# MULTIPLE TIMESCALE PERTURBATION THEORY OF THE VAN DER POL OSCILLATOR

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# ABSTRACT

#### **PROJECT FOCUS**

The project is an investigation of multiple timescale perturbation theory applied to solving the van der Pol oscillator.

#### Method

The project used Maple - the symbolic mathematical language, version 15.0 - to tackle the massive algebra generated by the perturbation expansion.

#### RESULTS

Multiple timescale perturbation theory approximates the van der Pol oscillator in simple periodic functions and calculates stability, limit cycle and frequency of the van der Pol oscillator.

#### CONCLUSIONS

The method adds value to analysis in that a pattern evolves in the solution to the van der Pol oscillator using multiple timescales perturbation theory. Exploiting this solution pattern, we can theoretically solve the van der Pol oscillator to any desired number of timescales.

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# PREFACE

This project came about when I approached Professor Thomas Prellberg and asked him to supervise me for my MSc Project. As an extension of the Dynamical Systems module beyond taught graduate level, Professor Prellberg suggested I read the article on the Long-term analysis of a swing using the two-timing method by Saebyok Bae. Next, he told me to study the two-timing perturbation method examples in Strogatz book: Nonlinear Dynamics and Chaos and to work out the examples in the Strogatz book by hand in detail showing a full understanding of perturbation theory. When he was satisfied that I understood the two timing perturbation theory method, Professor Prellberg suggested I tried the three timescale for the linear harmonic oscillator. By this time, computations involved were increasingly tedious and Professor Prellberg suggested using a symbolic language like Maple for the three timing perturbation method analysis. Professor Prellberg wrote the Maple script for the three timescale for the linear harmonic oscillator. I adapted his Maple script to higher timescales to solve the van der Pol oscillator. Finally, Professor Prellberg suggested that I studied the pattern that had evolved in my solutions of the multiple timescale perturbation method for the van der Pol to solve the oscillator to 20timescales.

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# INTRODUCTION

The van der Pol equation  $\ddot{x} + x + \varepsilon(x^2 - 1)\dot{x} = 0$ , is a self-exited oscillator that exhibits different time scales as it builds up to a limit cycle. Van der Pol discovered that no matter the initial conditions, the oscillator converged to a limit cycle of magnitude 2. However, for  $\varepsilon \ll 1$ , and for trajectories close to the origin, the amplitude of oscillation grows very slowly, each oscillation with a different amplitude and period. This behaviour gives rise to the concept of multiple timescale of oscillation. For  $\varepsilon \gg 1$ , the oscillator goes into important oscillations known as relaxation oscillations where a so called crawl is followed by a sudden discharge. In this project, we employ multiple timescale analysis in perturbation theory to investigate the approximate solution to the van der Pol oscillator. Firstly, multiple times scale is applied to the linear oscillator to prove the validity of perturbation theory, followed by the method applied to two, three and four timescale to approximate the solution to the van der Pol oscillator.

The method of Average Equations is shown as an alternative method to solving the van der Pol oscillator. The concept of non-linear period is inextricably linked to multiple scales, and we use this idea to obtain linearised solutions to the van der Pol to higher timescales.

Lastly, we show how this method is extended to linearised solutions to n-time scales. There is a very good conformity in the results obtained by numerical integration and multiple timescale perturbation theory.

# Chapter 1.0

# **1.0 PERTUBATION THEORY**

We begin by introducing the requisite mathematical concept of perturbation theory. Bender and Orszag (1991:317) wrote that 'perturbation theory is a collection of methods' we can use to analyse systematically the global behaviour of solutions to both difference and differential equations.

There are three steps to perturbation analysis. We list the three steps as defined verbatim in Bender and Orszag (1991:320).

- 1. 'Convert the original problem into a perturbation problem by introducing the small parameter  $\varepsilon$  .'
- 2. 'Assume an expression for the answer in the form of a perturbation series and compute the coefficients of that series.'
- 3. 'Recover the answer to the original problem by summing the perturbation series for the appropriate value of  $\varepsilon$ .'

By way of illustration of the perturbation method, let us consider the general form of equations, Strogatz (2001), referred to as, weakly nonlinear oscillator, given below as:

$$\ddot{x} + x + \varepsilon h(x, \dot{x}) = 0. \tag{1}$$

Where  $0 \le \varepsilon \ll 1$ , and  $h(x, \dot{x})$  is called 'an arbitrary smoothing constant', Strogatz (2001:215).

According to Strogatz (2001:215), equation (1) 'represents small perturbations of the linear oscillator  $\ddot{x} + x = 0$  and are therefore, called weakly nonlinear'.

We note in passing, that the Duffing equation: $\ddot{x} + x + \varepsilon x^3 = 0$ ,	(2)
and the van der Pol equation $\ddot{x} + x + \varepsilon (x^2 - 1)\dot{x} = 0$ ,	(3)
are both examples of weakly nonlinear oscillators.	

Equation (1), however, can only be solved in terms of elementary functions when  $\varepsilon = 0$ . According to perturbation theory, Bender and Orszag (1991) wrote that we should seek solution for  $\ddot{x} + x + \varepsilon h(x, \dot{x}) = 0$ . in the perturbation expansion form of  $x(t, \varepsilon) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + ...$  (4)

Equation (4) is called a perturbation series. Bender and Orszag (1991:317) wrote that it had the attractive feature that the left hand side (LHS) of equation (4) can be computed in terms of the right hand side (RHS) as long as the problem obtained by setting  $\varepsilon = 0$ , is soluble. That is, equation (1) is soluble when  $\varepsilon = 0$ . Indeed, we know this is the case because equation (1) simplifies to a linear oscillator when  $\varepsilon = 0$ , which is soluble in terms of elementary functions. In addition, Bender and Orszag (1991:317) informs us to notice that equation (4) is local in  $\varepsilon$  but global in  $x(t, \varepsilon)$ . Let us consider by way of example of the perturbation method, the perturbation approximation to the damped linear oscillator, whose equation, with initial conditions, is given below as:

$$\ddot{x} + x + 2\varepsilon \dot{x} = 0; \quad x(0) = 0, \quad \dot{x}(0) = 1.$$
 (5)

Equation (5) is of the form stated in equation (1) and hence it is a weakly non-linear oscillator. However, the equation of the damped linear oscillator in equation (5) is a homogenous linear equation with constant coefficients. So, we can just go ahead and solve it in the conventional way.

We assume that  $x = e^{rt}$ , is a solution to equation (5), where *r* is a constant. This means that when we find the value of *r* later and substitute  $x = e^{rt}$  into equation (5) it will be soluble. For now we shall substitute  $x = e^{rt}$  in equation (5) as it is and factorising we obtain:

 $e^{rt}(r^2+2\varepsilon r+1)=0$ , where we have substituted for  $\dot{x}=re^{rt}$ , and  $\ddot{x}=r^2e^{rt}$  in equation (5) and where  $\dot{x}$ , means that we differentiate with respect to t once and  $\ddot{x}$  means that we differentiate with respect to t, twice.

Now either  $e^{rt} = 0$  or  $r^2 + 2\varepsilon r + 1 = 0$ . We choose the later as it is a quadratic equation and it allows us to solve for the value of the *r*. Hence, we have:

$$r^2 + 2\varepsilon r + 1 = 0.$$

This form of the solution to equation (5) is called the auxiliary equation.

Completing the square and factorising we have:

 $(r+\varepsilon)^2 = i^2(1-\varepsilon^2)$ , where  $i^2 = -1$ .

Taking the square root of both sides and solving for the constant, r we have that:

$$r = -\varepsilon \pm i(1 - \varepsilon^2)^{1/2}$$

If we write  $\alpha = -\varepsilon$  as the real part and  $\beta = (1 - \varepsilon^2)^{1/2}$ , as the imaginary part, then we state without proof that the solution to equation (1) can be written in the general form of:

$$e^{\alpha t}\cos\beta t + e^{\alpha t}\sin\beta t.$$

Hence, 
$$x(t,\varepsilon) = Ae^{-\varepsilon t}\cos(1-\varepsilon^2)^{1/2}t + Be^{-\varepsilon t}\sin(1-\varepsilon^2)^{1/2}t.$$
 (6)

where A and B are constants.

We can find the values of the constants *A* and *B* from the initial conditions we were given. Substituting for the first initial condition,

x(0) = 0, we have that x(0) = A.1.1 + B.1.0 = 0, hence we have A = 0, hence we can write the unfinished solution as

$$x(t,\varepsilon) = Be^{-\varepsilon t} \sin\left[(1-\varepsilon^2)^{1/2}t\right].$$
(7)

Differentiating the LHS and RHS of equation (7) with respect to t, we obtain:

 $\dot{x}(t,\varepsilon) = -\varepsilon B e^{-\varepsilon t} \sin(1-\varepsilon^2)^{1/2} t + B e^{-\varepsilon t} \cos(1-\varepsilon^2)^{1/2} t (1-\varepsilon^2)^{1/2}$ . Substituting for the second initial condition:  $\dot{x}(0) = 1$ , in this equation,  $\dot{x}(t,\varepsilon) = 1$  on the LHS and t = 0 in the RHS, we get:

 $1 = B \cdot (1 - \varepsilon^2)^{1/2}$ . Solving for *B*, we have that,  $B = (1 - \varepsilon^2)^{-1/2}$ , hence equation (7) becomes:

$$x(t,\varepsilon) = (1-\varepsilon^2)^{-1/2} e^{-\varepsilon t} \sin(1-\varepsilon^2)^{1/2} t.$$
 (8)

In equation (8),  $(1-\varepsilon^2)^{-1/2}e^{-\varepsilon t}$  is the amplitude of oscillation of the damped linear oscillator and  $\left[(1-\varepsilon^2)^{1/2}\right]$  is the frequency of oscillation. We will call equation (8) the exact solution.

We shall now apply perturbation theory to solve the damped linear oscillator in equation (5). We assumed that in perturbation theory the solution is of the form:

$$x(t,\varepsilon) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \dots$$

Hence, substituting equation (4) into equation (5) we can now write equation (5) as:

$$= \frac{d^2}{dt^2} \Big( x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + ... \Big) + 2\varepsilon \frac{d}{dt} \Big( x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + ... \Big)$$
  
+  $x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + ...$ 

Next, we differentiate the above equation term by term with respect to t and we group terms according to the powers of  $\varepsilon$ , omitting all terms with coefficients of  $\varepsilon^2$  and higher.

Strogatz (2001: 219) advised that 'a subscript notation for differentiation is more compact', so we adopt the notation in his book from this point onwards. If we expand the equation above and collect like-terms we have the following:

$$\left(\partial_{tt}x_0 + x_0\right)\varepsilon^0 + \left(\partial_{tt}x_1 + 2\partial_tx_0 + x_1\right)\varepsilon^1 + 0(\varepsilon^2) = 0.$$
(9)

Since  $\varepsilon \neq 0$ , for the RHS of (9) to be equal to the LHS, the coefficient of each power of  $\varepsilon$  should be equal to zero.

The zeroth power of  $\varepsilon$  gives us:

$$o(1): \quad \partial_{\mu} x_0 + x_0 = 0. \tag{10}$$

and the first power of  $\varepsilon$  gives:

$$o(\varepsilon): \quad \partial_{tt} x_1 + 2\partial_t x_0 + x_1 = 0. \tag{11}$$

Next, we need to solve equation (10) and (11). Let us write equation (10) in compact form where differentiation is still with respect to t, hence equation (10) becomes:

$$\ddot{x}_0 + x_0 = 0.$$
 (12)

Equation (12) is indeed a homogenous linear differential equation with constant coefficients. As we did above, let us assume that  $x_0 = e^{rt}$  is a solution to equation (12) where *r* a constant. The auxiliary equation is:

$$r^2 + 1 = 0$$
, Hence, we have  $r^2 = -1 = i^2$ , which implies that  $r = \pm i$ .

Here, the real part,  $\alpha = 0$ , and the imaginary part,  $\beta = 1$ . If as we assumed from above that the general solution for  $x_0(t)$  is of the form:  $e^{\alpha t} \cos \beta t + e^{\alpha t} \sin \beta t$ , then we can write our solution to equation (12) in the convenient form:

$$x_0(t) = A\cos t + B\sin t \,. \tag{13}$$

Where *A* and *B* are constants to be determined.

If we utilised the initial conditions  $x_0(0) = 0$ ;  $\dot{x}_0(0) = 1$ , given to find the values of the constants A and B, we have by substituting these values in equation (13) that:

 $x_0(0) = A.1 + B.0 = 0$ . This implies that the value of A = 0. Hence Equation (13) becomes:

$$x_0(t) = B\sin t. \tag{14}$$

If we differentiate equation (14) with respect to *t* we have  $\dot{x}_0(t) = B\cos t$ . If we now apply the second initial condition, at t = 0, when  $\dot{x} = 1$ , we have that  $\dot{x}_0(0) = B.1 = 1$ , hence B = 1. Thus equation (14) is:

$$x_0(t) = \sin t. \tag{15}$$

From equation (11) we have therefore that  $\ddot{x}_1 + x_1 = -2\dot{x}_0$ . Substituting equation (15) in  $\ddot{x}_1 + x_1 = -2\dot{x}_0$ , we have:

$$\ddot{x}_1 + x_1 = -2\cos t. \tag{16}$$

Equation (16) is a typical case of resonance we will consider soon. For now, we state without showing the working that equation (16) is:

 $x_1(t) = -t\sin t.$ 

We only now have to substitute for,  $x_0$  and  $x_1$  in equation (4) to obtain the solution:

$$x(t,\varepsilon) = \sin t - \varepsilon t \sin t + o(\varepsilon^2). \tag{17}$$

We shall call equation (17) the perturbation theory solution. The graph of the exact solution and the perturbation theory solution is shown in Figure 1.0 below.



Figure 1.0 [Maple] plots the exact solution, equation (8), in red and the perturbation theory solution, equation (17), in blue, for  $\varepsilon = 0.1$ . It is evident from Figure 1.0, that the plot of the perturbation theory solution and the exact solution agree well for small values of *t*, In fact, it can be shown that the perturbation theory solution theory solution works reasonably well for

 $t \ll \frac{1}{\varepsilon}$ , that is  $t \ll 10$ , but it breaks down after that.

Let us explore equations (8) and (17) further briefly. In the exact solution of equation (8), the amplitude of oscillation,  $(1 - \varepsilon^2)^{-1/2} e^{-\varepsilon t}$  decays exponentially with time. In equation (17) of the perturbation theory solution, the amplitude of oscillation is *t* and this value we know increases as time, *t*, increases. The effect is a sine function that oscillates further and further away from the horizontal axis with time. In other words, the term,  $t \sin t$ , in equation (17), is the reason that  $x(t,\varepsilon)$ , (when  $\varepsilon = 0.1$ ) is unbounded for all *t* and grows with time. On the contrary, x(t) remains bounded for all *t* in the exact solution.

# Chapter 2.0 2.1 RESONANCE

To put this project in context, we need to talk about resonance. The phenomenon of resonance is well exemplified by the differential equation in equation (16). It's because of resonant interactions between consecutive orders that non-uniformity has appeared in the regular perturbation series. To see how this happens view the LHS of equation (16) as a simple harmonic oscillator with natural frequency 1, driven by a periodic, external, forcing frequency 1, on the RHS. The amplitude of oscillation for such a system is unbounded as  $t \rightarrow \infty$  because the oscillator continually absorbs energy from the periodic external force, thus, we say this system is in resonance with the external force. The solution, therefore, to such a system, represents this fact in the term  $t \sin t$  in equation (16). The term  $t \sin t$  whose amplitude grows with time is called a 'secular term'. Another way to explain equation (16) is that the so called secular term,  $t \sin t$  appeared in the solution in equation (17) because the inhomogeneous term,  $-2\cos t$  is itself a solution of the associated homogeneous equation:  $\ddot{x}_1 + x_1 = -2\cos t$ . In general, secular terms always appear whenever the inhomogeneous term is itself a solution of the associated homogeneous differential equation. According to Bender and Orszag (1991) a secular term always grows more faster than the corresponding solution of the homogeneous equation by at least a factor of t and the authors conclude: 'that the appearance of secular terms demonstrates the non-uniform validity of perturbation expansion for large t, Bender and Orszag (1991:545)

# 2.2 MULTIPLE TIMESCALE PERTURBATION THEORY

Let us now turn our attention to multiple scale analysis. Shit, Chattopadhyay, and Chaudhuri (2012) wrote : '...the method of multiple scale analysis is immensely popular as well as a very sophisticated and useful tool for constructing uniform or global approximate solutions for both small and large values of independent variables...' The authors commented further that: 'The general principle behind the method is that the dependent variables are uniformly expanded in terms of two or more independent variables, nominally referred to as scales. A consistent feature of all multiple scales analysis is the choice of ordering scheme and the form of the power series expansion'. Citing, Bender and Orszag (1978); Jordan and Smith (1977); Navfeh and Mook (1979); Dyke (1964); Andrianov and Manevitch (2002) and Cartmell, Ziegler and Forehand (2003), the authors conclude: 'multiple scale perturbation theory (MSPT) is a very effective technique among the approximate methods that can be applied, with varying degrees of success, to a huge range of problems in the field of physics and natural Sciences. However, in applying the method, Bender and Orszag (1991:549) warned that in the perturbation series expansion itself, 'secular terms appear in all orders except the o(1) (zeroth order) and violate the boundedness of the solution'. Bender and Orszag (1991:549), wrote that: 'a short cut for removing the most secular terms to all orders begins by introducing a new variable  $\tau = \varepsilon t$  such that t and  $\tau$  are assumed to be independent, and to seek approximate solutions of the form:

$$x(t,\varepsilon) = x_0(t,\tau) + \varepsilon x_1(t,\tau) + \dots$$
(18)

This is known as the method of multiple scales. Bender and Orszag (1991:544) wrote that, 'multiple-scale analysis is a useful technique for performing uniformly valid approximations to solutions of perturbation problems' and Strogatz (2001) concluded that multiple-scale method builds in the fact of multiple-time scales from the start to avoid lengthy calculations. Multiple scale analysis involving two variables as in  $\tau = \varepsilon t$  where t and  $\tau$  are assumed independent is called two-timing. In three timescales, we assumed three independent variables, e.g.:  $t, \tau$  and  $\sigma$  and the relationship between the variables are:  $\tau = \varepsilon t$  and  $\sigma = \varepsilon^2 t$ .



epsilon = 0.05 showing the concept of multiple timescale in perturbation theory

Figure 3.0 [Maple] shows a solution of the van der Pol oscillator in the  $(x, \dot{x})$  phase plane for  $\varepsilon = 0.05$  and the initial conditions  $x(0) = 0.1 \dot{x}(0) = 0.$ For initial those conditions close to the origin, the trajectory is a slowly winding spiral that takes many cycles for the amplitude to grow substantially. At least, the true solution exhibits two timescale, a fast time for the sinusoidal oscillations and a slow time over which the amplitude increase to reach the isolated periodic orbit. In fact, the true solution of the van der Pol oscillator is the sum of sine

and cosine functions. In perturbation theory, this means that each linearised solution,  $x_i$  is a periodic solution with a different amplitude and period.

# Chapter 3.0 3.1 TWO TIMING METHOD

We shall now go ahead to obtain a better solution to the damped linear oscillator defined in equation (5) than we obtained in equation (17). Here the emphasis is only to illustrate the method, so we shall seek only an approximate solution, omitting terms in  $\varepsilon^2$  and higher.

Since we are using the two timing method, we desire two time variables. We choose t and  $\tau$  and assumed they are independent variables and define  $\tau = \varepsilon t$ . The next step is to differentiate equation (18) with respect to t but treat both t and  $\tau$  as independent variables. Using chain rule in partial differentiation on the RHS with respect to t, (the LHS is full differentiation with respect to t) equation (18) becomes:

$$\dot{x}(t,\varepsilon) = \left(\frac{\partial x_0}{\partial t} + \frac{\partial x_0}{\partial \tau}\frac{d\tau}{dt}\right) + \varepsilon \left(\frac{\partial x_1}{\partial t} + \frac{\partial x_1}{\partial \tau}\frac{d\tau}{dt}\right) + \dots$$
(19)

From above, we have that  $\tau = \varepsilon t$ , hence if we differentiate  $\tau$  with respect to *t*, we have

$$\frac{d\tau}{dt} = \varepsilon.$$
(20)

Where *t* and  $\tau$  are the so called fast and slow time respectively, Strogatz (2001).

We adopt a subscript notation for differentiation in Strogatz (2001:219) and write equation (19) as:

$$\frac{dx}{dt} = \dot{x} = \partial_t x_0 + \varepsilon \left( \partial_\tau x_0 + \partial_t x_1 \right) + o(\varepsilon^2).$$
(21)

after grouping like terms and omitting terms in  $\varepsilon^2$ . Also, differentiating, with respect to *t* again, we have:

$$\ddot{x} = \frac{d^2 x}{dt^2} = \frac{d}{dt} \left( \frac{dx}{dt} \right) = \left( \partial_t + \varepsilon (\partial_\tau + \partial_t) \right) \left( \partial_t x_0 + \varepsilon (\partial_\tau x_0 + \partial_t x_1) \right) + o(\varepsilon^2) \text{ which gives}$$
$$= \partial_{tt} x_0 + \varepsilon \left( 2 \partial_{\tau t} x_0 + \partial_{tt} x_1 \right) + o(\varepsilon^2). \tag{22}$$

Here, we have multiplied out the brackets  $(\partial_t + \varepsilon(\partial_\tau + \partial_t))$  and  $(\partial_t x_0 + \varepsilon(\partial_\tau x_0 + \partial_t x_1))$ writing the product of  $\partial_t$  and  $\partial_t x_0$  as  $\partial_t x_0$ , and  $\partial_t$  and  $\varepsilon \partial_\tau x_0$  as  $\varepsilon \partial_{\tau t} x_0$  and so on. We group all the like terms together omitting terms in  $\varepsilon^2$ . Substituting equation (21) where is  $\dot{x}$  and equation (22), which is  $\ddot{x}$  into equation (5), we have:

$$\partial_{tt}x_{0} + \varepsilon(2\partial_{\tau t}x_{0} + \partial_{tt}x_{1}) + 2\varepsilon(\partial_{\tau}x_{0} + \varepsilon(\partial_{\tau}x_{0} + \partial_{t}x_{1})) + x_{0} + \varepsilon x_{1} + o(\varepsilon^{2}) = 0.$$
(23)

If we multiply out the brackets and collect like terms in powers of  $\varepsilon$ , omitting terms in  $\varepsilon^2$ , we have:

 $\partial_{u}x_{0} + x_{0} + \varepsilon (2\partial_{u}x_{0} + \partial_{u}x_{1} + 2\partial_{t}x_{0} + x_{1}) + o(\varepsilon^{2}) = 0$ . We know that  $\varepsilon \ll 1$  but  $\varepsilon \neq 0$ , hence for the LHS of this equation to be equal to zero, either

$$o(1): \partial_{tt} x_0 + x_0 = 0.$$
 (24)  
and

$$o(\varepsilon): \quad \partial_{tt} x_1 + x_1 + 2\partial_{\tau t} x_0 + 2\partial_t x_0 = 0. \tag{25}$$

The general solution for equation (24) is similar to that given by equation (13) which is  $x_0(t) = A\cos t + B\sin t$ . However, with one difference: that,  $x_0$  is a function of two variables t and  $\tau$  and further that the constants A and B are themselves functions of  $\tau$ , the slow time. Hence, we write the general solution to equation (24) as:

$$x_0(t,\tau) = A(\tau)\cos t + B(\tau)\sin t .$$
(26)

Differentiating equation (26) with respect to *t*, we have  $\frac{\partial x_0}{\partial t} = \partial_t x_0 = -A \sin t + B \cos t$ . Differentiating a second time with respect to  $\tau$ , this time, and remembering that *A* and *B* are functions of  $\tau$ , we have that,  $\frac{\partial^2 x_0}{\partial \tau \partial t} = \partial_{\tau t} x_0 = -A' \sin t + B' \cos t$ . Substituting for  $\partial_t x_0$  and  $\partial_{\tau t} x_0$  equation (25) becomes:  $\partial_t x_1 + x_1 = -2((-A \sin t + B \cos t)) - 2(-A' \sin t + B' \cos t))$ . Collecting like terms and factorising for  $\cos t$  and  $\sin t$ , we have:

$$\partial_{t} x_{1} + x_{1} = 2((A + A')\sin t - (B + B')\cos t).$$
(27)

Since  $\cos t$  and  $\sin t$  are solutions to the equation on the LHS of equation (27), they are resonant terms – to remove them, we set the coefficient of  $\cos t$  and  $\sin t$  to zero to avoid secular terms in the solution. Hence, we have the following equations:

$$A + A' = 0.$$
 (28)

$$B+B'=0.$$

Elementary working gives the solutions to equation (28) and equation (29) as:  $A(\tau) = A(0)e^{-\tau}$  and  $B(\tau) = B(0)e^{-\tau}$ . Where A(0) and B(0) mean the value of A and B when  $\tau = 0$ .

To determine A(0) and B(0), we use  $x(t, \varepsilon) = x_0(t) + \varepsilon x_1(t) + o(\varepsilon^2)$ , and the initial conditions x(0) = 0,  $\dot{x}(0) = 1$ , as follows:

 $x(0) = 0 = x_0(0,0) + \varepsilon x_1(0,0) + o(\varepsilon^2)$ . Since  $\varepsilon \neq 0$ , The RHS of this equation is only zero when  $x_0(0,0) = 0$ , and  $x_1(0,0) = 0$ . Similarly from equation (21), we have that  $\dot{x}(0) = 1 = \partial_t x_0(0,0) + \varepsilon (\partial_\tau x_0(0,0) + \partial_t x_1(0,0)) + o(\varepsilon^2)$ . This implies that  $\partial_t x_0(0,0) = 1$  since

the coefficient  $\varepsilon$ ,  $\partial_{\tau} x_0(0,0) + \partial_t x_1(0,0) = 0$ . We know from above that  $x_0(0,0) = 0$ , and hence  $x_0(t) = A(\tau) \cos t + B(\tau) \sin t$ , enables us to determine A(0). Substituting for t = 0, we have  $\tau = 0$ , and

$$A(0) = 0.$$
 (30)

This implies  $A(\tau) \equiv 0$ . This means that  $A(\tau) = 0$  for all values of the variable,  $\tau$ .

Similarly, combining  $x_0(t) = A(\tau)\cos t + B(\tau)\sin t$  and  $\partial_t x_0(0,0) = 1$ , we have:

$$1 = -A(\tau)\sin t + B(\tau)\cos t = 1$$
, substituting  $t = 0$ , here, we have:  
  $B(0) = 1$ . (31)

This implies  $B(\tau) = e^{-\tau}$ . Substituting for  $\tau = \varepsilon t$ , equation (26) becomes:  $x_0(t,\tau) = e^{-\tau} \sin t$ . Since we seek only an approximate solution and our aim is to show that perturbation theory works, we simply approximate equation (18) as:

$$x(t,\tau) = e^{-\varepsilon t} \sin t + o(\varepsilon). \tag{32}$$

Equation (32) is the approximate two-timing method solution. If  $\varepsilon \ll 1$ , the two-timing solution agrees well with the exact solution.



Figure 3.0 [Maple] show the approximate two timing perturbation theory method solution to the Exact solution, when  $\varepsilon = 0.1$ . Though, we have omitted expressions with powers in  $\varepsilon$  and higher, the two timing method still agree very well with the exact solution for  $t \le 10$  as predicted by theory.

Figure 3.0: Two timing perturbation theory method vs. Exact solution

#### Chapter 4.0 4.1 THE VAN DER POL OSCILLATOR

Now that we have shown how to apply multiple scales in perturbation theory to weakly non-linear oscillators, let us apply the technique to solve the van der Pol equation. But firstly, a little about, van der Pol, the man, and his accomplishments. We are told in Wikipedia (2012), that his full name was Balthazar van der Pol and he was a Dutch electrical engineer and physicist and that he proposed the van der Pol oscillator whilst working at Philips. It is related that van der Pol found stable oscillations in electrical circuits employing vacuum tubes. These stable oscillators are now known as limit cycles. Then, vacuum tubes were used to control the flow of electricity in the circuitry of transmitters and receivers.

Guckenheimer and Hoffman and Weckesser (2003) in the introduction to their paper, wrote on van der Pol and his accomplishments. The authors claimed that in the first half of the twentieth century, van der Pol pioneered the fields of radio and telecommunications. As proof, they cited the works of Stumpers (1960) and works by van der pol and van der Mark (1927). Guckenheimer, Hoffman and Weckesser (2003) wrote that 'van der Pol experimented with oscillations in a vacuum tube triode circuit and concluded that all initial conditions converged to the same periodic orbit of finite amplitude. This behaviour was different from the behavior of solutions of linear equations, hence van der Pol proposed a nonlinear differential equation',  $\ddot{x} + x + \varepsilon(x^2 - 1)\dot{x} = 0$ . Van der Pol (1920) referred to this equation as the unforced van der Pol equation.

He studied the equation for  $\varepsilon \gg 1$  and van der Pol (1926) discovered important oscillations now known as *relaxation oscillations*. According to Guckenheimer, Hoffman and Weckesser (2003) '...these oscillations have become the cornerstone of geometric singular perturbation theory...'.

Guckenheimer, Hoffman and Weckesser (2003) citing, McMurran and Tattersall (1996), wrote that van der Pol's work on nonlinear oscillations and circuit theory provided motivation for the works of notable scientists of his time. In summary, Guckenheimer, Hoffman and Weckesser (2003) wrote that since the 1920's, 'the van der Pol equation has been a prototype for systems with self-excited limit cycle oscillations. The equation has been studied from perturbations of harmonic motion to relaxation oscillations. In biology, the van der Pol equation has been used as the basis of a model of coupled neurons in the gastric mill circuit of the stomatogastric ganglion (citing Guckenheimer, Hoffman Weckesser (2000) and Rowat and Selverston (1996))'.

Forger and Kronauer (2002) citing Wever (1972) wrote Wever was the first to use the van der Pol equation as a model of the human circadian clock and since then it been used as an accurate model of the human circadian system; Forger and Kronauer (2002) cited Gundel and Spencer (1999) and Jewett, Forger, and Kronauer (1999) to name a few. Guckenheimer, Hoffman and Weckesser (2003) wrote further, that 'The Fitzhugh–Nagumo equation (citing FitzHugh (1961)) is a planar vector field that extends the van der Pol equation as a model for action potentials of neurons (citing Koch and Segev (1998). In seismology, the van der Pol equation has been used in the development of a model of the interaction of two plates in a geological fault (citing Cartwright and Eguiluz, et al, (1999))'.

# 4.2 Some Properties of the van der Pol Oscillator

We summarise the main points in the last section.

- 1. Van der Pol (1920) proposed the nonlinear differential equation in the form  $\ddot{x} + x + \varepsilon (x^2 1)\dot{x} = 0$  to describe self-excited, constants oscillations in a vacuum tube triode circuit. The van der Pol oscillator is an ordinary differential equation (ODE).
- 2. For  $\varepsilon \gg 1$  and van der Pol (1926) discovered important oscillations known as *relaxation oscillations*.
- 3. We state without proof that the energy of this ODE increases when the absolute value of x is less that 1 (|x| < 1) then decreases when the absolute

value of *x* is greater than 1, (|x| > 1).

- 4. The van der Pol oscillator has a single stable periodic orbit, or limit cycle.
- 5. The van der Pol equation is a simple model of a beating heart.
- 6. The van der Pol equation is used to model behaviour in the physical and biological sciences.

# Chapter 5.0 5.1 AMPLITUDE AND LIMIT CYCLE IN THE VAN DER POL OSCILLATOR: TWO-TIMING METHOD

Now that we have confirmed the validity of the two-timing method in perturbation theory, we will use perturbation to show that the van der Pol oscillator has a stable limit cycle with radius =  $2 + o(\varepsilon)$  and a frequency,  $\omega = 1 + o(\varepsilon)$ .

We recall again that the equation of the van der Pol oscillator is given as:  $\ddot{x} + x + \varepsilon (x^2 - 1)\dot{x} = 0.$  (33)

In perturbation theory, we assume that the solution is of the form:

 $x(t,\varepsilon) = x_0(t,\tau) + \varepsilon x_1(t,\tau) + o(\varepsilon^2)$ , where  $\tau = \varepsilon t$  and  $\frac{d\tau}{dt} = \varepsilon$ . For ease of writing, we will write  $x(t,\varepsilon)$  as x, as this form causes no confusion.

As before, we need to differentiate both sides of  $x = x_0(t, \tau) + \varepsilon x_1(t, \tau) + o(\varepsilon^2)$ , with respect to *t* and substitute for  $\dot{x}$  and  $\ddot{x}$  equation (33). As we have already derived  $\dot{x}$  and  $\ddot{x}$  for two timing - equations (21) and (22) - we will not show the working again. As before, after substituting and collecting like terms in powers of  $\varepsilon$ , we have

 $\partial_{tt} x_0 + x_0 + (\partial_{tt} x_1 + x_1 + 2\partial_{\tau t} x_0 + (x_0^2 - 1)\partial_t x_0)\varepsilon + 0(\varepsilon^2) = 0$ . Since  $\varepsilon \neq 0$ , for the LHS of this equation to be zero, the coefficients of the powers of  $\varepsilon$  has to be zero. Thus the equations:

$$o(1): \quad \partial_{tt} x_0 + x_0 = 0.$$
 (34)

$$o(\varepsilon): \quad \partial_{tt} x_1 + x_1 = -2\partial_{\tau t} x_0 - (x_0^2 - 1)\partial_t x_0.$$
(35)

Recall that *x* is a function of *t*, but that  $x_0$  and  $x_1$  are both functions of *t* and  $\tau$ . As above, the *o*(1) equation is a simple harmonic oscillator. The general solution can be written in the form of equation (26), which is  $x_0(t,\tau) = A(\tau)\cos t + B(\tau)\sin t$ . However, for ease in computation, we shall use the familiar, compact form of:

$$x_0 = r(\tau)\cos(t + \varphi(\tau)). \tag{36}$$

 $r(\tau)$  and  $\varphi(\tau)$  are the slowly-varying amplitude and phase of  $x_0$  respectively.

To find equations governing  $r(\tau)$  and  $\varphi(\tau)$ , we need to insert equation (36) into equation (35). From equation (36), we should also determine  $\partial_{\tau} x_0$  and  $\partial_{\tau \tau} x_0$ . Substituting for these in equation (35), we get:

$$\partial_{tt} x_{1} + x_{1} = -2 \Big[ r'(\tau) \sin(t + \varphi(\tau)) + r(\tau) \varphi'(\tau) \cos(t + \varphi(\tau)) \Big] - \Big[ r^{2}(\tau) \cos^{2}(t + \varphi(\tau)) - 1 \Big] \Big[ r(\tau) \sin(t + \varphi(\tau)) \Big].$$

Expanding the above equation, we arrive at the expression below:

$$= -2r(\tau)\varphi'(\tau)\cos(t+\varphi(\tau)) - 2r'(\tau)\sin(t+\varphi(\tau)) - r^{3}(\tau)\cos^{2}(t+\varphi(\tau))\sin(t+\varphi(\tau))$$
  
+ $r(\tau)\sin(t+\varphi(\tau)).$  (37)

We can simplify the notation in equation (37) if we let,

$$\theta = t + \varphi(\tau)$$
 and  $r(\tau) = r$ , and  $\varphi(\tau) = \varphi$ , hence we have:

$$\partial_{tt} x_{1} + x_{1} = -2r\phi'\cos\theta - 2r'\sin\theta - r^{3}\cos^{2}\theta\sin\theta + r\sin\theta.$$
(38)

We employ the trigonometric identity,  $\cos^2 \theta \sin \theta = \frac{1}{4} (\sin \theta + \sin 3\theta)$ , to simplify equation (38) further and we arrive at:

$$\partial_{tt} x_1 + x_1 = -2r\varphi'\cos\theta + (-2r' - \frac{1}{4}r^3 + r)\sin\theta - \frac{1}{4}r^3\sin3\theta.$$
(39)

Recall that both  $\cos\theta$  and  $\sin\theta$  are resonant terms in equation (39). That is, to avoid secular terms in our solution, we must set the coefficients of  $\cos\theta$  and  $\sin\theta$  to zero on the RHS of equation (39) as these are themselves solutions to the associated homogeneous differential equation on the LHS of equation (39).

Hence we have that:

$$-2r' - \frac{1}{4}r^3 + r = 0. \tag{40}$$

$$-2r\varphi' = 0. \tag{41}$$

We can rewrite equation (40) as:

$$r' = \frac{1}{8}r(4 - r^2). \tag{42}$$

Strogatz (2001) stated that we can write equation (42) as a vector field. Professor Prellberg stated in lecture notes, Autumn 2011, that to solve such an equation for fixed points, that is those points for which r' = 0; and writing r' = f(r), we can write equation (42) as:

 $f(r) = \frac{1}{8}r(4-r^2) = 0$ . This means that  $r(4-r^2) = 0$ . Solving for fixed points, we have  $r^* = 0$  or  $r^* = \pm 2$ . However, we are only interested in the half-line  $r \ge 0$ .  $r^* = 0$  is an unstable fixed point and  $r^* = 2$  is a stable fixed point.



clearly shows the phase plot of the vector field. All trajectories starting above 2 converge to 2, and trajectories starting close to the origin converge to 2. Thus,  $r^* = 0$  is an unstable fixed point whilst  $r^* = 2$  is a stable fixed point. Hence,  $r(\tau) \rightarrow 2$  as From equation (41), we have that  $\varphi' = 0$ so  $\varphi(\tau) = \varphi_0$ , where  $\varphi_0$  is a constant.  $x_0(t,\tau) \rightarrow 2\cos(t+\varphi_0)$ and therefore

$$x(t) \rightarrow 2\cos(t + \varphi_0) + o(\varepsilon).$$
  
as  $t \rightarrow \infty$ 

Thus x(t) approaches a stable limit cycle of radius  $2+o(\varepsilon)$ . Thus we conclude that the van der Pol oscillator has a stable limit cycle of radius 2.

Recall from above that  $\theta = t + \varphi(\tau)$ , the angular frequency,  $\omega = \frac{d\theta}{d\tau}$ .  $\frac{d\theta}{dt} = \frac{d}{dt} \left( t + \varphi(\tau) \right) = 1 + \frac{d\varphi(\tau)}{dt}.$ 

However,  $\frac{d\varphi(\tau)}{dt} = \frac{d\varphi(\tau)}{d\tau} \cdot \frac{d\tau}{dt}$ . Remembering that  $\tau = \varepsilon t$ , we have  $\varepsilon = \frac{d\tau}{dt}$ , and therefore,  $\frac{d\varphi(\tau)}{dt} = \varepsilon \frac{d\varphi(\tau)}{d\tau}$ .

Hence, we have that

$$\omega = 1 + \varepsilon \varphi'. \tag{44}$$

But  $\frac{d\varphi(\tau)}{d\tau} = \varphi' = 0$ , through first order in  $\varepsilon$ . Thus,  $\omega = 1 + o(\varepsilon^2)$ .

Hence, we conclude that the van der Pol has angular frequency,  $\omega = 1$ .

# Chapter 6.0 6.1 THREE TIMING IN THE VAN DER POL OSCILLATOR

Here, we define the van der Pol oscillator with initial conditions below as:

$$\ddot{x} + x + \varepsilon (1 - x^2) \dot{x} = 0 \quad x(0) = 2 \quad \dot{x}(0) = 0.$$
(45)

As in the example of two timing, our aim is to remove the most secular terms to all orders. However, as this is three timing, we must introduce an extra timescale,  $\sigma$ , as according to perturbation theory, this is the best way to remove the most secular terms. We define the three timescale with  $\tau = \varepsilon t$  and  $\sigma = \varepsilon^2 t$ , assuming that t,  $\tau$  and  $\sigma$  are independent variables.

The perturbation theory expansion is in the form given below:

$$x(t,\varepsilon) = x_0(t,\tau,\sigma) + \varepsilon x_1(t,\tau,\sigma) + \varepsilon^2 x_2(t,\tau,\sigma) + o(\varepsilon^3).$$
(46)

If we obtain  $\dot{x}$  and  $\ddot{x}$  from equation (46) by differentiating *x* with respect to *t* once and twice, respectively, and substituting for  $\dot{x}$  and  $\ddot{x}$  in equation (45), we obtain the following equation:

$$\partial_{tt} x_{0} + x_{0} + (x_{1} + 2\partial_{t\tau} x_{0} + \partial_{tt} x_{1} + (x_{0}^{2} - 1)\partial_{t} x_{0})\varepsilon + (2\partial_{t\sigma} x_{0} + \partial_{tt} x_{2} + 2\partial_{t\tau} x_{1} + x_{2} + \partial_{\tau\tau} x_{0} + (x_{0}^{2} - 1)(\partial_{\tau} x_{0} + \partial_{t} x_{1}) + 2x_{0} x_{1} \partial_{t} x_{0})\varepsilon^{2} + o(\varepsilon^{3}) = 0.$$

$$(47)$$

The computation is a bit tedious to do by hand and is obtained by maple and it can he checked in Appendix 1.

As usual, for the LHS of equation (47) to be zero, the coefficient of powers of  $\varepsilon$  must be equal to zero. The zeroth order of epsilon gives  $\partial_{tt}x_0 + x_0 = 0$  As stated earlier, the zeroth order equation is a simple harmonic oscillator. The general solution equation is:

$$x_0(t,\tau,\sigma) = F1(\tau,\sigma)\cos t + F2(\tau,\sigma)\sin t.$$
(48)

Equation (48) is sometimes written in the equivalent compact from:

$$x_0(t,\tau,\sigma) = A(\tau,\sigma)\cos(t-\varphi(\tau,\sigma)).$$
(49)

where  $A(\tau, \sigma)$  and  $\varphi(\tau, \sigma)$ , are the slowly-varying amplitude and phase of  $x_0$  respectively.

Similarly, we write the coefficient of the first power of  $\varepsilon$  equals to zero as:

$$x_1 + 2\partial_{t\tau} x_0 + \partial_{tt} x_1 + (x_0^2 - 1)\partial_t x_0 = 0.$$
(50)

Note that from equation (49), we have:  $x_0 = A(\tau, \sigma) \cos(t + \varphi(\tau, \sigma))$ . As before, we only need to compute  $\partial_t x_0$ , and  $\partial_{t\tau} x_0$  from here and substitute them into equation (50).

Maple does the hard work so we don't have to. Hence, equation (51) simplifies in Maple to:

$$(-2\partial_{\tau}A - \frac{1}{4}A^{3} + A)\sin(t+\varphi) + x_{1} + 2A\cos(t+\varphi)\partial_{t}\varphi + \partial_{tt}x_{1} - \frac{1}{4}A^{3}\sin(3t+3\varphi) = 0.$$
(51)

Where, *A* and  $\varphi$  are  $A(\tau, \sigma)$  and  $\varphi(\tau, \sigma)$  respectively in equation (51). To solve equation (51) we must remove all secular terms. This means that we equate the coefficients of  $\cos(t + \varphi)$  and  $\sin(t + \varphi)$  to zero. We obtain the following equations below:

$$-2\partial_{\tau}A - \frac{1}{4}A^3 + A = 0.$$
 (52)

$$2A\partial_{\tau}\varphi = 0. \tag{53}$$

Since  $A \neq 0$ , equation (53) appear to imply that  $\varphi$  does not depend on  $\tau$ . Indeed, solving equations (52) and (53) simultaneously in Maple gives:

$$\varphi = F_2(\sigma). \tag{54}$$

$$A(\tau,\sigma) = -\frac{2}{\left(1 + 4e^{-\tau}F1(\sigma)\right)^{1/2}}.$$
(55)

Equation (55) imply that as  $t \to \infty$ ,  $(1+4e^{-\tau}F1(\sigma))^{1/2} \to 1$ , and therefore  $A(\tau,\sigma) \to 2$ . (Recall that  $\tau = \varepsilon t$ .)Hence,

$$A(\tau,\sigma) = 2. \tag{56}$$

Equation (49), simplify further to:

$$x_0 = 2\cos(t - \varphi(\sigma)). \tag{57}$$

Substituting these new values of  $x_0$  in equation (56) and A = 2 in equation (57) back into equation (50), we get

$$x_1 + \partial_{tt} x_1 - 2\sin\left(3t - 3\varphi(\sigma)\right) = 0.$$
(58)

Thus we see that all terms leading to secular terms in the solution have been removed. The solution to equation (58) in Maple is given below as:

$$x_1(t,\tau,\sigma) = F1(\tau,\sigma)\cos t + F2(\tau,\sigma)\sin t - \frac{1}{4}\sin(3t+3\varphi(\sigma)).$$
(59)

Thus we equation (59) reduces to:

$$x_{1} = -\frac{1}{4}\sin(3t + 3\varphi(\sigma)).$$
 (60)

Similarly, if we equate the coefficients of

 $\varepsilon^2$  to zero, we have below:

$$2\partial_{t\sigma}x_{0} + \partial_{t\tau}x_{2} + 2\partial_{t\tau}x_{1} + x_{2} + 2x_{0}x_{1}\partial_{\tau}x_{0} + \partial_{\tau\tau}x_{0} + (x_{0}^{2} - 1)(\partial_{\tau}x_{0} + \partial_{\tau}x_{1}).$$
(61)

Maple automatically substitutes for  $x_0$ ,  $x_1$  and all the partial differentiations to give:

$$\frac{1}{4}(-1+16\partial_{\sigma}\varphi\cos(t-\varphi) - \frac{5}{4}\cos(5t-5\varphi) + \partial_{tt}x_{2} + x_{2} - \frac{3}{4}\cos(3t-3\varphi) = 0,$$
 (62)  
where  $\varphi = \varphi(\sigma)$ .

As before, to solve equation (62), we equate the coefficients of resonant terms to zero, hence:

$$-\frac{1}{4} + 4\partial_{\sigma}\varphi = 0. \tag{63}$$

Solving for  $\varphi$  in equation (63), we that

$$\varphi(\sigma) = \frac{1}{16}\sigma + \phi. \tag{64}$$

where  $\phi$  is a constant. Substituting equation (64) into equations (57) and (60),

equation (57) becomes

$$x_0 = 2\cos(-t + \frac{1}{16}\sigma + \phi).$$
(65)

and equation (60) becomes

$$x_1 = -\frac{1}{4}\sin(-3t + \frac{3}{16}\sigma + 3\phi).$$
(66)

Substituting the new values of  $x_0$  and  $x_1$  and  $\varphi$  back into equation (61), Maple computes equation (61) as:

$$-\frac{5}{4}\cos(-5t+\frac{5}{16}\sigma+5\phi)+\partial_{tt}x_{2}+x_{2}-\frac{3}{4}\cos(-3t+\frac{3}{16}\sigma+3\phi)=0.$$
(67)

Equation (67) is now devoid of all secular terms in the solutions and solving equation (67), we obtain:

$$x_2 = -\frac{5}{96}\cos(-5t + \frac{5}{16}\sigma + 5\phi) - \frac{3}{32}\cos(-3t + \frac{3}{16}\sigma + 3\phi).,$$
(68)

Hence, we can now write equation (46) as

$$x = 2\cos(-t + \frac{1}{16}\varepsilon^{2}t + \phi) + \frac{1}{4}\varepsilon\sin(-3t + \frac{3}{16}\varepsilon^{2}t + 3\phi) - \left(\frac{5}{96}\cos(-5t + \frac{5}{16}\varepsilon^{2}t + 5\phi) - \frac{3}{16}\cos(-3t + \frac{3}{16}\varepsilon^{2}t + 3\phi) + o\right)\varepsilon^{2}.$$
(69)

We can solve for the constant  $\phi$  in equation (69) utilising the initial conditions, x(0) = 2  $\dot{x}(0) = 0$ . Substituting the value  $\varepsilon = 0.5$ , Maple computes the value  $\phi = 0.1396823540$ . Substituting  $\phi = 0.1396823540$  into equation (69), the three timing perturbation theory approximates the solution to equation (45) as:

$$x(t) = 2\cos(-\frac{63}{64}t + 0.1396823540) + \frac{1}{8}\sin(-\frac{189}{64}t + 0.4190470620)$$
  
$$-\frac{5}{384}\cos(-\frac{315}{64}t + 0.6984117700) - \frac{3}{128}\cos(-\frac{189}{64}t + 0.4190470620).$$
 (70)

Note, however that  $\phi$  is an arbitrary constant determined by initial conditions.



Figure 5.0(a) numerical integration of the van der Pol in red vs approximate solution in green by the 3-timing method for phi = 0.2 and epsilon = 0.5

Figure 5.0(a) [Maple] shows the approximate 3timing plot in green against numerical integration in red for  $\varepsilon = 0.5$  and  $\phi = 0.2$ . We have plotted x(t) on the vertical axis against t on the horizontal axis for t = 0...50. There is a very agreement between the results obtained by numerical integration and multiple timescale perturbation method.



red for and we have plotted  $\dot{x}(t)$  on the vertical axis against x(t) on the horizontal axis. The phase plot of limit cycle obtained by numerical integration and agrees well with that obtained by multiple timescale perturbation method.

Figure 5.0(a) [Maple] shows the phase plot for the approximate 3-timing

plot in green against

numerical integration in

Figure 5.0(b) phase plot of the numerical integration of the van der Pol in red vs approximate solution in green by the 3-timing method for phi = 0.2 and epsilon = 0.5

# Chapter 7.0 7.1 FOUR TIMESCALE IN THE VAN DER POL OSCILLATOR

The computation involved in the four timescale for the van der Pol oscillator is tedious and more demanding than in the three timing. Again, we have used Maple to cope with the rigor of computation. The details for the four timescale are in Appendix 2.

As before, we use the same equation for the van der Pol as defined in equation (45) above, namely:

 $\ddot{x} + x + \varepsilon (1 - x^2) \dot{x} = 0$  x(0) = 2  $\dot{x}(0) = 0.$ 

As above, we chose four variables this time,  $t, \tau, \sigma$  and v and we assumed they are independent. We define the variables as follows:  $\tau = \varepsilon t$ ,  $\sigma = \varepsilon^2 t$  and  $v = \varepsilon^3 t$ . *t* is called the fast time,  $\tau$ , the slow time,  $\sigma$ , the super slow time and v, the super slowest time. From multiple scales perturbation theory, the solution should be of the form:

$$x(t,\varepsilon) = x_0(t,\tau,\sigma,\upsilon) + \varepsilon x_1(t,\tau,\sigma,\upsilon) + \varepsilon^2 x_2(t,\tau,\sigma,\upsilon) + o(\varepsilon^3).$$

The procedure is the same as before. We need to derive  $\dot{x}$  and  $\ddot{x}$  in the above equation and substitute these into equation (45).

The resulting equation is far too cumbersome to list all of its terms here. However, we list the coefficient of each power of  $\varepsilon$  and equate each coefficient to zero. The following shows the hierarchy of equations:

$$o(1): \quad \partial_{tt} x_0(\mathbf{t}, \tau, \sigma, \upsilon) + x_0(\mathbf{t}, \tau, \sigma, \upsilon) = 0. \tag{71}$$

$$o(\varepsilon): \quad x_1(t,\tau,\sigma,\upsilon) + 2\partial_{\tau\tau}x_0(t,\tau,\sigma,\upsilon) + (x_0(t,\tau,\sigma,\upsilon)^2 - 1)(\partial_{\tau}x_0(t,\tau,\sigma,\upsilon) + \partial_{\tau}x_1(t,\tau,\sigma,\upsilon) = 0.$$
(72)

$$o(\varepsilon^{2}): \quad x_{2}(t,\tau,\sigma,\upsilon) + \left(x_{0}(t,\tau,\sigma,\upsilon)^{2} - 1)(\partial_{\tau}(x_{0})(t,\tau,\sigma,\upsilon) + \partial_{t}(x_{1})(t,\tau,\sigma,\upsilon)\right) + 2x_{0}(t,\tau,\sigma,\upsilon)x_{1}(t,\tau,\sigma,\upsilon)\partial_{t}(x_{0})(t,\tau,\sigma,\upsilon) + 2\partial_{t\sigma}(x_{0})(t,\tau,\sigma,\upsilon) + \partial_{\tau\tau}(x_{0})(t,\tau,\sigma,\upsilon) + 2\partial_{t\tau}(x_{1})(t,\tau,\sigma,\upsilon) + \partial_{t\tau}(x_{2})(t,\tau,\sigma,\upsilon) = 0.$$
(73)

$$o(\varepsilon^{3}): 2\partial_{t\nu}x_{0}(t,\tau,\sigma,\upsilon) + x_{3}(t,\tau,\sigma,\upsilon) + \partial_{tt}(x_{3})(t,\tau,\sigma,\upsilon) + 2\partial_{\tau\sigma}(x_{0})(t,\tau,\sigma,\upsilon) + 2\partial_{\tau\sigma}(x_{0})(t,\tau,\sigma,\upsilon) + 2\partial_{\tau\sigma}(x_{1})(t,\tau,\sigma,\upsilon) + \partial_{\tau\tau}(x_{1})(t,\tau,\sigma,\upsilon) + 2x_{0}(t,\tau,\sigma,\upsilon)x_{1}(t,\tau,\sigma,\upsilon)(\partial_{\tau}(x_{0})(t,\tau,\sigma,\upsilon) + (x_{0}(t,\tau,\sigma,\upsilon)^{2}-1)(\partial_{\sigma}(x_{0})(t,\tau,\sigma,\upsilon) + \partial_{\tau}(x_{1})(t,\tau,\sigma,\upsilon) + \partial_{t}(x_{2})(t,\tau,\sigma,\upsilon)) + (2x_{0}(t,\tau,\sigma,\upsilon)(x_{2})(t,\tau,\sigma,\upsilon) + x_{1}(t,\tau,\sigma,\upsilon)^{2})\partial_{\tau}x_{0}(t,\tau,\sigma,\upsilon) + \partial_{t}(x_{1})(t,\tau,\sigma,\upsilon) + 2\partial_{\tau\tau}(x_{2})(t,\tau,\sigma,\upsilon) + 2\partial_{\tau}(x_{2})(t,\tau,\sigma,\upsilon) + 2\partial_{\tau\tau}(x_{2})(t,\tau,\sigma,\upsilon) + 2\partial_{\tau}(x_{2})(t,\tau,\sigma,\upsilon) + 2\partial_{\tau}(x_{2})(t,\tau,\sigma,\upsilon)$$

The solution of the o(1) equation as before is:

$$F1(\tau,\sigma,\upsilon)\sin(t) + F2(\tau,\sigma,\upsilon)\cos(t) = 0.$$
(75)

Or we can write this in the compact form as before as:

$$x_0(t,\tau,\sigma,\upsilon) = A(\tau,\sigma,\upsilon)\cos(t - \Phi(\tau,\sigma,\upsilon)).$$
(76)

Where A and  $\Phi$  are functions of  $\tau, \sigma, \upsilon$ . Next, we need to substitute the solution  $x_0$  in equation (76) into the  $o(\varepsilon)$  equation, equation (72) above. In Maple this yield:

$$\begin{bmatrix} \left(-2\partial_{\tau}(A)(\tau,\sigma,\upsilon)\right) - \frac{1}{4}(A)(\tau,\sigma,\upsilon)^{3} + A(\tau,\sigma,\upsilon) \end{bmatrix} \sin\left(t - \Phi(\tau,\sigma,\upsilon)\right) + x_{1}(t,\tau,\sigma,\upsilon) + 2A(\tau,\sigma,\upsilon)\cos\left(t - \Phi(\tau,\sigma,\upsilon)\right)\partial_{\tau}(\Phi)(\tau,\sigma,\upsilon) - \frac{1}{4}(A)(\tau,\sigma,\upsilon)^{3}\sin\left(3t - 3\Phi(\tau,\sigma,\upsilon)\right) + \partial_{t}(x_{1})(t,\tau,\sigma,\upsilon) = 0. \tag{77}$$

If we equate all the resonant terms in equation (77) to zero that is, the coefficients of  $\sin(t - \Phi(\tau, \sigma, \upsilon))$ , and  $\cos(t - \Phi(\tau, \sigma, \upsilon))$ , we have the equations:

$$2A(\tau,\sigma,\upsilon)\partial_{\tau}(\Phi)(\tau,\sigma,\upsilon) = 0$$
  
-2 $\partial_{\tau}(A)(\tau,\sigma,\upsilon) - \frac{1}{4}(A)(\tau,\sigma,\upsilon)^{3} + A(\tau,\sigma,\upsilon) = 0.$  (78)

Solving the two equations in (78) in Maple, we obtain the following results:

$$A(\tau,\sigma,\nu) = -\frac{2}{\left(1 + 4e^{-\tau} - F1(\sigma,\nu)\right)^{1/2}}.$$
(79)

and

$$\Phi(\tau,\sigma,\upsilon) = F2(\sigma,\upsilon). \tag{80}$$

Since, we already know that the van der Pol oscillator has a stable limit cycle of 2, we set  $A(\tau, \sigma, \upsilon) = 2$ .

Equation (80) imply that  $\Phi$  does not depend on  $\tau$  and is a function of  $\sigma$  and v only. Equation (76) is therefore further refined as:

$$x_0 = 2\cos(t - \Phi(\sigma, \upsilon)). \tag{81}$$

If we substitute  $x_0$  in equation (81) into equation (72) again, we obtain:

$$\partial_{tt}(x_1)(t,\tau,\sigma,\upsilon) + x_1(t,\tau,\sigma,\upsilon) - 2\sin(3t - 3\Phi(\tau,\sigma,\upsilon)) = 0.$$
(82)

Equation (82) has no secular terms, so we can solve for  $x_1$ . Solving for  $x_1$  in equation (82) in Maple, we have:

$$x_{1}(\tau,\sigma,\upsilon) = -\frac{1}{4}\sin(3t - 3\Phi(\sigma,\upsilon)) + F_{2}(\tau,\sigma,\upsilon)\sin(t) + F_{3}(\tau,\sigma,\upsilon)\cos(t).$$
(83)

Hence 
$$x_1 = -\frac{1}{4}\sin((3t - 3\Phi(\sigma, \upsilon))).$$
 (84)

Next, we substitute  $x_0$  and  $x_1$  in equations (81) and (82) respectively into equation (73), the  $o(\varepsilon^2)$  equation. Doing the computation in Maple, we obtain the following result:

$$\frac{1}{4} \left( 16\partial_{\sigma} \Phi(\sigma, \upsilon) - 1 \right) \cos\left(t - \Phi(\sigma, \upsilon)\right) - \frac{5}{4} \cos\left(5t - 5\Phi(\sigma, \upsilon)\right) - \frac{3}{4} \cos\left(3t - 3\Phi(\sigma, \upsilon)\right) + x_2(t, \tau, \sigma, \upsilon) + \partial_{tt}(x_2)(t, \tau, \sigma, \upsilon) = 0.$$
(85)

Again, we equate all the resonant terms in equation (85) to zero, that is the coefficients of  $\cos(t - \Phi(\sigma, \upsilon))$  and  $\sin(t - \Phi(\sigma, \upsilon))$ . The coefficient of  $\sin(t - \Phi(\sigma, \upsilon))$  is zero, however the coefficient of  $\cos(t - \Phi(\sigma, \upsilon))$ , yield:

$$\frac{1}{4} \left( 16\partial_{\sigma} \Phi(\sigma, \upsilon) - 1 \right) = 0. \tag{86}$$

Solving for  $\Phi$  in equation (86) in Maple, we have:

$$\Phi(\sigma,\upsilon) = \frac{1}{16}\sigma + F_1(\upsilon).$$
(87)

From equation (87), we again update results for  $x_0$  and  $x_1$  as follows:

$$x_{0}(t,\tau,\sigma,\upsilon) = 2\cos\left(t - \frac{1}{16}\sigma - F_{1}(\upsilon)\right).$$
(88)

and

$$x_{1}(t,\tau,\sigma,\upsilon) = -\frac{1}{4}\sin\left(3t - \frac{3}{16}\sigma - 3_{-}F1(\upsilon)\right).$$
(89)

respectively.

Again, we substitute equations (87), (88) and (89) back into equation (73). The Maple output is:

$$x_{2}(t,\tau,\sigma,\upsilon) - \frac{5}{4}\cos\left(5t - \frac{5}{16}\sigma - 5_{F1}(\upsilon)\right) - \frac{3}{4}\cos\left(3t - \frac{3}{16}\sigma - 3_{F1}(\upsilon)\right) + \partial_{u}(x_{2})(t,\tau,\sigma,\upsilon) = 0.$$
(90)

Since equation (90) contain no secular terms we go ahead and solve for  $x_2$ . The solution to equation (90) in Maple is:

$$x_{2}(t,\tau,\sigma,\upsilon) = F_{3}(\tau,\sigma,\upsilon)\sin(t) + F_{2}(\tau,\sigma,\upsilon)\cos(t) - \frac{5}{96}\cos\left(5t - \frac{5}{16}\sigma - 5F_{1}(\upsilon)\right) - \frac{3}{32}\cos\left(3t - \frac{3}{16}\sigma - 3F_{1}(\upsilon)\right).$$
(91)

Hence,

$$x_{2} = -\frac{5}{96}\cos\left(5t - \frac{5}{16}\sigma - 5_{F1}(\upsilon)\right) - \frac{3}{32}\cos\left(3t - \frac{3}{16}\sigma - 3_{F1}(\upsilon)\right).$$
 (92)

Lastly, we need to solve for  $x_3$  in equation (74), the  $o(\varepsilon^3)$  equation. We substitute equations (87), (88), (89) and equation (92) into equation (74). The answer in Maple is:

$$4\cos\left(t - \frac{1}{16}\sigma - F_{1}(\upsilon)\right)\partial_{\upsilon}(F_{1})(\upsilon) + x_{3}(t,\tau,\sigma,\upsilon) + \partial_{u}x_{3}(t,\tau,\sigma,\upsilon) + \frac{1}{32}\sin\left(t - \frac{1}{16}\sigma - F_{1}(\upsilon)\right) + \frac{9}{32}\sin\left(3t - \frac{3}{16}\sigma - 3_{-}F_{1}(\upsilon)\right) + \frac{7}{12}\sin\left(7t - \frac{7}{16}\sigma - 7_{-}F_{1}(\upsilon)\right) + \frac{85}{96}\sin\left(5t - \frac{5}{16}\sigma - 5_{-}F_{1}(\upsilon)\right).$$
(93)

Again, we need to equate resonant terms in equation (93) to zero and we arrive at:

$$4\partial_{\nu}(-F1)(\nu) = 0. \tag{94}$$

The solution to  $_F1$  in equation (94) indicate that  $_F1(\upsilon) = _C$  where  $_C = cons \tan t$ .

Substituting for  $\_F1$  back into (93), we removed all resonant terms and we have:

$$x_{3}(t,\tau,\sigma,\upsilon) + \partial_{u}x_{3}(t,\tau,\sigma,\upsilon) - \frac{9}{32}\sin\left(-3t + \frac{3}{16}\sigma + 3_{-}C\right) - \frac{7}{12}\sin\left(-7t + \frac{7}{16}\sigma + 7_{-}C\right) - \frac{85}{96}\sin\left(-5t + \frac{5}{16}\sigma + 5_{-}C\right).$$
(95)

Note that we have set the term  $\frac{1}{32}\sin\left(t-\frac{1}{16}\sigma-F_1(\upsilon)\right)=0$ . as this is a secular term. The solution to equation (95) in Maple is shown below:

$$x_{3}(t,\tau,\sigma,\upsilon) = _F3(\tau,\sigma,\upsilon)\sin(t) + _F2(\tau,\sigma,\upsilon)\cos(t) - \frac{85}{2304}\sin\left(5t - \frac{5}{16}\sigma - 5_{-}C\right) + \frac{9}{256}\sin\left(3t - \frac{3}{16}\sigma - 3_{-}C\right) + \frac{7}{576}\sin\left(7t - \frac{7}{16}\sigma - 7_{-}C\right).$$
(96)

And hence,

$$x_{3} = -\frac{85}{2304} \sin\left(-5t + \frac{5}{16}\sigma + 5_{-}C\right) - \frac{9}{256} \sin\left(-3t + \frac{3}{16}\sigma + 3_{-}C\right) - \frac{7}{576} \sin\left(-7t + \frac{7}{16}\sigma + 7_{-}C\right).$$
(97)

Therefore the full 4-timescale perturbation theory solution to equation (45) is:

$$x(t,\varepsilon) = 2\cos\left(t - \frac{1}{16}\sigma - C\right) - \frac{1}{4}\sin\left(3t - \frac{3}{16}\sigma - 3 - C\right)\varepsilon - \left[\frac{5}{96}\cos\left(5t - \frac{5}{16}\sigma - 5 - C\right) - \frac{3}{32}\cos\left(3t - \frac{3}{16}\sigma - 3 - C\right)\right]\varepsilon^{2} -$$

$$\left[\frac{85}{2304}\sin\left(5t - \frac{5}{16}\sigma - 5 - C\right) + \frac{9}{256}\sin\left(3t - \frac{3}{16}\sigma - 3 - C\right)\right]\varepsilon^{3}.$$

$$\left[+\frac{7}{576}\sin\left(7t - \frac{7}{16}\sigma - 7 - C\right)\right]\varepsilon^{3}.$$
(98)

The computation is contained in Appendix 2.0.



Figure 6.0(b), [Maple] phase plot  $(\dot{x}, x)$  of van der Pol numerical integration in red vs. approximate solution in green.



Figure 6.0(b) phase plot of the numerical integration of the van der Pol in red vs approximate solution in green by the 4-timing method for phi = 0.2 and epsilon = 0.5

# Chapter 8.0 8.1 AVERAGED EQUATIONS

As the reader may have noticed the same steps keep recurring in the solutions to the van der Pol. According to Strogatz (2000:223) this is true of weakly nonlinear oscillators in general. We can therefore speed up things by deriving some general formulas. The best way to illustrate the method is to work through an example. We will use the 4-timescale and the van der Pol oscillator.

Consider again the weakly non-linear equation for the van der Pol again given below as

 $\ddot{x} + x + \varepsilon (1 - x^2) \dot{x} = 0$  x(0) = 2  $\dot{x}(0) = 0.$ 

Recall that Strogatz (2001:215) wrote that the general form of weakly nonlinear equations is:  $\ddot{x} + x + \varepsilon h(x, \dot{x}) = 0$ , defined in equation (1) above. For the van der Pol oscillator, *h* are all terms in the perturbation expansion affected by the quantity  $\varepsilon(1-x^2)\dot{x}$ . Hence, the usual 4-timing equations can now be written as:

$$o(1): \quad \partial_{tt} x_0(\mathbf{t}, \tau, \sigma, \upsilon) + x_0(\mathbf{t}, \tau, \sigma, \upsilon) = 0.$$
(99)

$$o(\varepsilon): \quad x_1(t,\tau,\sigma,\upsilon) + 2\partial_{t\tau}(\mathbf{x}_0)(t,\tau,\sigma,\upsilon) + h + \partial_t(\mathbf{x}_1)(t,\tau,\sigma,\upsilon) = 0.$$
(100)

where 
$$h = (x_0(t,\tau,\sigma,\upsilon)^2 - 1)\partial_t(x_0)(t,\tau,\sigma,\upsilon).$$
 (101)

Similarly, the second and third order of  $\varepsilon$  equations are:

$$o(\varepsilon^{2}): \quad x_{2}(t,\tau,\sigma,\upsilon) + h + 2x_{0}(t,\tau,\sigma,\upsilon)x_{1}(t,\tau,\sigma,\upsilon)\partial_{t}(x_{0})(t,\tau,\sigma,\upsilon) + 2\partial_{t\sigma}(x_{0})(t,\tau,\sigma,\upsilon) + \partial_{\tau\tau}(x_{0})(t,\tau,\sigma,\upsilon) + 2\partial_{t\tau}(x_{1})(t,\tau,\sigma,\upsilon) + \partial_{tt}(x_{2})(t,\tau,\sigma,\upsilon) = 0.$$
(102)

where 
$$h = (x_0(t,\tau,\sigma,\upsilon)^2 - 1) (\partial_{\tau}(x_0)(t,\tau,\sigma,\upsilon) + \partial_{\tau}(x_1)(t,\tau,\sigma,\upsilon))$$
 (103)

$$o(\varepsilon^{3}): \quad 2\partial_{t\nu}x_{0}(t,\tau,\sigma,\upsilon) + x_{3}(t,\tau,\sigma,\upsilon) + \partial_{t\tau}(x_{3})(t,\tau,\sigma,\upsilon) + 2\partial_{\tau\sigma}(x_{0})(t,\tau,\sigma,\upsilon) + 2\partial_{t\sigma}(x_{0})(t,\tau,\sigma,\upsilon) + 2\partial_{\tau\tau}(x_{1})(t,\tau,\sigma,\upsilon) + 2x_{0}(t,\tau,\sigma,\upsilon)x_{1}(t,\tau,\sigma,\upsilon)(\partial_{\tau}(x_{0})(t,\tau,\sigma,\upsilon) - (104)) + \partial_{t\tau}(x_{1})(t,\tau,\sigma,\upsilon) + 2\partial_{t\tau}(x_{2})(t,\tau,\sigma,\upsilon) + h = 0.$$

where :

$$h = \left(x_0(t,\tau,\sigma,\upsilon)^2 - 1\right) \left(\partial_{\sigma}(x_0)(t,\tau,\sigma,\upsilon) + \partial_{\tau}(x_1)(t,\tau,\sigma,\upsilon) + \partial_{t}(x_2)(t,\tau,\sigma,\upsilon)\right) + \left(2x_0(t,\tau,\sigma,\upsilon)(x_2)(t,\tau,\sigma,\upsilon) + x_1(t,\tau,\sigma,\upsilon)^2\right) \partial_{t}x_0(t,\tau,\sigma,\upsilon).$$
(105)

The solution of the o(1) equation is given in equation (76) as:

$$x_0(t,\tau,\sigma,\upsilon) = A(\tau,\sigma,\upsilon) \cos(t - \varphi(\tau,\sigma,\upsilon)).$$

Substituting equation (76) into equation (100), the  $o(\varepsilon)$ , we have that:

$$x_{1}(t,\tau,\sigma,\upsilon) - 2\partial_{\tau}(A)(\tau,\sigma,\upsilon)\sin(t - \Phi(\tau,\sigma,\upsilon)) + 2A(\tau,\sigma,\upsilon)\cos(t - \Phi(\tau,\sigma,\upsilon))$$
  

$$\partial_{\tau}(\Phi)(\tau,\sigma,\upsilon) + \partial_{tt}(x_{1})(t,\tau,\sigma,\upsilon) + h = 0.$$
(106)

To extract the terms in *h*, proportional to  $\cos(t - \Phi(\tau, \sigma, \upsilon))$ , and  $\sin(t - \Phi(\tau, \sigma, \upsilon))$ , we borrow some ideas from Fourier series.

Notice that *h* is a  $2\pi$  - periodic function of  $t - \Phi(\tau, \sigma, \upsilon)$ . Let  $\theta = t - \Phi(\tau, \sigma, \upsilon)$ .

From Fourier analysis,  $h(\theta)$ , can be written as a Fourier series

$$h(\theta) = \sum_{k=0}^{\infty} a_k \cos k\theta + \sum_{k=1}^{\infty} b_k \sin k\theta.$$
 (107)

Where the Fourier coefficients are given by:

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} h(\theta) d\theta.$$
(108)

$$a_k = \frac{1}{\pi} \int_0^{2\pi} h(\theta) \cos k\theta d\theta \quad k \ge 1.$$
(109)

$$b_k = \frac{1}{\pi} \int_0^{2\pi} h(\theta) \sin k\theta d\theta \quad k \ge 1.$$
(110)

Substituting  $\theta = t - \Phi(\tau, \sigma, \upsilon)$ , in equation (106), it reduces to:

$$x_{1}(t,\tau,\sigma,\upsilon) - 2\partial_{\tau}(A)(\tau,\sigma,\upsilon)\sin(\theta) + 2A(\tau,\sigma,\upsilon)\cos(\theta)\partial_{\tau}(\Phi)(\tau,\sigma,\upsilon) + \partial_{tt}(x_{1})(t,\tau,\sigma,\upsilon) + h = 0.$$
(111)

Considering only resonant terms in equation (111) we have:

$$-2\partial_{\tau}(A)(\tau,\sigma,\upsilon)\sin(\theta) + 2A(\tau,\sigma,\upsilon)\cos(\theta)\partial_{\tau}(\Phi)(\tau,\sigma,\upsilon) + h = 0.$$
(112)

and substituting for h in equation (107), equation (112) gives:

$$-2\partial_{\tau}(A)(\tau,\sigma,\upsilon)\sin(\theta) + 2A(\tau,\sigma,\upsilon)\cos(\theta)\partial_{\tau}(\Phi)(\tau,\sigma,\upsilon) + \sum_{k=0}^{\infty} a_{k}\cos k\theta$$

$$+\sum_{k=1}^{\infty} b_{k}\sin k\theta = 0.$$

$$-2\partial_{\tau}(A)(\tau,\sigma,\upsilon)\sin(\theta) + 2A(\tau,\sigma,\upsilon)\cos(\theta)\partial_{1}(\Phi)(\tau,\sigma,\upsilon) + (a_{0} + a_{1}\cos\theta + \sum_{k=2}^{\infty} a_{k}\cos k\theta)$$

$$+(b_{1}\sin\theta + \sum_{k=2}^{\infty} b_{k}\sin k\theta) = 0.$$
(113)

We discard all terms in  $\sum_{k=2}^{\infty} a_k \cos k\theta$ , and  $\sum_{k=2}^{\infty} b_k \sin k\theta$ , as there are no resonant terms in them.

Hence the only resonant terms in equation (114) are:

 $-2\partial_{\tau}(A)(\tau,\sigma,\upsilon)\sin(\theta) + a_{1}\cos\theta + 2A(\tau,\sigma,\upsilon)\cos(\theta)\partial_{\tau}(\Phi)(\tau,\sigma,\upsilon) + b_{1}\sin\theta = 0.$  Grouping like terms in  $\sin(\theta)$ , and  $\cos(\theta)$ , we have:

$$\left(-2\partial_{\tau}(A)(\tau,\sigma,\upsilon)+b_{1}\right)\sin(\theta)+\left(2A(\tau,\sigma,\upsilon)\partial_{\tau}(\Phi)(\tau,\sigma,\upsilon)+a_{1}\right)\cos(\theta)=0.$$

Therefore to avoid secular terms in the solution, we have

$$-2\partial_{\tau}(A)(\tau,\sigma,\upsilon) + b_1 = 0. \tag{115}$$

$$2A(\tau,\sigma,\upsilon)\partial_{\tau}(\Phi)(\tau,\sigma,\upsilon) + a_1 = 0.$$
(116)

Equation (115) gives

$$\partial_{\tau}(A)(\tau,\sigma,\upsilon) = \frac{b_1}{2}.$$
(117)

and equation (116) gives:

$$A(\tau,\sigma,\upsilon)\partial_{\tau}(\Phi)(\tau,\sigma,\upsilon) = \frac{-a_1}{2}.$$
(118)

If we substitute for  $b_1$  in equation (117), from equation (110), the third Fourier coefficient equation, we have that:

$$\partial_{\tau}(A)(\tau,\sigma,\upsilon) = \frac{b_1}{2} = \frac{1}{2\pi} \int_{0}^{2\pi} h(\theta) \sin\theta d\theta \equiv \langle h\sin\theta \rangle.$$
(119)

Similarly, substituting for  $a_1$  in equation (118) from equation (109), the second Fourier coefficient equation, we obtain

$$A(\tau,\sigma,\upsilon)\partial_{\tau}(\Phi)(\tau,\sigma,\upsilon) = -\frac{a_1}{2} = -\frac{1}{2\pi} \int_{0}^{2\pi} h(\theta)\cos\theta d\theta \equiv \langle h\cos\theta \rangle.$$
(120)

From equation (101),  $h = h(\theta) = (x_0(t, \tau, \sigma, \upsilon)^2 - 1)\partial_t(x_0)(t, \tau, \sigma, \upsilon)$ . Substituting for  $x_0 = A(\tau, \sigma, \upsilon)\cos(t - \varphi(\tau, \sigma, \upsilon))$  in equation (101), Maple gives the following output:

$$h = \left(-\frac{1}{4}(A)(\tau,\sigma,\upsilon)^3 + A(\tau,\sigma,\upsilon)\right)\sin(\theta) - \frac{1}{4}(A)(\tau,\sigma,\upsilon)^3\sin(3\theta).$$
(121)

We are only interested in terms in  $sin(\theta)$  in equation (121) as the term in  $sin(3\theta)$  do not contribute to secular terms. Therefore, combining equations (119) and (121), we have:

$$\partial_{\tau}(A)(\tau,\sigma,\upsilon) = \left\langle h\sin(\theta) \right\rangle = \left( -\frac{1}{4} (A)(\tau,\sigma,\upsilon)^3 + A(\tau,\sigma,\upsilon) \right) \left\langle \sin(\theta)^2 \right\rangle.$$
(122)

Where the angle bracket  $\langle . \rangle$  denote an average over one cycle of  $\theta$ .

From tables, the averaged value of  $\langle sin(\theta)^2 \rangle = \frac{1}{2}$ , hence

$$\partial_{\tau}(A)(\tau,\sigma,\upsilon) = \frac{1}{2}A(\tau,\sigma,\upsilon) - \frac{1}{8}(A)(\tau,\sigma,\upsilon)^{3}.$$
(123)

We can contract equation (123) to:

$$\partial_{\tau}(A)(\tau,\sigma,\nu) = \frac{1}{2}A - \frac{1}{8}A^3.$$
 (124)

where  $A = A(\tau, \sigma, \upsilon)$ .

Similarly,

$$A(\tau,\sigma,\upsilon)\partial_{\tau}(\Phi)(\tau,\sigma,\upsilon) = \langle h\cos(\theta) \rangle = \left(-\frac{1}{4}(A)(\tau,\sigma,\upsilon)^3 + A(\tau,\sigma,\upsilon)\right) \langle \sin(\theta)\cos(\theta) \rangle.$$
(125)

From tables, the value of the average  $\langle sin(\theta) \cos(\theta) \rangle = 0$ , hence,

$$A(\tau,\sigma,\upsilon)\partial_{\tau}(\Phi)(\tau,\sigma,\upsilon) = 0.$$
(126)

To solve equations (124) and equations (126), we write  $\partial_{\tau}(A)(\tau, \sigma, \upsilon)$ , to mean  $\frac{\partial A}{\partial \tau}$ , where  $A = A(\tau, \sigma, \upsilon)$ , and hence equation (124) becomes:

$$\frac{\partial A}{\partial \tau} = \frac{1}{2}A - \frac{1}{8}A^3. \tag{127}$$

The differential equation (127) separates as:

$$\int \partial \tau = \int \frac{8\partial A}{A(4-A^2)}.$$
(128)

Expanding the right hand side of equation (128) into partial equations yields the following expression:

$$8\int \left(\frac{1}{4A} + \frac{1}{8(2-A)} - \frac{1}{8(2+A)}\right) \partial A.$$
 (129)

$$=\int \frac{2}{A}\partial A + \int \frac{1}{(2-A)}\partial A - \int \frac{1}{(2+A)}\partial A.$$
(130)

$$= 2\ln|A| - \ln|2 - A| - \ln|2 + A|.$$
(131)

Hence, equation (128) becomes:

$$\ln \left| \frac{A^2}{(2-A)(2+A)} \right| = \tau + \cos \tan t.$$

The initial conditions x(0) = 1;  $\dot{x}(0) = 0$  imply that A(0) = 1.

Hence, we have that the  $\ln\left(\frac{1}{3}\right) = cons \tan t$ ; hence,  $cons \tan t = -\ln 3$ .

This implies that 
$$\ln \left| \frac{A^2}{(2-A)(2+A)} \right| = \tau - \ln |3| = \ln \left| \frac{3A^2}{(2-A(2+A))} \right| = \tau$$
, hence

$$e^{\tau} = \frac{3A^2}{(4-A^2)} = e^{\tau}(4-A^2) = 3A^2.$$
  
=  $3A^2 = 4e^{\tau} - A^2e^{\tau}.$   
=  $A^2(3+e^{\tau}) = 4e^{\tau}.$ 

Solving for *A*, we have:

$$= A = \left(\frac{4e^{\tau}}{(3+e^{\tau})}\right)^{1/2}.$$

$$A(\tau,\sigma,\upsilon) = \frac{2}{(1+3e^{-\tau})^{1/2}}.$$
(132)

$$x_0(t,\tau,\sigma,\upsilon) = \frac{2}{(1+3e^{-\varepsilon\tau})^{1/2}} \cos\left(t - \Phi(\tau,\sigma,\upsilon)\right) + o(\varepsilon).$$
(133)

To simplify  $\Phi(\tau, \sigma, \upsilon)$ , we guess from:  $A(\tau, \sigma, \upsilon)\partial_{\tau}(\Phi)(\tau, \sigma, \upsilon) = 0$ , that

since 
$$A(\tau, \sigma, \upsilon) \neq 0$$
, then  $\partial_{\tau}(\Phi)(\tau, \sigma, \upsilon) = 0$ , or that  $\frac{\partial \Phi}{\partial \tau} = 0$ .

This implies that the function  $\Phi$  does not depend on  $\tau$ . Also,  $(\Phi)(0) = 0$  and we have equation (133) as:

$$x_0(t,\tau,\sigma,\upsilon) = 2\cos(t - \Phi(\sigma,\upsilon)) + o(\varepsilon).$$
(134)

where  $A \rightarrow 2$  in equation (132) as before, as  $t \rightarrow \infty$ . This is again the stable limit cycle of the van der Pol oscillator derived by the method of averaged equations.

Substituting equation (134) into the  $o(\varepsilon)$  equation above, we have:

$$x_1(t,\tau,\sigma,\upsilon) + \partial_{tt}(x_1)(t,\tau,\sigma,\upsilon) - 2sin(3\theta) = 0.$$

Solving equation (135) for  $x_1$ , we obtain the solution:

$$x_{1} = F_{2}(\tau, \sigma, \upsilon)\sin(t) + F_{3}(\tau, \sigma, \upsilon)\cos(t) - \frac{1}{4}\sin(3\theta).$$
(135)

and we simply write this result as:

$$x_1 = -\frac{1}{4}\sin(3\theta) = -\frac{1}{4}\sin\left((3t - 3\Phi(\sigma, \upsilon))\right).$$
 where  $\theta = t - \Phi(\sigma, \upsilon)$ 

Similarly, for the  $o(\varepsilon^2)$ , we have:

$$o(\varepsilon^{2}): x_{2}(t,\tau,\sigma,\upsilon) + 2x_{0}(t,\tau,\sigma,\upsilon)x_{1}(t,\tau,\sigma,\upsilon)\partial_{t}(x_{0})(t,\tau,\sigma,\upsilon) + 2\partial_{t\sigma}(x_{0})(t,\tau,\sigma,\upsilon) + \partial_{t\tau}(x_{0})(t,\tau,\sigma,\upsilon) + 2\partial_{t\tau}(x_{1})(t,\tau,\sigma,\upsilon) + \partial_{tt}(x_{2})(t,\tau,\sigma,\upsilon) + h = 0.$$
(136)

Substituting for  $x_0$  and  $x_1$  in equation (136), Maple simplifies the expression to:

$$=\frac{1}{2}(1+8\partial_{\sigma}\Phi(\sigma,\upsilon))\cos(t-\Phi(\sigma,\upsilon))+x_{2}(t,\tau,\sigma,\upsilon)-\frac{1}{2}\cos(5t-5\Phi(\sigma,\upsilon))$$
  
+ $\partial_{u}(x_{2})(t,\tau,\sigma,\upsilon)+h.$  (137)

The resonant term is only:  $\frac{1}{2}(1+8\partial_{\sigma}\Phi(\sigma,\upsilon))\cos(\theta)$  and we have  $h = a_1\cos(\theta)$ . combining the two cosine functions, equation (137) can be written as:

$$=\frac{1}{2}(1+8\partial_{\sigma}\Phi(\sigma,\nu))\cos(\theta)+a_{1}\cos(\theta)=0, \text{ where } \theta=t-\Phi(\sigma,\nu).$$

Again, to avoid secular terms in the solution, we equate the coefficients of  $cos(\theta) = 0$ , hence,

$$\partial_{\sigma}\Phi(\sigma,\nu) = \frac{\partial\Phi}{\partial\sigma} = \frac{1}{8}\left(-2a_1 - 1\right) = \left(\frac{-a_1}{4} - \frac{1}{8}\right).$$
(138)

We need to find  $a_1$  from:

 $a_k = \frac{1}{\pi} \int_0^{2\pi} h(\theta) \cos k\theta d\theta$   $k \ge 1$ ; and substitute this into equation (138). After a little manipulation, equation (138) simplifies to:

$$\frac{\partial \Phi}{\partial \sigma} = -\frac{\left\langle h\cos(\theta) \right\rangle}{2} - \frac{1}{8}.$$
(139)

In Maple, equation (103), is computed as:

$$h = -\frac{3}{4}\cos(t - \Phi(\sigma, \nu)) - \frac{3}{4}\cos(5t - 5\Phi(\sigma, \nu)) - \frac{3}{4}\cos(3t - 3\Phi(\sigma, \nu)).$$
(140)

As before, we are only interested in the coefficient of *h*, in equation (140) that leads to secular terms  $=-\frac{3}{4}\cos(t-\Phi(\sigma,\upsilon))$ .

Equation (139) can now be written as:

$$\frac{\partial \Phi}{\partial \sigma} = \frac{3}{8} \left\langle \cos^2(\theta) \right\rangle - \frac{1}{8}, \text{ where the averaged value } \left\langle \cos^2(\theta) \right\rangle = \frac{1}{2}, \text{ hence}$$
$$\frac{\partial \Phi}{\partial \sigma} = \frac{3}{16} - \frac{1}{8} = \frac{1}{16}. \text{ Thus we have:}$$
$$\int \partial \Phi = \frac{1}{16} \int \partial \sigma, \text{ hence,}$$
$$\Phi(\sigma, \upsilon) = \frac{1}{16} \sigma + _F 1(\upsilon). \tag{141}$$

To solve for  $x_2$ , we realised that for our solution to remain bounded for all t, we only need solve the differential equation:

$$-\frac{3}{4}\cos(5t - 5\Phi(\sigma, \upsilon)) - \frac{3}{4}\cos(3t - 3\Phi(\sigma, \upsilon)) + x_2(t, \tau, \sigma, \upsilon) - \frac{1}{2}\cos(5t - 5\Phi(\sigma, \upsilon))$$
(142)  
+ $\partial_u(x_2)(t, \tau, \sigma, \upsilon) = 0.$   
= $-\frac{5}{4}\cos(5t - 5\Phi(\sigma, \upsilon)) - \frac{3}{4}\cos(3t - 3\Phi(\sigma, \upsilon)) + x_2(t, \tau, \sigma, \upsilon) + \partial_u(x_2)(t, \tau, \sigma, \upsilon) = 0.$ (143)

Substituting for  $\Phi(\sigma, \upsilon) = \frac{1}{16}\sigma + F_1(\upsilon)$ , from equation (141), equation (143)

$$= -\frac{5}{4}\cos\left(5t - \frac{5}{16}\sigma - 5_F I(\upsilon)\right) - \frac{3}{4}\cos\left(3t - \frac{3}{16}\sigma - 3_F I(\upsilon)\right) + x_2(t,\tau,\sigma,\upsilon)$$
(144)  
+ $\partial_u(x_2)(t,\tau,\sigma,\upsilon) = 0.$ 

Solving equation (144), we have that:

$$x_{2}(t,\tau,\sigma,\upsilon) = _F1(\tau,\sigma,\upsilon)\sin(t) + _F2(\tau,\sigma,\upsilon)\cos(t) - \frac{5}{96}\cos\left(5t - \frac{5}{16}\sigma - 5_F1(\upsilon)\right) - \frac{3}{32}\cos\left(3t - \frac{3}{16}\sigma - 3_F1(\upsilon)\right).$$

Hence,

$$x_{2} = -\frac{5}{96}\cos\left(5t - \frac{5}{16}\sigma - 5_{F1}(\upsilon)\right) - \frac{3}{32}\cos\left(3t - \frac{3}{16}\sigma - 3_{F1}(\upsilon)\right).$$

To solve for  $x_3$  we substitute for the values already calculated for  $x_0$ ,  $x_1$  and  $x_2$  in the  $o(\varepsilon^3)$  equation above to obtain:

$$4\cos\left(t - \frac{1}{16}\sigma - F_{1}(\upsilon)\right)\partial_{\upsilon}(F_{1})(\upsilon) + x_{3}(t,\tau,\sigma,\upsilon) + \partial_{tt}x_{3}(t,\tau,\sigma,\upsilon) - \frac{9}{32}\sin\left(3t - \frac{3}{16}\sigma - 3_{-}F_{1}(\upsilon)\right) + \frac{3}{16}\sin\left(7t - \frac{7}{16}\sigma - 7_{-}F_{1}(\upsilon)\right) + \frac{3}{16}\sin\left(5t - \frac{5}{16}\sigma - 5_{-}F_{1}(\upsilon)\right) + h.$$
(145)

If we let 
$$\theta = \left(t - \frac{1}{16}\sigma - F_1(\upsilon)\right)$$
, the above equation becomes:  

$$= 4\cos(\theta)\partial_{\upsilon}(F_1)(\upsilon) + x_3(t,\tau,\sigma,\upsilon) + \partial_{u}x_3(t,\tau,\sigma,\upsilon) - \frac{9}{32}\sin(3\theta) + \frac{3}{16}\sin(7\theta) + \frac{3}{16}\sin(5\theta) + h = 0.$$
(146)

As we did above, we can write the resonant terms in  $\cos(\theta)$  as:

$$4\cos(\theta)\partial_{\nu}(F1)(\nu) + a_{1}\cos(\theta) = 0.$$
(147)

To solve equation (147) we equate the coefficients of  $cos(\theta)$  to zero and we have that

 $4\partial_{\nu}(\_F1)(\upsilon) + a_1 = 0$ , hence, similarly:

$$\partial_{\nu}(F1)(\nu) = \frac{-a_1}{4} = -\frac{1}{2} \langle h\cos(\theta) \rangle.$$
(148)

From equation (105) we can evaluate in Maple that:

$$h = \frac{9}{16}\sin(3\theta) + \frac{1}{32}\sin(\theta) + \frac{19}{48}\sin(7\theta) + \frac{67}{96}\sin(5\theta).$$
 (149)

Again we are only interested in those terms in h which give rise to secular terms, thus

$$\partial_{\nu}(F1)(\nu) = -\frac{1}{64} \langle \sin(\theta) \cos(\theta) \rangle$$
. From tables, the average  $\langle \sin(\theta) \cos(\theta) \rangle = 0$  hence,

$$F1(\upsilon) = C$$
, where  $C = cons \tan t$  and  $\Phi = \frac{1}{16}\sigma + C$ .

To solve  $x_3$ , we solve the differential equation

$$x_{3}(t,\tau,\sigma,\upsilon) + \partial_{u}x_{3}(t,\tau,\sigma,\upsilon) - \frac{9}{32}\sin\left(3t - \frac{3}{16}\sigma - 3_{-}C\right) + \frac{3}{16}\sin\left(7t - \frac{7}{16}\sigma - 7_{-}C\right) + \frac{3}{16}\sin\left(5t - \frac{5}{16}\sigma - 5_{-}C\right) + \frac{9}{16}\sin\left(3t - \frac{3}{16}\sigma - 3_{-}C\right) + \frac{19}{48}\sin\left(7t - \frac{7}{16}\sigma - 7_{-}C\right)$$
(150)  
+  $\frac{67}{96}\sin\left(5t - \frac{5}{16}\sigma - 5_{-}C\right) = 0.$ 

If we let  $\theta = t - \frac{1}{16}\sigma - C$ , equation (150) becomes:

$$x_{3}(t,\tau,\sigma,\upsilon) + \partial_{tt}x_{3}(t,\tau,\sigma,\upsilon) - \frac{9}{32}\sin(3\theta) + \frac{3}{16}\sin(7\theta) + \frac{3}{16}\sin(5\theta) + \frac{9}{16}\sin(3\theta) + \frac{19}{48}\sin(7\theta) + \frac{67}{96}\sin(5\theta) = 0.$$
(151)

Adding like terms we have:

$$x_{3}(t,\tau,\sigma,\upsilon) + \partial_{tt}x_{3}(t,\tau,\sigma,\upsilon) + \frac{9}{32}\sin(3\theta) + \frac{7}{12}\sin(7\theta) + \frac{85}{96}\sin(5\theta) = 0.$$
(152)

Solving for  $x_3$ , we have:

$$x_{3}(t,\tau,\sigma,\upsilon) = F_{3}(\tau,\sigma,\upsilon)\sin(t) + F_{2}(\tau,\sigma,\upsilon)\cos(t) - \frac{85}{2304}\sin\left(5t - \frac{5}{16}\sigma - 5 - C\right) + \frac{9}{256}\sin\left(3t - \frac{3}{16}\sigma - 3 - C\right) + \frac{7}{576}\sin\left(7t - \frac{7}{16}\sigma - 7 - C\right).$$
(153)

Hence, we write:

$$x_{3} = -\frac{85}{2304} \sin\left(5t - \frac{5}{16}\sigma - 5_{-}C\right) + \frac{9}{256} \sin\left(3t - \frac{3}{16}\sigma - 3_{-}C\right) + \frac{7}{576} \sin\left(7t - \frac{7}{16}\sigma - 7_{-}C\right).$$
(154)

Thus the values for  $x_0$ ,  $x_1$ ,  $x_2$  and  $x_3$  obtained in the averaged equations method are the same as we obtained in the 4-timescales for the van der Pol.

# Chapter 9.0 9.1 PERIOD AND MULTIPLE TIMESCALE IN THE VAN DER POL OSCILATOR

In this chapter, we will combine period, angular frequency and multiple timescale of the van der Pol.

We recall from equation (3) above, that, the van der Pol oscillator can be written as

 $\ddot{x} + x + \varepsilon (1 - x^2) \dot{x} = 0.$ 

We can define the period, an independent variable in this case as:

 $T = \omega t \tag{155}$ 

Our aim is to rewrite equation (3) in terms of the new independent variable, *T*. We know from multiple timescale perturbation theory that each linearised solution,  $x_j$  for j = 0,1... is a periodic solution with a different amplitude and period. Then, we can write  $x_j$  as a function of the independent variable, *T*, where  $x_j = x_j(T)$  is periodic of period  $2\pi$  in *T* and does not depend on  $\varepsilon$ .

We define the angular frequency,  $\omega$ , of the van der Pol oscillator as the stretched time variable, where:

$$\omega = 1 + \varepsilon \omega_1 + \varepsilon^2 \omega_2 + \varepsilon^3 \omega_3 + \dots$$
 (156)

is a real positive constant to be determined later.

Perturbation Theory still requires that we seek solutions of the form:

$$x(T) = x_0(T) + \varepsilon x_1(T) + \varepsilon^2 x_2(T) + \varepsilon^3 x_3(T)...$$
(157)

Rewriting the equation of the van der Pol oscillator:  $\ddot{x} + x + \varepsilon(1 - x^2)\dot{x} = 0$  via:

$$\frac{d}{dt} = \omega \frac{d}{dT}.$$
(158)

we obtain:

$$\omega^2 x''(T) + x(T) + \omega \varepsilon (x^2(T) - 1) x'(T).$$
(159)

Equation (159) is the form of the van der Pol oscillator that contains the nonlinear period *T*, defined in equation (155), as the stretched time variable, and the angular frequency,  $\omega$ , as defined in equation (156).

# 9.2 7-TIMESCALE OF THE VAN DER POL OSCILLATOR

The way we solve equation (159) is the same as we solved  $\ddot{x} + x + \varepsilon(1 - x^2)\dot{x} = 0$  above before. We are required to substitute equations (156) and (157) into equation (159). This will give us an equation with too many terms. We have done the tedious computation in Maple. As usual, we must collect like terms in powers of  $\varepsilon$  and equate the coefficients of each power of  $\varepsilon$  to zero. The full Maple script showing the entire computation and results is contained in Appendix 3.0.

As above, the coefficient of the zeroth power of  $\varepsilon$  is:

$$\frac{d^2}{dT^2}x_0(T) - x_0(T).$$
(160)

The solution is of the general form:  $_C2\cos(T)$  where  $_C2$  is a constant to be determined. For the van der Pol oscillator, we have shown before that  $_C2 = 2$ . Our aim is not to go through the laborious working systematically, but rather to show the linearised solutions in terms of the period, *T*, and the angular frequency,  $\omega$ .

Following from the above examples, to approximate to 7-timescale, we mean to compute  $x_0(T) \dots x_6(T)$  omitting terms in  $\varepsilon^7$ , and higher. We list only the first 5 solutions for illustration. The complete solution is in Appendix 3.0

$$x_0(T) = 2\cos(T); \qquad \omega_1 = 0.$$
 (161)

$$x_1(T) = -\frac{1}{4}\sin(3T); \quad \omega_2 = -\frac{1}{16}.$$
 (162)

$$x_2(T) = \frac{1}{64}\cos(T) - \frac{3}{32}\cos(3T) - \frac{5}{96}\cos(5T); \quad \omega_3 = 0.$$
 (163)

$$x_3(T) = \frac{15}{512}\sin(3T) + \frac{85}{2304}\sin(5T) + \frac{7}{576}\sin(7T); \quad \omega_4 = \frac{17}{3072}.$$
 (164)

$$\begin{aligned} x_4(T) &= -\frac{23}{49152}\cos(T) + \frac{101}{12288}\cos(3T) + \frac{1865}{110592}\cos(5T) + \frac{1379}{110592}\cos(7T) \\ &+ \frac{61}{20480}\cos(9T); \quad \omega_5 = 0. \end{aligned}$$
(165)

It is easy to see from equations (161) – (165) that a solution pattern emerges.  $x_0(T)$  is a cosine function in *T*, and  $x_1(T)$  is a sine term in *T*. For  $j \ge 2$ , there are j+1 cosine terms in  $x_j$ . The first term,  $\cos(T)$ , the second term,  $\cos(3T)$ , and so on. For example when j = 2, there are 3 cosine terms. The three terms are  $\cos(T)$ ,  $\cos(3T)$ , and  $\cos(5T)$ , and  $\omega_j = cons \tan t$ . When j is odd, for  $j \ge 2$ ,  $x_j(T)$  has only *j* sine terms. The first term,  $\sin(3T)$ , the second,  $\sin(5T)$ , the third  $\sin(7T)$ , and so on and  $\omega_i = 0$ .

The solutions above can be written in the general forms below:

$$\begin{split} x_0(T) &= \lambda_{0,1} \cos(T); \quad \omega_1 = 0. \\ x_1(T) &= \lambda_{1,1} \sin(3T); \quad \omega_2 = \lambda_1. \\ x_2(T) &= \lambda_{2,1} \cos(T) + \lambda_{2,2} \cos(3T) + \lambda_{2,3} \cos(5T), \quad \omega_3 = 0. \\ x_3(T) &= \lambda_{3,1} \sin(3T) + \lambda_{3,2} \sin(5T) + \lambda_{3,3} \sin(7T); \quad \omega_4 = \lambda_2. \end{split}$$

Where  $\lambda_{m,n}$  are constants to be determine for m = 0, 1... and n = 1, 2...Suppose, we desire the approximate solution  $x(t, \varepsilon) = x_0(T) + x_1(T) + x_2(T) + o(\varepsilon^3)$ . We have determined from above that  $\omega_1 = \omega_3 = 0$ , hence,  $T = t + \varepsilon^2 t \omega_2$ . Similarly, if we desire the solution  $x(t, \varepsilon) = x_0(T) + x_1(T) + x_2(T) + x_3(T) + x_4(T) + x_5(T) + o(\varepsilon^6)$ , then  $T = t + \varepsilon^2 t \omega_2 + \varepsilon^4 t \omega_4$ . since  $\omega_1 = \omega_3 = \omega_5 = 0$ , and so on.

Hence, strictly,  $\omega = 1 + \varepsilon^2 \omega_2 + \varepsilon^4 \omega_4 + \dots$  (166)

# Chapter 10.0 10.1 N-TIMESCALE OF THE VAN DER POL OSCILATOR

We shall combine all the concepts we have learnt so far to approximate the van der Pol oscillator to 16-timescale. Following the example above, this means we wish to compute  $x_0(T) \dots x_{15}(T)$ . The working is shown in the Maple script in Appendix 4.0. The method is the same as above:

From equation (157), our solution should be of the form:

$$x(T) = \lambda_{0,1} \cos(T) + \lambda_{1,1} \sin(3T)\varepsilon + (\lambda_{2,1} \cos(T) + \lambda_{2,2} \cos(3T) + \lambda_{2,3} \cos(5T))\varepsilon^{2} + \dots \begin{pmatrix} \lambda_{15,1} \sin(3T) + \lambda_{15,2} \sin(5T) + \lambda_{15,3} \sin(7T) + \lambda_{15,4} \sin(9T) + \lambda_{15,5} \sin(11T) + \lambda_{15,6} \sin(13T) + \lambda_{15,7} \sin(15T) + \lambda_{15,8} \sin(19T) \\ + \lambda_{15,9} \sin(21T) + \lambda_{15,10} \sin(23T) + \lambda_{15,11} \sin(25T) + \lambda_{15,12} \sin(27T) \\ + \lambda_{15,13} \sin(29T) + \lambda_{15,14} \sin(31T) + \lambda_{15,15} \end{pmatrix} \varepsilon^{15} + o(\varepsilon^{16}).$$
(167)

And  $T = t + \varepsilon^2 t \omega_2 + \dots \varepsilon^{14} t \omega_{14}$ .

Substituting equations (155), (166) and (167) into equation (5) and collect like terms in powers of  $\varepsilon$  and equating coefficients to zero, the first term to solve is:

$$\left(-\frac{1}{4}\lambda_{0,1}^{3}+\lambda_{0,1}\right)\sin(t)+\left(-8\lambda_{1,1}-\frac{1}{4}\lambda_{0,1}^{3}\right)\sin(3t)=0.$$
(168)

To solve equation (168), we note that the LHS of equation (168) is zero only if the coefficients of the trigonometric terms are zero, hence we compute:

$$\lambda_{0,1} = 2; \quad \lambda_{1,1} = -\frac{1}{4}.$$
 (169)

and we solve iteratively. If we equate coefficients of the second power of  $\varepsilon$  to zero as well and solve for constants, we have:

$$\lambda_1 = -\frac{1}{32}; \quad \lambda_{2,2} = -\frac{3}{32}; \quad \lambda_{2,3} = -\frac{5}{96}$$
 (170)

Similarly, equating coefficients of the third power of  $\varepsilon$  to zero and solving for constants we:

$$\lambda_{2,1} = \frac{1}{64}; \quad \lambda_{3,1} = \frac{15}{512}; \quad \lambda_{3,2} = \frac{85}{2304}; \quad \lambda_{3,3} = \frac{7}{576}.$$
 (171)

Substituting for constants in equation (167) up to the third power of  $\varepsilon$ , we have:

$$x(T) = 2\cos(T) - \frac{1}{4}\sin(3T)\varepsilon + \left(\frac{1}{64}\cos(T) - \frac{3}{32}\cos(3T) - \frac{5}{96}\cos(5T)\right)\varepsilon^{2} + \left(\frac{15}{512}\sin(3T) + \frac{85}{2304}\sin(5T) + \frac{7}{576}\sin(7T)\right)\varepsilon^{3} + \dots$$
(172)

# Chapter 11.0

# 11.1 CONCLUSION

The van der Pol is a weakly nonlinear equation. It cannot be solved in explicit form. The method of multiple scales perturbation theory is able to produce a very good approximation to the solution of the van der Pol oscillator. We can use multiple timescale perturbation theory to calculate limit cycle, stability and frequency of the van der Pol oscillator.

However, the computation involved is tedious and the algebra quickly becomes huge. Above two timing, we require a computer and software, to tackle the algebra involved easily. In addition, the method of Averaged Equations provides an equally powerful alternative method to approximating solutions to weakly nonlinear equations like the van der Pol oscillator.

At higher timescale, we learn something useful about the behaviour of the van der Pol oscillator; however, an untoward consequence is the massive amount of algebra the expansion produces. In practice, we must consider whether multiple 'timescales' might complicate the algebra for no real gain at all. We found, for example, in the case of the linear oscillator, that we could obtain a good approximation to the exact solution without the need to go above two timing. Nonetheless, in multiple timescale perturbation, we learnt something useful about the solution, even though the symbolic computation grew rapidly at higher timescale.

Nonetheless, multiple timescale perturbation remains a useful and powerful mathematical method for finding approximate solutions to differential equations like the van der Pol that cannot be solved exactly.

Chapter 12.0

# 12.1 APPENDIX

- Appendix 1 Maple worksheet 3- timescale.

- Appendix 2 Maple worksheet 4- timescale. Appendix 3 Maple worksheet 7- timescale. Appendix 4 Maple worksheet 16 timescale.

Chapter 13.0

# 13.1 GLOSSARY

# Amplitude

The maximum displacement of a vibrating system from its rest position.

# Dynamical System

A dynamical system is a system that evolves with time.

# **Electronic oscillator**

An electronic circuit that produces a repeating and vibrating electronic signal.

#### Fixed points:

For a vector field on a line, points where the vector is zero and there is no flow.

# Frequency

The number of oscillations in unit time.

# Harmonic oscillator

A system that, when displaced from its equilibrium position, will experience a restoring force proportional to the displacement.

#### **Isolated periodic cycle**

A stable limit cycle.

#### Maple

A symbolic mathematical language created by maplesoft.

# **Ordinary differential equation (ODE)**

An equation containing a function of one independent variable and its derivatives.

#### Oscillations

A displacement that repeats itself in a regular manner.

#### **Perturbation theory**

A method to find an approximate solution to a problem which cannot be solved exactly, but close to one whose exact solution we know.

#### Phase plane

A phase plane is a 2-dimensional plot of certain characteristics of certain kinds of differential equations.

#### **Phase portrait**

A geometric depiction of the trajectories of a dynamical system in the phase plane.

# **Relaxation oscillations**

Stable oscillations also called, limit cycles.

#### Resonance

When the forcing frequency is equal to the natural frequency of an oscillation system resonance occurs.

# Trajectories

A time-order set of states of a dynamical system.

#### Vector field

The vector field in the x-y plane, is a set of arrows with magnitude and direction each representing a point in the plane.

Chapter	14.0

#### 14.1 REFERENCES

Andrianov, I. V., Manevitch, L. I. (2002) *Asymptotology, Ideas Methods and Applications*. Dordrecht: Kluwer Academic.

Bae, S. (2008) *Long-term analysis of a swing using the two-timing method* European Journal of Physics 26 (2005) 791–801

Bender, C. M., Orszag, S. A. (1978) *Advanced Mathematical Methods for Scientists and Engineers*. NY : McGraw-Hill.

Bender, M. C., Orszag, A. S. (1991) *Advanced Mathematical Methods for Scientists and Engineers*: Asymptotic Methods and Perturbation Theory: New York, NY: Springer-Velag

C. Koch, C., I. Segev, I. (1998) *Methods in Neuronal Modeling*, MIT Press, Cambridge, MA.

Cartmell, M.P., Ziegler, S.W., Khanin, R., Forehand, D.I.M. (2003) *Multiple scales analyses of the dynamics of weakly nonlinear mechanical systems*. Transactions of ASME, Applied Mechanics Reviews, 56 (5), pp. 455-492.

Cartwright, J., Eguiluz, V., Hernandez-Garcia, E., Piro, O. (1999) *Dynamics of elastic excitable media*, International Journal of Bifurcation, 9, pp. 2197–2202.

Cartwright, M. L., *Van der Pol, B.* (1960) Journal of the London Mathematical Society 35, pp. 367–376.

FitzHugh, R.(1961) *Impulses and physiological states in models of nerve membrane*, Biophysics Journal, 1, pp. 445–466.

Forger, D.B., Jewett, M. E., Kronauer, R. E. (1999) *A simpler model of the human circadian pacemaker*, Journal of Biological Rhythms, 14, pp. 532-537.

Forger, D.B., Richard E. Kronauer, R.E. (2002) *Reconciling Mathematical Models of Biological Clocks by Averaging on Approximate Manifolds*. Journal on Applied Mathematics, Vol. 62, No. 4, pp. 1281-1296

Guckenheimer, J., Hoffman, K., Weckesser, W. (2000) *Numerical computation of canards*, International Journal for Bifurcation and Chaos, 10, pp. 2669–2687.

Guckenheimer, *J.*, Hoffman, K., Weckesser, W. (2003) *The Forced van der Pol Equation I: The Slow Flow and Its Bifurcations*. Journal of Applied Dynamical Systems, vol. 2, No. 1, pp. 1–35.

Gundel, A., Spencer, M. B. (1999) *A circadian oscillator model based on empirical data*, Journal of Biological Rhythms, 14, pp. 516-523.

Jordan, D. W., Smith, P. (1977) *Nonlinear Ordinary Differential Equations*, Oxford Applied Mathematics and Computing Science Series. Oxford, UK: Oxford University Press.

Lynch, S. (2000) *Dynamical Systems with Applications using Maple*. 2<sup>nd</sup> edition. Boston. Basel. Berlin. Birkhauser

McMurran, S., Tattersall, J. (1996) *The mathematical collaboration of M. L. Cartwright and J. E. Littlewood*, Amer. Math. Monthly, 103, pp. 833–845.

Nayfeh, A. H., Mook, D. T. (1970) Nonlinear Oscillations New York: Wiley

Prellberg, T. (2012) MTH744P, *First-order differential equations (one-dimensional flows): linear and nonlinear equations, graphical solutions, bifurcations*, Queen Mary University, UCL: School of Mathematical Studies.

Rowat, P. F., Selverston, A. I.(1993) *Modeling the gastric mill central pattern generator of the lobster with a relaxation-oscillator network*, J. Neurophysiology, 70, pp. 1030–1053.

Shit, A., Chattopadhyay, S., Chaudhuri, R. J. (2012) *Time-independent description of rapidly driven systems in the presence of friction: Multiple scale perturbation approach*. American Institute of Physics: an interdisciplinary journal of Nonlinear Science. Volume 21, Issue 1 (5 pages)

Strogatz, S. H. (2001) *Nonlinear Dynamics and Chaos: with applications to Physics, Biology, Chemistry and Engineering*. Cambridge, MA: Westview Press: a member of Perseus Book Publishing.

Stumpers, F. L. H. M. (1960) Balthasar *van der Pol's work on nonlinear circuits*, IRE Trans. Circuit Theory, *7*, pp. 366–367.

Van der Pol, B. (1920) *A theory of the amplitude of free and forced triode vibrations*, Radio Review, 1, pp. 701–710, 754–762.

Van der Pol, B. (1926) On "Relaxation Oscillations" I, Phil. Mag., 2, pp. 978–992.

Van der Pol, B., Van der Mark, J. (1927) *Frequency demultiplication*, Nature, 120, pp. 363–364

Van Dyke, M. D. (1964) *Perturbation Methods in Fluid Mechanics* NY: Academic.

Wever, R. (1972) Virtual synchronization towards the limits of the range of entrainment, Journal of Theoretical Biology, 36, pp. 119-132.

Wikipedia (2012) *Van der Pol oscillator* [Online], Available: <u>http://en.wikipedia.org/wiki/Van der Pol oscillator [15</u> August 2012]