}_s, .)$ cannot tend to $-e^{\pi}$. We find a contradiction in this proof and conclude that Kostov's theorem is indeed true.

4.2 Possible further work including visual exploration

It might be interesting to conduct further work and see if any generalisation of Kostov's Theorem and/or related work can be made for the β -family of functions

$$\theta_{\beta}(q,x) := \sum_{j=0}^{\infty} q^{\binom{j+1+\beta}{2+\beta}} x^{j}$$

for $\beta \geq 0$, where $\beta = 0$ has already been considered by Kostov himself.

You may be wondering why do this? and why this specific β -family of functions?

Originally, it was purely imaginative thinking which gave me the idea of considering Kostov's paper for higher powers of q. However, after much exploration (using Maple) I found some rather interesting results on how the spectral numbers, \tilde{q}_j 's and their corresponding y_j values changed in comparison to the case $\beta = 0$ considered by Kostov himself in his paper for $\theta_{\beta}(q, x)$ when considering $\beta > 0$.

I then conjectured the possibility that Kostov's theorem could be generalised $\forall \beta \geq 0$, not just for $\beta = 0$.

The following maple code and resulting graphs show some of my findings when considering $\beta > 0$. Firstly, I construct my β -family of functions using Maple. I then plot its truncated forms in the same way as previously done for $\beta = 0$. Following this, I discuss what these graphs represent in relation to Kostov's theorem for changing values for $\beta \ge 0$.

> theta:=(x,q,beta,N) -> sum(x^n*y^binomial(n+1+beta,2+beta),n=0..N);

 $\theta := (x, q, N) \to \sum_{n=0}^{N} x^n y^{\binom{n}{n} + 1 + \beta, 2 + \beta}$

> with(plots):plt0:=implicitplot(theta(x,q,0,46),q=0..1,x=-30..30,gridrefine=5, view=[0..1,-30..0]):

> plt05:=implicitplot(theta(x,q,1/2,46),q=0..1,x=-30..30,gridrefine=5,

view=[0..1,-30..0]):

- > plt1:=implicitplot(theta(x,q,1,46),q=0..1,x=-30..30,gridrefine=5,view=[0..1,-30..0]):
- > plt2:=implicitplot(theta(x,q,2,50),q=0..1,x=-30..30,gridrefine=5,view=[0..1,-30..0]):
- > plt3:=implicitplot(theta(x,q,3,50),q=0..1,x=-30..30,gridrefine=5,view=[0..1,-30..0]):
- > plt4:=implicitplot(theta(x,q,4,50),q=0..1,x=-30..30,gridrefine=5,view=[0..1,-30..0]):
- > display(array([plt0,plt05,plt1]));



Remark 4.1. Graphs of $\theta_0(q, x)$ (top), $\theta_{\frac{1}{2}}(q, x)$ (middle), $\theta_1(q, x)$ (bottom), truncated to order N = 46.

> display(array([plt2,plt3,plt4]));



Remark 4.2. Graphs of $\theta_2(q, x)$ (top), $\theta_3(q, x)$ (middle), $\theta_4(q, x)$ (bottom), with order N = 50.

Kostov's paper considers the case of $\theta_{\beta}(q, x) := \sum_{j=0}^{\infty} q^{\binom{j+1+\beta}{2+\beta}} x^j$ for $\beta = 0$ and we have: $\tilde{q}_j = 1 - (\frac{\pi}{2j}) + o(\frac{1}{j})$ and $\lim_{j \to \infty} y_j = -e^{\pi} = -23.1407 \cdots$. This can be seen visually in the graph of $\theta_0(q, x)$.

Now, regarding how changing values of β affects these above two quantities, the interesting thing to notice is that as soon as we consider $\beta > 0$ i.e. $\beta = \frac{1}{2}$, there is a dramatic change in $\lim_{j \to \infty} y_j$ (appears to be more than halved in value). For $\beta = 1$, again there is a change, however it is not as dramatic as for $\beta = \frac{1}{2}$.

The spectral numbers \tilde{q}_j for $\beta > 0$ changes quite dramatically too. For example; for $\beta = 0$ we have $\tilde{q}_1 \approx 0.306$ and $\tilde{q}_2 \approx 0.5192$, whereas for $\beta = 1$, $\tilde{q}_1 \approx 0.5 \approx \tilde{q}_2 \approx 0.5192$.

For $\beta > 1$, there is also change in $\lim_{j \to \infty} y_j$, but it is much less dramatic than $\beta = 0 \to \beta = 1$ change (see graphs previously for a clear geometrical visualisation of these points).

After much exploration, I suggest that the limits in Kostov's Theorem would be different $\forall \beta \geq 0$, and that the path and speed of convergence to their limit point may differ also. I leave the reader with a question in which if I had more time I would try to answer it myself.

This question is- At each β and as we increase β , what is $\lim_{i \to \infty} y_j$?.

Following this, if $\lim_{j\to\infty} y_j$ and $y'_j s$ are calculated, we could then possibly also calculate their corresponding $\tilde{q}_j's$ and formulate a generalised asymptotic approximation for the spectral numbers $\forall \beta \geq 0$.

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