> *APPENDIX 1.0 THREE TIMESCALE*

> *restart*; *#causes the maple kernel to clear its internal memory so that Maple acts (almost) as* **if** *just started.*

> $deqn := diff(x(t), t, t) + x(t) + \text{epsilon}\cdot(x(t)^2 - 1)\cdot diff(x(t), t);$
 # the equation of the van der Pol oscillator

$$deqn := \frac{d^2}{dt^2}\, x(t) + x(t) + \varepsilon\,(x(t)^2 - 1)\left(\frac{d}{dt}\, x(t)\right) \tag{1}$$

> $dsolve(deqn, x(t));$
 # we are not able to solve the van der Pol equation in simple functions in maplesoft

$$x(t) = \_a \;\&\text{where}\; \left[\left\{\left(\frac{d}{d\_a}\,\_b(\_a)\right)\_b(\_a) + \_a + \varepsilon\,\_b(\_a)\,\_a^2 - \varepsilon\,\_b(\_a) = 0\right\}, \left\{\_a = x(t),\, \_b(\_a) = \frac{d}{dt}\, x(t)\right\}, \left\{t = \int \frac{1}{\_b(\_a)}\, d\_a + \_C1,\, x(t) = \_a\right\}\right] \tag{2}$$

> $xansatz := t \rightarrow X[0](t, \text{epsilon} * t, \text{epsilon}\verb|^|2 * t) + \text{epsilon} * X[1](t, \text{epsilon} * t, \text{epsilon}\verb|^|2 * t) + \text{epsilon}\verb|^|2 * X[2](t, \text{epsilon} * t, \text{epsilon}\verb|^|2 * t);$ *# pertubation series expansion*

$$xansatz := t \rightarrow X_0(t, \varepsilon\, t, \varepsilon^2\, t) + \varepsilon\, X_1(t, \varepsilon\, t, \varepsilon^2\, t) + \varepsilon^2\, X_2(t, \varepsilon\, t, \varepsilon^2\, t) \tag{3}$$

> $deqnseries := series(subs(\text{epsilon}\verb|^|2 * t = \text{sigma}, subs(\text{epsilon} * t = \text{tau}, eval(subs(x = xansatz, deqn)))), \text{epsilon}, 3);$
 # substitutes equation (3) into equation (1) and collect like terms in powers of epsilon

$$\begin{aligned} deqnseries := {} & D_{1,1}(X_0)(t, \tau, \sigma) + X_0(t, \tau, \sigma) + \Big(X_1(t, \tau, \sigma) + 2\, D_{1,2}(X_0)(t, \tau, \sigma) \\ & + D_{1,1}(X_1)(t, \tau, \sigma) + \big(X_0(t, \tau, \sigma)^2 - 1\big)\, D_1(X_0)(t, \tau, \sigma)\Big)\,\varepsilon + \Big(2\, D_{1,3}(X_0)(t, \tau, \sigma) \\ & + D_{1,1}(X_2)(t, \tau, \sigma) + 2\, D_{1,2}(X_1)(t, \tau, \sigma) + X_2(t, \tau, \sigma) + D_{2,2}(X_0)(t, \tau, \sigma) \\ & + \big(X_0(t, \tau, \sigma)^2 - 1\big)\big(D_2(X_0)(t, \tau, \sigma) + D_1(X_1)(t, \tau, \sigma)\big) + 2\, X_0(t, \tau, \sigma)\, X_1(t, \tau, \\ & \sigma)\, D_1(X_0)(t, \tau, \sigma)\Big)\,\varepsilon^2 + O(\varepsilon^3) \end{aligned} \tag{4}$$

> $deqn0 := coeff(deqnseries, \text{epsilon}, 0);$ *# lists the coefficients of the zeroth power of epsilon*

$$deqn0 := D_{1,1}(X_0)(t, \tau, \sigma) + X_0(t, \tau, \sigma) \tag{5}$$

> $deqn1 := coeff(deqnseries, \text{epsilon}, 1);$ *# lists the coefficients of the first power of epsilon*

$$\begin{aligned} deqn1 := {} & X_1(t, \tau, \sigma) + 2\, D_{1,2}(X_0)(t, \tau, \sigma) + D_{1,1}(X_1)(t, \tau, \sigma) + \big(X_0(t, \tau, \sigma)^2 \\ & - 1\big)\, D_1(X_0)(t, \tau, \sigma) \end{aligned} \tag{6}$$

> $deqn2 := coeff(deqnseries, \text{epsilon}, 2);$ *# lists the coefficients of the second power of epsilon*

$$\begin{aligned} deqn2 := {} & 2\, D_{1,3}(X_0)(t, \tau, \sigma) + D_{1,1}(X_2)(t, \tau, \sigma) + 2\, D_{1,2}(X_1)(t, \tau, \sigma) + X_2(t, \tau, \sigma) \\ & + D_{2,2}(X_0)(t, \tau, \sigma) + \big(X_0(t, \tau, \sigma)^2 - 1\big)\big(D_2(X_0)(t, \tau, \sigma) + D_1(X_1)(t, \tau, \sigma)\big) \\ & + 2\, X_0(t, \tau, \sigma)\, X_1(t, \tau, \sigma)\, D_1(X_0)(t, \tau, \sigma) \end{aligned} \tag{7}$$

> $deqn0;$ *# we call the zeroth power coefficient*

$$D_{1,1}(X_0)(t, \tau, \sigma) + X_0(t, \tau, \sigma) \tag{8}$$

```
> dsolve(deqn0, X[0](t, tau, sigma)); # we solve equation (8)
```

$$X_0(t,\tau,\sigma)=\_F1(\tau,\sigma)\sin(t)+\_F2(\tau,\sigma)\cos(t) \tag{9}$$

```
> X[0] := (t, tau, sigma) → A(tau, sigma) * cos(t-Phi(tau, sigma));
    # we assume that the first linearized X_0 solution is a periodic solution with a given amplitude
    and period.
```

$$X_0 := (t,\tau,\sigma)\rightarrow A(\tau,\sigma)\cos(t-\Phi(\tau,\sigma)) \tag{10}$$

```
> deqn0; # we show that equation (10) solves equation (8)
```

$$0 \tag{11}$$

```
> deqn1temp := collect(combine(deqn1, trig), [sin, cos]); # we substitute for X_0 in equation (6)
    and obtain
```

$$deqn1temp := \left(-2\,D_1(A)(\tau,\sigma)-\frac{1}{4}A(\tau,\sigma)^3+A(\tau,\sigma)\right)\sin(t-\Phi(\tau,\sigma))+X_1(t,\tau,\sigma)+2A(\tau,\sigma)\cos(t-\Phi(\tau,\sigma))\,D_1(\Phi)(\tau,\sigma)+D_{1,1}(X_1)(t,\tau,\sigma)-\frac{1}{4}A(\tau,\sigma)^3\sin(3t-3\Phi(\tau,\sigma)) \tag{12}$$

```
> secular_terms := {coeff(deqn1temp, sin(t-Phi(tau, sigma))), coeff(deqn1temp, cos(t
    -Phi(tau, sigma)))} # we collect secular terms in equation (12)
```

$$secular\_terms := \left\{2A(\tau,\sigma)\,D_1(\Phi)(\tau,\sigma),\ -2\,D_1(A)(\tau,\sigma)-\frac{1}{4}A(\tau,\sigma)^3+A(\tau,\sigma)\right\} \tag{13}$$

```
> dsolve(secular_terms, {Phi(tau, sigma), A(tau, sigma)});
    # we equate secular terms to zero and solve for Φ and A
```

$$\{A(\tau,\sigma)=0,\Phi(\tau,\sigma)=\Phi(\tau,\sigma)\},\ \left\{A(\tau,\sigma)=-\frac{2}{\sqrt{1+4\,e^{-\tau}\,\_F1(\sigma)}},\Phi(\tau,\sigma)=\_F2(\sigma)\right\} \tag{14}$$

```
> X[0] := (t, tau, sigma) → 2·cos(t-Phi(sigma)); # as t tends to infinity, A(τ, σ) tends
    to the value 2; the solution Φ(τ, σ) = _F2(σ), means Φ is a function of σ only.
```

$$X_0 := (t,\tau,\sigma)\rightarrow 2\cos(t-\Phi(\sigma)) \tag{15}$$

```
> deqn1temp := map(simplify, collect(combine(deqn1, trig), [sin, cos]));
    # equation (12) simplifies to:
```

$$deqn1temp := X_1(t,\tau,\sigma)+D_{1,1}(X_1)(t,\tau,\sigma)-2\sin(3t-3\Phi(\sigma)) \tag{16}$$

```
> dsolve(deqn1temp, X[1](t, tau, sigma));
    # as equation (16) has no secular terms, we next solve for the linearized solution, X_1
```

$$X_1(t,\tau,\sigma)=\sin(t)\,\_F2(\tau,\sigma)+\cos(t)\,\_F1(\tau,\sigma)-\frac{1}{4}\sin(3t-3\Phi(\sigma)) \tag{17}$$

```
> X[1] := (t, tau, sigma) → − 1/4 sin(3 t − 3 Φ(σ));
    # the second linearized X_1 solution is a periodic solution with amplitude and period
    and written as :
```

$$X_1 := (t,\tau,\sigma)\rightarrow -\frac{1}{4}\sin(3t-3\Phi(\sigma)) \tag{18}$$

```
> collect(combine(deqn1, trig), [sin, cos]); # we show that the linearised solutions X_0
    and X_1 solves equation (6)
```

$$0 \tag{19}$$

> deqn2temp := map(simplify, collect(combine(deqn2, trig), [sin, cos]));
we substitute the values of X_0 **and** X_1 **in** equation (7) **and** obtain :

$$deqn2temp := \frac{1}{4}\left(-1+16\,\mathrm{D}(\Phi)(\sigma)\right)\cos(t-\Phi(\sigma)) - \frac{5}{4}\cos(5\,t-5\,\Phi(\sigma)) + \mathrm{D}_{1,\,1}(X_2)(t,\tau,\sigma) + X_2(t,\tau,\sigma) - \frac{3}{4}\cos(3\,t-3\,\Phi(\sigma)) \quad \textbf{(20)}$$

> secular_terms := {coeff(deqn2temp, sin(t-Phi(sigma))), coeff(deqn2temp, cos(t-Phi(sigma)))}; # we collect secular terms as we did before

$$secular\_terms := \left\{0, -\frac{1}{4}+4\,\mathrm{D}(\Phi)(\sigma)\right\} \quad \textbf{(21)}$$

```
> secular_terms:= 4*diff(Phi(sigma), sigma) = 1/4;
```

$$secular\_terms := 4\left(\frac{\mathrm{d}}{\mathrm{d}\sigma}\,\Phi(\sigma)\right) = \frac{1}{4} \quad \textbf{(22)}$$

> dsolve(secular_terms); # we solve for *Φ* in equation (22)

$$\Phi(\sigma) = \frac{1}{16}\,\sigma + \_C1 \quad \textbf{(23)}$$

> Phi := sigma → $\frac{1}{16}$ σ + phi; # we write *Φ* as a function of σ and ϕ

$$\Phi := \sigma \rightarrow \frac{1}{16}\,\sigma + \phi \quad \textbf{(24)}$$

> X[0](t, tau, sigma);

$$2\cos\left(-t+\frac{1}{16}\,\sigma+\phi\right) \quad \textbf{(25)}$$

> X[1](t, tau, sigma);

$$\frac{1}{4}\sin\left(-3\,t+\frac{3}{16}\,\sigma+3\,\phi\right) \quad \textbf{(26)}$$

> deqn2temp := map(factor, map(simplify, collect(combine(deqn2, trig), [sin, cos])));
substituting for *Φ* in equation (20) above, we have:

$$deqn2temp := \mathrm{D}_{1,\,1}(X_2)(t,\tau,\sigma) + X_2(t,\tau,\sigma) - \frac{5}{4}\cos\left(-5\,t+\frac{5}{16}\,\sigma+5\,\phi\right) - \frac{3}{4}\cos\left(-3\,t+\frac{3}{16}\,\sigma+3\,\phi\right) \quad \textbf{(27)}$$

> dsolve(deqn2temp, X[2](t, tau, sigma)); # solving equation (27) for the linearized solution X_2, we have :

$$X_2(t,\tau,\sigma) = \sin(t)\,\_F2(\tau,\sigma) + \cos(t)\,\_F1(\tau,\sigma) - \frac{5}{96}\cos\left(-5\,t+\frac{5}{16}\,\sigma+5\,\phi\right) - \frac{3}{32}\cos\left(-3\,t+\frac{3}{16}\,\sigma+3\,\phi\right) \quad \textbf{(28)}$$

> X[2] := (t, tau, sigma) → $-\frac{5}{96}\cos\left(-5\,t+\frac{5}{16}\,\sigma+5\,\phi\right) - \frac{3}{32}\cos\left(-3\,t+\frac{3}{16}\,\sigma+3\,\phi\right)$;
the third linearized X_2 solution is a periodic solution with amplitude **and** period **and** written as

$$X_2 := (t,\tau,\sigma) \rightarrow -\frac{5}{96}\cos\left(-5\,t+\frac{5}{16}\,\sigma+5\,\phi\right) - \frac{3}{32}\cos\left(-3\,t+\frac{3}{16}\,\sigma+3\,\phi\right) \quad \textbf{(29)}$$

```
> collect(combine(deqn2, trig), [sin, cos]); # we show that substituting for X_2
    in equation (27) solves that equation
```

$$0 \tag{30}$$

```
> xansatz(t); # hence the solution or xansatz, as a function of t only is:
```

$$2\cos\left(-t+\frac{1}{16}\varepsilon^2 t+\phi\right)+\frac{1}{4}\varepsilon\sin\left(-3t+\frac{3}{16}\varepsilon^2 t+3\phi\right)+\varepsilon^2\left(-\frac{5}{96}\cos\left(-5t+\frac{5}{16}\varepsilon^2 t+5\phi\right)-\frac{3}{32}\cos\left(-3t+\frac{3}{16}\varepsilon^2 t+3\phi\right)\right) \tag{31}$$

```
> foo := dsolve([subs(epsilon = 1/2, deqn), x(0) = 2.0, D(x)(0) = 0], range = 0. ..100, numeric);
    # numerical integration of the van der Pol with initial conditions
```

$$foo := \textbf{proc}(x\_rkf45)\ \ldots\ \textbf{end proc} \tag{32}$$

```
> # tmp:=fsolve(subs(epsilon=1/2,{xansatz(0)=2.0})); # we set
  epsilon=0.5 and solve for phi in equation (31) with initial
  conditions
> tmp := {phi = 0.2}
```

$$tmp := \{\phi = 0.2\} \tag{33}$$

```
> xapprox:=subs(epsilon=1/2,subs(tmp,xansatz(t))); # substituting
  epsilon =0.5 and phi = .1396823540, the xansantz is
```

$$xapprox := 2\cos\left(-\frac{63}{64}t+0.2\right)+\frac{1}{8}\sin\left(-\frac{189}{64}t+0.6\right)-\frac{5}{384}\cos\left(-\frac{315}{64}t+1.0\right)-\frac{3}{128}\cos\left(-\frac{189}{64}t+0.6\right) \tag{34}$$

```
> with(plots): # we call the plot command
> pnum:=odeplot(foo,t=0..50,refine=1,color=red): # plot numerical
  integration of the van der Pol in red
> pexp:=plot(xapprox,t=0..50,color=green): # plot the linearized
  solutions in green
> display(pexp,pnum); # display both graphs
```

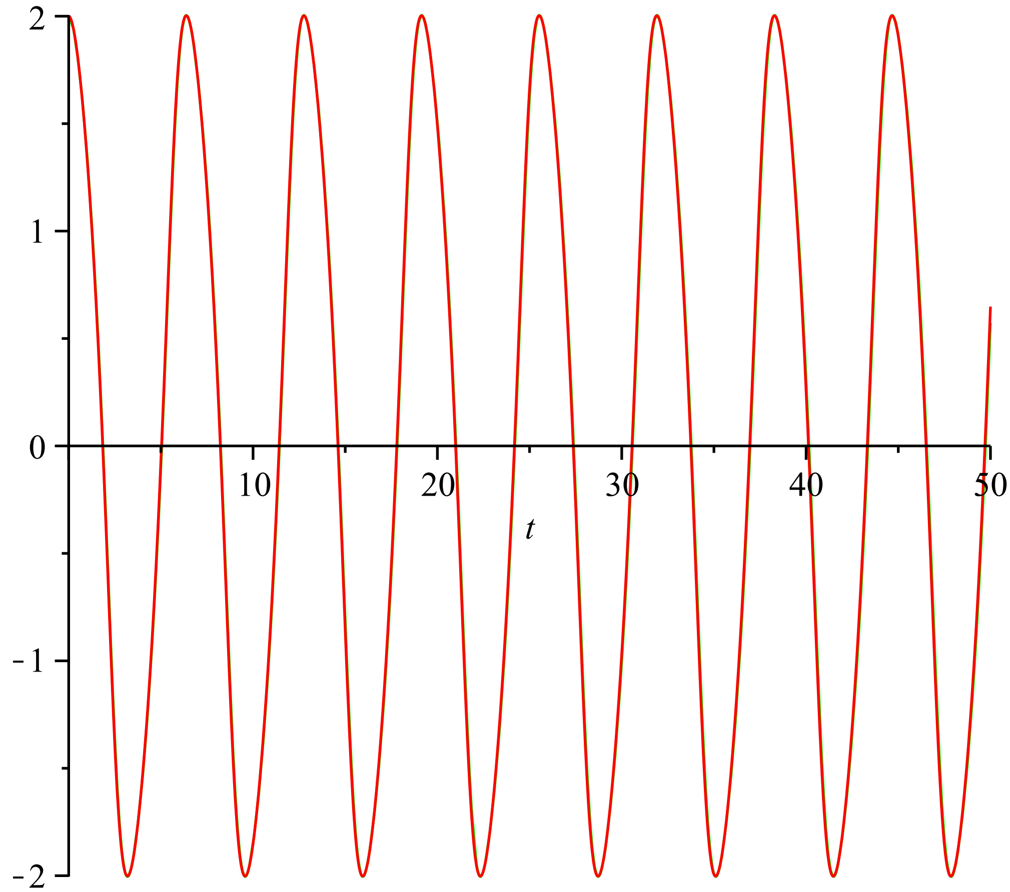

Figure 5.0(a) numerical integration of the van der Pol in red vs approximate solution in green by the 3-timing method for phi = 0.2 and epsilon = 0.5

```
> pnum2:=odeplot(foo,[x(t),D(x)(t)],t=0..10,refine=1,color=red):#
  phase plot of numerical integration of the van der Pol in red
> pexp2:=plot([xapprox,diff(xapprox,t),t=0..10],color=green): #
  phase plot of linearized solutions in green
> display(pnum2,pexp2); # display graphs
```

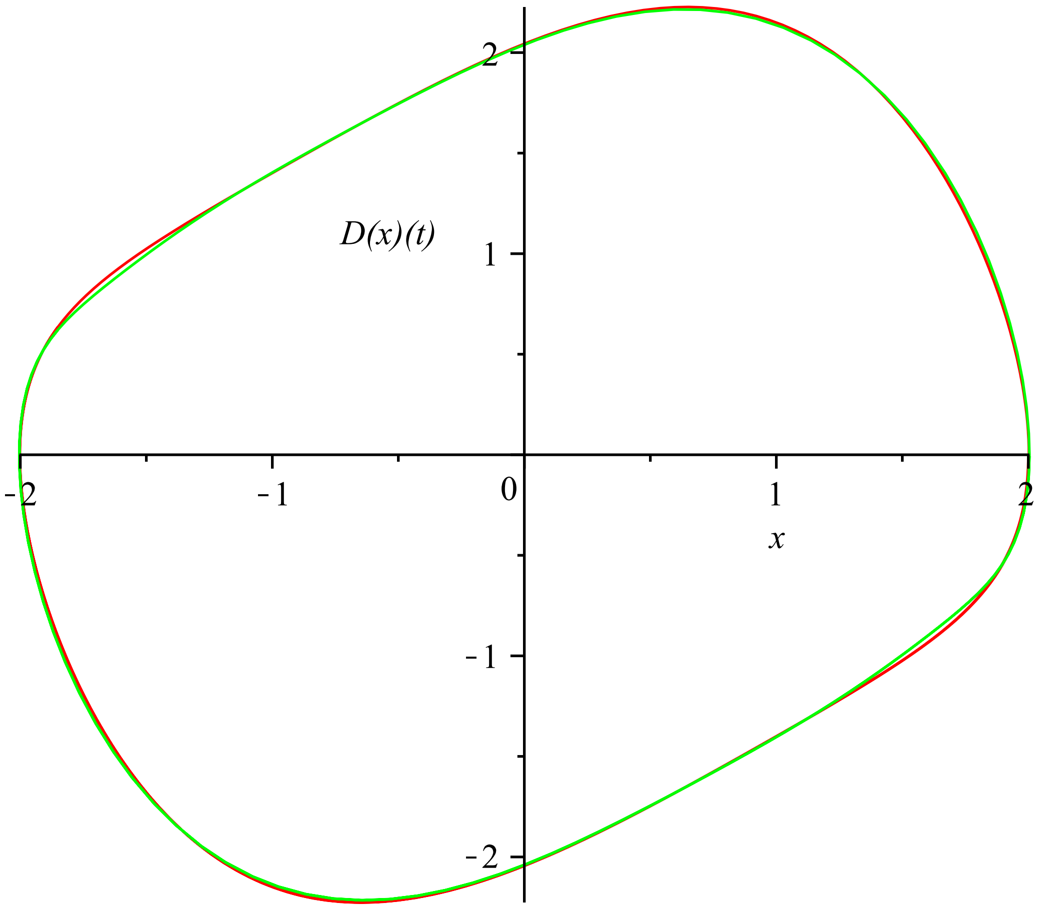

Figure 5.0(b) phase plot of the numerical integration of the van der Pol in red vs approximate solution in green by the 3-timing method for phi = 0.2 and epsilon = 0.5