APPENDIX 3.0 - 7-TIMESCALE

```
> restart :
> n := 6 :
> deqn := phi^2·diff(x(t), t, t) + x(t) +  phi·epsilon·(x(t)^2 - 1)·diff(x(t), t) :
> phi := 1 + sum(ε^j·omega[j], j = 1..n);
```

$$\phi := 1 + \varepsilon\,\omega_1 + \varepsilon^2\,\omega_2 + \varepsilon^3\,\omega_3 + \varepsilon^4\,\omega_4 + \varepsilon^5\,\omega_5 + \varepsilon^6\,\omega_6 \tag{1.1}$$

```
> xansatz := t → (sum(ε^j·X[j](t), j = 0..n)) :
> deqnseries := series(subs( t = T, eval(subs(x = xansatz, deqn))), epsilon, n + 2);
```

$$\begin{aligned}
deqnseries := {} & \frac{d}{dT}\left(\frac{d}{dT}X_0(T)\right) + X_0(T) + \left(X_1(T) + \left(X_0(T)^2 - 1\right)\left(\frac{d}{dT}X_0(T)\right)\right. \\
& \left. + \frac{d}{dT}\left(\frac{d}{dT}X_1(T)\right) + 2\,\omega_1\left(\frac{d}{dT}\left(\frac{d}{dT}X_0(T)\right)\right)\right)\varepsilon + \left(\left(X_0(T)^2\right.\right. \\
& \left. - 1\right)\left(\frac{d}{dT}X_1(T)\right) + \left(2\,X_0(T)\,X_1(T) + \omega_1\left(X_0(T)^2 - 1\right)\right)\left(\frac{d}{dT}X_0(T)\right) \\
& + \frac{d}{dT}\left(\frac{d}{dT}X_2(T)\right) + 2\,\omega_1\left(\frac{d}{dT}\left(\frac{d}{dT}X_1(T)\right)\right) + \left(2\,\omega_2 + \right. \\
& \left.\left. \omega_1^2\right)\left(\frac{d}{dT}\left(\frac{d}{dT}X_0(T)\right)\right) + X_2(T)\right)\varepsilon^2 + \left(X_3(T) + \left(X_0(T)^2 - 1\right)\left(\frac{d}{dT}X_2(T)\right)\right. \\
& + \left(2\,X_0(T)\,X_1(T) + \omega_1\left(X_0(T)^2 - 1\right)\right)\left(\frac{d}{dT}X_1(T)\right) + \left(2\,X_0(T)\,X_2(T) + X_1(T)^2\right. \\
& \left. + \omega_2\left(X_0(T)^2 - 1\right) + 2\,\omega_1\,X_0(T)\,X_1(T)\right)\left(\frac{d}{dT}X_0(T)\right) \\
& + 2\,\omega_1\left(\frac{d}{dT}\left(\frac{d}{dT}X_2(T)\right)\right) + \frac{d}{dT}\left(\frac{d}{dT}X_3(T)\right) + \left(2\,\omega_1\,\omega_2\right. \\
& \left.\left. + 2\,\omega_3\right)\left(\frac{d}{dT}\left(\frac{d}{dT}X_0(T)\right)\right) + \left(2\,\omega_2 + \omega_1^2\right)\left(\frac{d}{dT}\left(\frac{d}{dT}X_1(T)\right)\right)\right)\varepsilon^3 \\
& + \left(\left(2\,X_0(T)\,X_1(T) + \omega_1\left(X_0(T)^2 - 1\right)\right)\left(\frac{d}{dT}X_2(T)\right) + \left(X_0(T)^2\right.\right. \\
& \left. - 1\right)\left(\frac{d}{dT}X_3(T)\right) + \left(\omega_1\left(2\,X_0(T)\,X_2(T) + X_1(T)^2\right) + 2\,X_1(T)\,X_2(T)\right. \\
& \left. + 2\,X_0(T)\,X_3(T) + 2\,\omega_2\,X_0(T)\,X_1(T) + \omega_3\left(X_0(T)^2 - 1\right)\right)\left(\frac{d}{dT}X_0(T)\right) \\
& + \left(2\,X_0(T)\,X_2(T) + X_1(T)^2 + \omega_2\left(X_0(T)^2 - 1\right) + 2\,\omega_1\,X_0(T)\,X_1(T)\right)\left(\frac{d}{dT}X_1(T)\right) \\
& + X_4(T) + \frac{d}{dT}\left(\frac{d}{dT}X_4(T)\right) + 2\,\omega_1\left(\frac{d}{dT}\left(\frac{d}{dT}X_3(T)\right)\right) + \left(2\,\omega_2 + \right. \\
& \left.\omega_1^2\right)\left(\frac{d}{dT}\left(\frac{d}{dT}X_2(T)\right)\right) + \left(2\,\omega_1\,\omega_2 + 2\,\omega_3\right)\left(\frac{d}{dT}\left(\frac{d}{dT}X_1(T)\right)\right) + \left(2\,\omega_4\right. \\
& \left.\left. + 2\,\omega_1\,\omega_3 + \omega_2^2\right)\left(\frac{d}{dT}\left(\frac{d}{dT}X_0(T)\right)\right)\right)\varepsilon^4 + \left(\left(\omega_1\left(2\,X_0(T)\,X_2(T) + X_1(T)^2\right)\right.\right.
\end{aligned} \tag{1.2}$$

$$+2 X_1(T) X_2(T) + 2 X_0(T) X_3(T) + 2 \omega_2 X_0(T) X_1(T) + \omega_3 \left(X_0(T)^2 - 1\right)\Big)$$

$$\left(\frac{d}{dT} X_1(T)\right) + \left(2 X_0(T) X_1(T) + \omega_1 \left(X_0(T)^2 - 1\right)\right) \left(\frac{d}{dT} X_3(T)\right) + \Big(X_0(T)^2$$

$$-1\Big) \left(\frac{d}{dT} X_4(T)\right) + \Big(2 \omega_3 X_0(T) X_1(T) + \omega_1 \left(2 X_1(T) X_2(T) + 2 X_0(T) X_3(T)\right)$$

$$+2 X_0(T) X_4(T) + 2 X_1(T) X_3(T) + X_2(T)^2 + \omega_4 \left(X_0(T)^2 - 1\right)$$

$$+\omega_2 \left(2 X_0(T) X_2(T) + X_1(T)^2\right)\Big) \left(\frac{d}{dT} X_0(T)\right) + \Big(2 X_0(T) X_2(T) + X_1(T)^2$$

$$+\omega_2 \left(X_0(T)^2 - 1\right) + 2 \omega_1 X_0(T) X_1(T)\Big) \left(\frac{d}{dT} X_2(T)\right) + X_5(T)$$

$$+\frac{d}{dT}\left(\frac{d}{dT} X_5(T)\right) + \left(2 \omega_4 + 2 \omega_1 \omega_3 + \omega_2^2\right) \left(\frac{d}{dT}\left(\frac{d}{dT} X_1(T)\right)\right) + \Big(2 \omega_2 +$$

$$\omega_1^2\Big) \left(\frac{d}{dT}\left(\frac{d}{dT} X_3(T)\right)\right) + \left(2 \omega_5 + 2 \omega_4 \omega_1 + 2 \omega_2 \omega_3\right) \left(\frac{d}{dT}\left(\frac{d}{dT} X_0(T)\right)\right)$$

$$+2 \omega_1 \left(\frac{d}{dT}\left(\frac{d}{dT} X_4(T)\right)\right) + \left(2 \omega_1 \omega_2 + 2 \omega_3\right) \left(\frac{d}{dT}\left(\frac{d}{dT} X_2(T)\right)\right)\Bigg) \varepsilon^5$$

$$+\Bigg(X_6(T) + \left(X_0(T)^2 - 1\right) \left(\frac{d}{dT} X_5(T)\right) + \Big(\omega_1 \left(2 X_0(T) X_2(T) + X_1(T)^2\right)$$

$$+2 X_1(T) X_2(T) + 2 X_0(T) X_3(T) + 2 \omega_2 X_0(T) X_1(T) + \omega_3 \left(X_0(T)^2 - 1\right)\Big)$$

$$\left(\frac{d}{dT} X_2(T)\right) + \left(2 X_0(T) X_1(T) + \omega_1 \left(X_0(T)^2 - 1\right)\right) \left(\frac{d}{dT} X_4(T)\right)$$

$$+\Big(2 \omega_3 X_0(T) X_1(T) + \omega_1 \left(2 X_1(T) X_2(T) + 2 X_0(T) X_3(T)\right) + 2 X_0(T) X_4(T)$$

$$+2 X_1(T) X_3(T) + X_2(T)^2 + \omega_4 \left(X_0(T)^2 - 1\right) + \omega_2 \left(2 X_0(T) X_2(T) + X_1(T)^2\right)\Big)$$

$$\left(\frac{d}{dT} X_1(T)\right) + \Big(2 X_0(T) X_2(T) + X_1(T)^2 + \omega_2 \left(X_0(T)^2 - 1\right)$$

$$+2 \omega_1 X_0(T) X_1(T)\Big) \left(\frac{d}{dT} X_3(T)\right) + \Big(2 \omega_4 X_0(T) X_1(T) + \omega_3 \Big(2 X_0(T) X_2(T)$$

$$+X_1(T)^2\Big) + \omega_1 \left(2 X_0(T) X_4(T) + 2 X_1(T) X_3(T) + X_2(T)^2\right) + \omega_2 \Big(2 X_1(T) X_2(T)$$

$$+2 X_0(T) X_3(T)\Big) + 2 X_0(T) X_5(T) + 2 X_4(T) X_1(T) + 2 X_2(T) X_3(T) + \omega_5 \Big(X_0(T)^2$$

$$-1\Big)\Big) \left(\frac{d}{dT} X_0(T)\right) + 2 \omega_1 \left(\frac{d}{dT}\left(\frac{d}{dT} X_5(T)\right)\right) + \Big(2 \omega_4 + 2 \omega_1 \omega_3 +$$

$$\omega_2^2\Big) \left(\frac{d}{dT}\left(\frac{d}{dT} X_2(T)\right)\right) + \left(2 \omega_2 + \omega_1^2\right) \left(\frac{d}{dT}\left(\frac{d}{dT} X_4(T)\right)\right) + \Big(2 \omega_5 + 2 \omega_4 \omega_1$$

$$+2 \omega_2 \omega_3\Big) \left(\frac{d}{dT}\left(\frac{d}{dT} X_1(T)\right)\right) + \frac{d}{dT}\left(\frac{d}{dT} X_6(T)\right) + \Big(2 \omega_1 \omega_2$$

$$+2 \omega_3\Big) \left(\frac{d}{dT}\left(\frac{d}{dT} X_3(T)\right)\right) + \Big(2 \omega_1 \omega_5 + 2 \omega_4 \omega_2 + 2 \omega_6 +$$

$$\omega_3^2\Big)\left(\frac{d}{dT}\left(\frac{d}{dT}X_0(T)\right)\right)\varepsilon^6+\Bigg(2\,\omega_1\left(\frac{d^2}{dT^2}X_6(T)\right)+\big(2\,\omega_1\,\omega_6+2\,\omega_4\,\omega_3$$

$$+2\,\omega_5\,\omega_2\big)\left(\frac{d^2}{dT^2}X_0(T)\right)+\big(2\,\omega_1\,\omega_2+2\,\omega_3\big)\left(\frac{d^2}{dT^2}X_4(T)\right)+\big(2\,\omega_4+2\,\omega_1\,\omega_3+$$

$$\omega_2^2\big)\left(\frac{d^2}{dT^2}X_3(T)\right)+\big(2\,\omega_1\,\omega_5+2\,\omega_4\,\omega_2+2\,\omega_6+\omega_3^2\big)\left(\frac{d^2}{dT^2}X_1(T)\right)+\big(2\,\omega_5$$

$$+2\,\omega_4\,\omega_1+2\,\omega_2\,\omega_3\big)\left(\frac{d^2}{dT^2}X_2(T)\right)+\big(2\,\omega_2+\omega_1^2\big)\left(\frac{d^2}{dT^2}X_5(T)\right)+\big(X_0(T)^2$$

$$-1\big)\left(\frac{d}{dT}X_6(T)\right)+\Big(\omega_6\,\big(X_0(T)^2-1\big)+\omega_3\,\big(2\,X_1(T)\,X_2(T)+2\,X_0(T)\,X_3(T)\big)$$

$$+\omega_2\,\big(2\,X_0(T)\,X_4(T)+2\,X_1(T)\,X_3(T)+X_2(T)^2\big)+\omega_4\,\big(2\,X_0(T)\,X_2(T)+X_1(T)^2\big)$$

$$+2\,X_1(T)\,X_5(T)+2\,X_4(T)\,X_2(T)+2\,X_0(T)\,X_6(T)+X_3(T)^2+2\,\omega_5\,X_0(T)\,X_1(T)$$

$$+\omega_1\,\big(2\,X_0(T)\,X_5(T)+2\,X_4(T)\,X_1(T)+2\,X_2(T)\,X_3(T)\big)\Big)\left(\frac{d}{dT}X_0(T)\right)$$

$$+\Big(2\,X_0(T)\,X_2(T)+X_1(T)^2+\omega_2\,\big(X_0(T)^2-1\big)+2\,\omega_1\,X_0(T)\,X_1(T)\Big)\left(\frac{d}{dT}X_4(T)\right)$$

$$+\Big(\omega_1\,\big(2\,X_0(T)\,X_2(T)+X_1(T)^2\big)+2\,X_1(T)\,X_2(T)+2\,X_0(T)\,X_3(T)$$

$$+2\,\omega_2\,X_0(T)\,X_1(T)+\omega_3\,\big(X_0(T)^2-1\big)\Big)\left(\frac{d}{dT}X_3(T)\right)+\Big(2\,\omega_4\,X_0(T)\,X_1(T)$$

$$+\omega_3\,\big(2\,X_0(T)\,X_2(T)+X_1(T)^2\big)+\omega_1\,\big(2\,X_0(T)\,X_4(T)+2\,X_1(T)\,X_3(T)+X_2(T)^2\big)$$

$$+\omega_2\,\big(2\,X_1(T)\,X_2(T)+2\,X_0(T)\,X_3(T)\big)+2\,X_0(T)\,X_5(T)+2\,X_4(T)\,X_1(T)$$

$$+2\,X_2(T)\,X_3(T)+\omega_5\,\big(X_0(T)^2-1\big)\Big)\left(\frac{d}{dT}X_1(T)\right)+\Big(2\,\omega_3\,X_0(T)\,X_1(T)$$

$$+\omega_1\,\big(2\,X_1(T)\,X_2(T)+2\,X_0(T)\,X_3(T)\big)+2\,X_0(T)\,X_4(T)+2\,X_1(T)\,X_3(T)+X_2(T)^2$$

$$+\omega_4\,\big(X_0(T)^2-1\big)+\omega_2\,\big(2\,X_0(T)\,X_2(T)+X_1(T)^2\big)\Big)\left(\frac{d}{dT}X_2(T)\right)$$

$$+\Big(2\,X_0(T)\,X_1(T)+\omega_1\,\big(X_0(T)^2-1\big)\Big)\left(\frac{d}{dT}X_5(T)\right)\Bigg)\varepsilon^7+\mathrm{O}\big(\varepsilon^8\big)$$

> *deqn0* := *coeff*(*deqnseries*, epsilon, 0);

$$deqn0 := \frac{d^2}{dT^2}X_0(T)+X_0(T) \qquad \textbf{(1.3)}$$

> *deqn1* := *coeff*(*deqnseries*, epsilon, 1) :

> *deqn2* := *coeff*(*deqnseries*, epsilon, 2) :

> *deqn3* := *coeff*(*deqnseries*, epsilon, 3) :

> *deqn4* := *coeff*(*deqnseries*, epsilon, 4) :

> *deqn5* := *coeff*(*deqnseries*, epsilon, 5) :

> $deqn6 := coeff(deqnseries, \text{epsilon}, 6):$

> $deqn7 := coeff(deqnseries, \text{epsilon}, 7):$

> $deqn0;$

$$\frac{d^2}{dT^2} X_0(T) + X_0(T) \tag{1.4}$$

> $dsolve(deqn0, X[0](\mathrm{T})):$

> $X[0] := (\mathrm{T}) \rightarrow \_C2 \cos(T);$

$$X_0 := T \rightarrow \_C2 \cos(T) \tag{1.5}$$

> $deqn0;$

$$0 \tag{1.6}$$

> $deqn1temp := collect(combine(deqn1, trig), [\sin, \cos]):$

> $secular\_terms := \{coeff(deqn1temp, \sin(\mathrm{T})), coeff(deqn1temp, \cos(\mathrm{T}))\}:$

> $solve(secular\_terms, \{\_C2, \text{omega}[1]\});$

$$\{\_C2 = 0, \omega_1 = \omega_1\}, \{\_C2 = 2, \omega_1 = 0\}, \{\_C2 = -2, \omega_1 = 0\} \tag{1.7}$$

> $\_C2 := 2;$

$$\_C2 := 2 \tag{1.8}$$

> $\omega_1 := 0;$

$$\omega_1 := 0 \tag{1.9}$$

> $deqn1temp := map(simplify, collect(combine(deqn1, trig), [\sin, \cos])):$

> $dsolve(deqn1temp, X[1](\mathrm{T}));$

$$X_1(T) = \sin(T)\ \_C3 + \cos(T)\ \_C1 - \frac{1}{4} \sin(3\ T) \tag{1.10}$$

> $X_1 := (\mathrm{T}) \rightarrow \cos(T)\ \_C1 - \frac{1}{4} \sin(3\ T);$

$$X_1 := T \rightarrow \cos(T)\ \_C1 - \frac{1}{4} \sin(3\ T) \tag{1.11}$$

> $deqn2temp := map(simplify, collect(combine(deqn2, trig), [\sin, \cos])):$

> $secular\_terms := \{coeff(deqn2temp, \sin(\mathrm{T})), coeff(deqn2temp, \cos(\mathrm{T}))\};$

$$secular\_terms := \left\{-2\ \_C1, -\frac{1}{4} - 4\ \omega_2\right\} \tag{1.12}$$

> $solve(secular\_terms, \{\_C1, \text{omega}[2]\});$

$$\left\{\_C1 = 0, \omega_2 = -\frac{1}{16}\right\} \tag{1.13}$$

> $\_C1 := 0;$

$$\_C1 := 0 \tag{1.14}$$

> $\omega_2 := -\frac{1}{16};$

$$\omega_2 := -\frac{1}{16} \tag{1.15}$$

> $deqn2temp := map(factor, map(simplify, collect(combine(deqn2, trig), [\sin, \cos]))):$

> $odesol[2] := dsolve(deqn2temp, X[2](\mathrm{T})):$

> $sol[2] := collect(combine(odesol[2], trig), [\sin, \cos]):$

> $X[2] := (T) \rightarrow C3\cos(T) - \frac{3}{32}\cos(3\,T) - \frac{5}{96}\cos(5\,T);$

$$X_2 := T \rightarrow C3\cos(T) - \frac{3}{32}\cos(3\,T) - \frac{5}{96}\cos(5\,T) \quad \textbf{(1.16)}$$

> $deqn3temp := map(factor, map(simplify, collect(combine(deqn3, trig), [\sin, \cos])))$:

> $secular\_terms := \{coeff(deqn3temp, \sin(T)), coeff(deqn3temp, \cos(T))\}$:

> $solve(secular\_terms, \{C3, \text{omega}[3]\})$:

> $C3 := \frac{1}{64}$;

$$C3 := \frac{1}{64} \quad \textbf{(1.17)}$$

> $\omega_3 := 0;$

$$\omega_3 := 0 \quad \textbf{(1.18)}$$

> $combine(deqn2)$:

> $deqn3temp := map(factor, map(simplify, collect(combine(deqn3, trig), [\sin, \cos])))$:

> $dsolve(deqn3temp, X[3](T));$

$$X_3(T) = \sin(T)\,\_C4 + \_C3\cos(T) + \frac{85}{2304}\sin(5\,T) + \frac{15}{512}\sin(3\,T) + \frac{7}{576}\sin(7\,T) \quad \textbf{(1.19)}$$

> $X_3 := (T) \rightarrow \_C3\cos(T) + \frac{1}{4608}\,(170\sin(5\,T) + 135\sin(3\,T) + 56\sin(7\,T));$

$$X_3 := T \rightarrow \_C3\cos(T) + \frac{85}{2304}\sin(5\,T) + \frac{15}{512}\sin(3\,T) + \frac{7}{576}\sin(7\,T) \quad \textbf{(1.20)}$$

> $deqn4temp := map(factor, map(simplify, collect(combine(deqn4, trig), [\sin, \cos])))$:

> $secular\_terms := \{coeff(deqn4temp, \sin(T)), coeff(deqn4temp, \cos(T))\};$

$$secular\_terms := \left\{-2\,\_C3, \frac{17}{768} - 4\,\omega_4\right\} \quad \textbf{(1.21)}$$

> $solve(secular\_terms, \{\_C3, \text{omega}[4]\});$

$$\left\{\_C3 = 0, \omega_4 = \frac{17}{3072}\right\} \quad \textbf{(1.22)}$$

> $\omega_4 := \frac{17}{3072}$;

$$\omega_4 := \frac{17}{3072} \quad \textbf{(1.23)}$$

> $\_C3 := 0$

$$\_C3 := 0 \quad \textbf{(1.24)}$$

> $combine(deqn3);$

$$0 \quad \textbf{(1.25)}$$

> $deqn4temp := map(factor, map(simplify, collect(combine(deqn4, trig), [\sin, \cos])));$

$$deqn4temp := X_4(T) + \frac{d^2}{dT^2}X_4(T) + \frac{1379}{2304}\cos(7\,T) + \frac{101}{1536}\cos(3\,T) + \frac{61}{256}\cos(9\,T) + \frac{1865}{4608}\cos(5\,T) \quad \textbf{(1.26)}$$

> $odesol[4] := dsolve(deqn4temp, X[4](T));$

$$odesol_4 := X_4(T) = \sin(T)\ \_C5 + \_C4 \cos(T) + \frac{7}{192}\cos(T)^3 + \frac{1843}{11520}\cos(T)^5 - \frac{991}{1080}\cos(T)^7 + \frac{61}{80}\cos(T)^9 \quad \textbf{(1.27)}$$

> *sol*[4] := *collect*(*combine*(*odesol*[4], *trig*), [sin, cos]);

$$sol_4 := X_4(T) = \sin(T)\ \_C5 + \left(\_C4 + \frac{113}{138240}\right)\cos(T) + \frac{1865}{110592}\cos(5\,T) + \frac{61}{20480}\cos(9\,T) + \frac{101}{12288}\cos(3\,T) + \frac{1379}{110592}\cos(7\,T) \quad \textbf{(1.28)}$$

> $X[4] := (T) \rightarrow \left(\_C4 + \frac{113}{138240}\right)\cos(T) + \frac{101}{12288}\cos(3\,T) + \frac{1379}{110592}\cos(7\,T) + \frac{61}{20480}\cos(9\,T) + \frac{1865}{110592}\cos(5\,T);$

$$X_4 := T \rightarrow \left(\_C4 + \frac{113}{138240}\right)\cos(T) + \frac{101}{12288}\cos(3\,T) + \frac{1379}{110592}\cos(7\,T) + \frac{61}{20480}\cos(9\,T) + \frac{1865}{110592}\cos(5\,T) \quad \textbf{(1.29)}$$

> *deqn5temp* := *map*(*factor*, *map*(*simplify*, *collect*(*combine*(*deqn5*, *trig*), [sin, cos])));

$$deqn5temp := -\frac{1}{1105920}(2211840\ \_C4 + 2843)\sin(T) - \frac{1}{122880}(368640\ \_C4 + 1123)\sin(3\,T) + X_5(T) + \frac{d^2}{dT^2}X_5(T) - \frac{99967}{276480}\sin(7\,T) - \frac{9791}{30720}\sin(9\,T) - \frac{8095}{55296}\sin(5\,T) - \frac{5533}{61440}\sin(11\,T) - 4\,\omega_5\cos(T) \quad \textbf{(1.30)}$$

> *secular_terms* := {*coeff*(*deqn5temp*, sin(T)), *coeff*(*deqn5temp*, cos(T))};

$$secular\_terms := \left\{-4\,\omega_5, -2\ \_C4 - \frac{2843}{1105920}\right\} \quad \textbf{(1.31)}$$

> *solve*(*secular_terms*, {_*C4*, omega[5]});

$$\left\{\_C4 = -\frac{2843}{2211840}, \omega_5 = 0\right\} \quad \textbf{(1.32)}$$

> $\omega_5 := 0;$

$$\omega_5 := 0 \quad \textbf{(1.33)}$$

> $\_C4 := -\frac{2843}{2211840};$

$$\_C4 := -\frac{2843}{2211840} \quad \textbf{(1.34)}$$

> *X*[4](*T*);

$$-\frac{23}{49152}\cos(T) + \frac{101}{12288}\cos(3\,T) + \frac{1379}{110592}\cos(7\,T) + \frac{61}{20480}\cos(9\,T) + \frac{1865}{110592}\cos(5\,T) \quad \textbf{(1.35)}$$

> *combine*(*deqn4*);

$$0 \quad \textbf{(1.36)}$$

> $deqn5temp := map(factor, map(simplify, collect(combine(deqn5, trig), [\sin, \cos])));$

$$deqn5temp := X_5(T) + \frac{d^2}{dT^2} X_5(T) - \frac{779}{147456} \sin(3\,T) - \frac{8095}{55296} \sin(5\,T) - \frac{99967}{276480} \sin(7\,T) - \frac{9791}{30720} \sin(9\,T) - \frac{5533}{61440} \sin(11\,T) \tag{1.37}$$

> $dsolve(deqn5temp, X[5](T));$

$$X_5(T) = \sin(T)\ \_C6 + \cos(T)\ \_C5 - \frac{8095}{1327104} \sin(5\,T) - \frac{779}{1179648} \sin(3\,T) - \frac{99967}{13271040} \sin(7\,T) - \frac{9791}{2457600} \sin(9\,T) - \frac{5533}{7372800} \sin(11\,T) \tag{1.38}$$

> $X_5 := T \rightarrow \cos(T)\ \_C5 - \frac{1}{265420800} (1619000 \sin(5\,T) + 175275 \sin(3\,T) + 199188 \sin(11\,T) + 1057428 \sin(9\,T) + 1999340 \sin(7\,T));$

$$X_5 := T \rightarrow \cos(T)\ \_C5 - \frac{8095}{1327104} \sin(5\,T) - \frac{779}{1179648} \sin(3\,T) - \frac{5533}{7372800} \sin(11\,T) - \frac{9791}{2457600} \sin(9\,T) - \frac{99967}{13271040} \sin(7\,T) \tag{1.39}$$

> $deqn6temp := map(factor, map(simplify, collect(combine(deqn6, trig), [\sin, \cos])));$

$$deqn6temp := -2 \sin(T)\ \_C5 - 3\ \_C5 \sin(3\,T) - \frac{1}{221184} \left(-35 + 884736\, \omega_6\right) \cos(T) + X_6(T) + \frac{d^2}{dT^2} X_6(T) - \frac{715247}{22118400} \cos(13\,T) - \frac{10923199}{66355200} \cos(7\,T) - \frac{1769369}{7372800} \cos(9\,T) + \frac{24061}{3538944} \cos(3\,T) - \frac{328835}{10616832} \cos(5\,T) - \frac{409871}{2764800} \cos(11\,T) \tag{1.40}$$

> $secular\_terms := \{coeff(deqn6temp, \sin(T)), coeff(deqn6temp, \cos(T))\};$

$$secular\_terms := \left\{-2\ \_C5, \frac{35}{221184} - 4\,\omega_6\right\} \tag{1.41}$$

> $solve(secular\_terms, \{\_C5, omega[6]\});$

$$\left\{\_C5 = 0, \omega_6 = \frac{35}{884736}\right\} \tag{1.42}$$

> $\omega_6 := \frac{35}{884736};$

$$\omega_6 := \frac{35}{884736} \tag{1.43}$$

> $\_C5 := 0$

$$\_C5 := 0 \tag{1.44}$$

> $combine(deqn5);$

$$0 \tag{1.45}$$

> $deqn6temp := map(factor, map(simplify, collect(combine(deqn6, trig), [\sin, \cos])));$

$$deqn6temp := X_6(T) + \frac{d^2}{dT^2} X_6(T) - \frac{715247}{22118400} \cos(13\,T) - \frac{10923199}{66355200} \cos(7\,T) \tag{1.46}$$

$$-\frac{1769369}{7372800}\cos(9T)+\frac{24061}{3538944}\cos(3T)-\frac{328835}{10616832}\cos(5T)-\frac{409871}{2764800}\cos(11T)$$

> odesol[6] := dsolve(deqn6temp, X[6](T));

$$odesol_6 := X_6(T)=\sin(T)\ \_C7+\_C6\cos(T)-\frac{715247}{907200}\cos(T)^{13}+\frac{7846093}{6048000}\cos(T)^{11}-\frac{20281}{4423680}\cos(T)^3+\frac{3850711}{132710400}\cos(T)^5-\frac{423961}{8709120}\cos(T)^7-\frac{35269}{71680}\cos(T)^9 \quad (1.47)$$

> sol[6] := collect(combine(odesol[6], trig), [sin, cos]);

$$sol_6 := X_6(T)=\sin(T)\ \_C7+\left(\frac{101123153}{111476736000}+\_C6\right)\cos(T)-\frac{328835}{254803968}\cos(5T)-\frac{715247}{3715891200}\cos(13T)-\frac{1769369}{589824000}\cos(9T)+\frac{24061}{28311552}\cos(3T)-\frac{409871}{331776000}\cos(11T)-\frac{10923199}{3185049600}\cos(7T) \quad (1.48)$$

> phi;

$$1-\frac{1}{16}\varepsilon^2+\frac{17}{3072}\varepsilon^4+\frac{35}{884736}\varepsilon^6 \quad (1.49)$$

> $X_6 := (T) \to \left(\_C6+\frac{101123153}{111476736000}\right)\cos(T)-\frac{328835}{254803968}\cos(5T)-\frac{1769369}{589824000}\cos(9T)-\frac{10923199}{3185049600}\cos(7T)-\frac{409871}{331776000}\cos(11T)-\frac{715247}{3715891200}\cos(13T)+\frac{24061}{28311552}\cos(3T);$

$$X_6 := T\to\left(\frac{101123153}{111476736000}+\_C6\right)\cos(T)-\frac{328835}{254803968}\cos(5T)-\frac{1769369}{589824000}\cos(9T)-\frac{10923199}{3185049600}\cos(7T)-\frac{409871}{331776000}\cos(11T)-\frac{715247}{3715891200}\cos(13T)+\frac{24061}{28311552}\cos(3T) \quad (1.50)$$

> deqn7temp := map(factor, map(simplify, collect(combine(deqn7, trig), [sin, cos])));

$$deqn7temp := -\frac{1}{222953472000}(539992487+445906944000\ \_C6)\sin(T)-\frac{1}{24772608000}(222357577+74317824000\ \_C6)\sin(3T)+\frac{21731177}{344064000}\sin(13T)+\frac{415949513}{7962624000}\sin(7T)+\frac{117258703}{884736000}\sin(9T)-\frac{1252495}{254803968}\sin(5T)+\frac{1657839733}{12386304000}\sin(11T)+\frac{138697}{12386304}\sin(15T) \quad (1.51)$$

> secular_terms := {coeff(deqn7temp, sin(T)), coeff(deqn7temp, cos(T))};

$$secular\_terms := \left\{0,-\frac{539992487}{222953472000}-2\ \_C6\right\} \quad (1.52)$$

> solve(secular_terms, {_C6, omega[7]});

$$\left\{\_C6 = -\frac{539992487}{445906944000}, \omega_7 = \omega_7\right\} \tag{1.53}$$

> $\_C6 := -\frac{539992487}{445906944000};$

$$\_C6 := -\frac{539992487}{445906944000} \tag{1.54}$$

> $combine(deqn6);$

$$0 \tag{1.55}$$

> $T := t\cdot\text{phi};$

$$T := t\left(1 - \frac{1}{16}\varepsilon^2 + \frac{17}{3072}\varepsilon^4 + \frac{35}{884736}\varepsilon^6\right) \tag{1.56}$$

> $xansatz(T);$

$$\begin{aligned}
&2\cos\left(t\left(1-\frac{1}{16}\varepsilon^2+\frac{17}{3072}\varepsilon^4+\frac{35}{884736}\varepsilon^6\right)\right)-\frac{1}{4}\varepsilon\sin\left(3t\left(1-\frac{1}{16}\varepsilon^2+\frac{17}{3072}\varepsilon^4\right.\right.\\
&\quad\left.\left.+\frac{35}{884736}\varepsilon^6\right)\right)+\varepsilon^2\left(\frac{1}{64}\cos\left(t\left(1-\frac{1}{16}\varepsilon^2+\frac{17}{3072}\varepsilon^4+\frac{35}{884736}\varepsilon^6\right)\right)\right.\\
&\quad-\frac{3}{32}\cos\left(3t\left(1-\frac{1}{16}\varepsilon^2+\frac{17}{3072}\varepsilon^4+\frac{35}{884736}\varepsilon^6\right)\right)-\frac{5}{96}\cos\left(5t\left(1-\frac{1}{16}\varepsilon^2\right.\right.\\
&\quad\left.\left.\left.+\frac{17}{3072}\varepsilon^4+\frac{35}{884736}\varepsilon^6\right)\right)\right)+\varepsilon^3\left(\frac{85}{2304}\sin\left(5t\left(1-\frac{1}{16}\varepsilon^2+\frac{17}{3072}\varepsilon^4\right.\right.\right.\\
&\quad\left.\left.+\frac{35}{884736}\varepsilon^6\right)\right)+\frac{15}{512}\sin\left(3t\left(1-\frac{1}{16}\varepsilon^2+\frac{17}{3072}\varepsilon^4+\frac{35}{884736}\varepsilon^6\right)\right)\\
&\quad\left.+\frac{7}{576}\sin\left(7t\left(1-\frac{1}{16}\varepsilon^2+\frac{17}{3072}\varepsilon^4+\frac{35}{884736}\varepsilon^6\right)\right)\right)+\varepsilon^4\left(-\frac{23}{49152}\cos\left(t\left(1\right.\right.\right.\\
&\quad\left.\left.-\frac{1}{16}\varepsilon^2+\frac{17}{3072}\varepsilon^4+\frac{35}{884736}\varepsilon^6\right)\right)+\frac{101}{12288}\cos\left(3t\left(1-\frac{1}{16}\varepsilon^2+\frac{17}{3072}\varepsilon^4\right.\right.\\
&\quad\left.\left.+\frac{35}{884736}\varepsilon^6\right)\right)+\frac{1379}{110592}\cos\left(7t\left(1-\frac{1}{16}\varepsilon^2+\frac{17}{3072}\varepsilon^4+\frac{35}{884736}\varepsilon^6\right)\right)\\
&\quad+\frac{61}{20480}\cos\left(9t\left(1-\frac{1}{16}\varepsilon^2+\frac{17}{3072}\varepsilon^4+\frac{35}{884736}\varepsilon^6\right)\right)+\frac{1865}{110592}\cos\left(5t\left(1\right.\right.\\
&\quad\left.\left.\left.-\frac{1}{16}\varepsilon^2+\frac{17}{3072}\varepsilon^4+\frac{35}{884736}\varepsilon^6\right)\right)\right)+\varepsilon^5\left(-\frac{8095}{1327104}\sin\left(5t\left(1-\frac{1}{16}\varepsilon^2\right.\right.\right.\\
&\quad\left.\left.+\frac{17}{3072}\varepsilon^4+\frac{35}{884736}\varepsilon^6\right)\right)-\frac{779}{1179648}\sin\left(3t\left(1-\frac{1}{16}\varepsilon^2+\frac{17}{3072}\varepsilon^4\right.\right.\\
&\quad\left.\left.+\frac{35}{884736}\varepsilon^6\right)\right)-\frac{5533}{7372800}\sin\left(11t\left(1-\frac{1}{16}\varepsilon^2+\frac{17}{3072}\varepsilon^4+\frac{35}{884736}\varepsilon^6\right)\right)\\
&\quad-\frac{9791}{2457600}\sin\left(9t\left(1-\frac{1}{16}\varepsilon^2+\frac{17}{3072}\varepsilon^4+\frac{35}{884736}\varepsilon^6\right)\right)\\
&\quad\left.-\frac{99967}{13271040}\sin\left(7t\left(1-\frac{1}{16}\varepsilon^2+\frac{17}{3072}\varepsilon^4+\frac{35}{884736}\varepsilon^6\right)\right)\right)+\varepsilon^6\left(\right.\\
&\quad-\frac{51619}{169869312}\cos\left(t\left(1-\frac{1}{16}\varepsilon^2+\frac{17}{3072}\varepsilon^4+\frac{35}{884736}\varepsilon^6\right)\right)\\
&\quad-\frac{328835}{254803968}\cos\left(5t\left(1-\frac{1}{16}\varepsilon^2+\frac{17}{3072}\varepsilon^4+\frac{35}{884736}\varepsilon^6\right)\right)
\end{aligned} \tag{1.57}$$

$$- \frac{1769369}{589824000} \cos\left(9\, t\, \left(1 - \frac{1}{16}\, \varepsilon^2 + \frac{17}{3072}\, \varepsilon^4 + \frac{35}{884736}\, \varepsilon^6\right)\right)$$
$$- \frac{10923199}{3185049600} \cos\left(7\, t\, \left(1 - \frac{1}{16}\, \varepsilon^2 + \frac{17}{3072}\, \varepsilon^4 + \frac{35}{884736}\, \varepsilon^6\right)\right)$$
$$- \frac{409871}{331776000} \cos\left(11\, t\, \left(1 - \frac{1}{16}\, \varepsilon^2 + \frac{17}{3072}\, \varepsilon^4 + \frac{35}{884736}\, \varepsilon^6\right)\right)$$
$$- \frac{715247}{3715891200} \cos\left(13\, t\, \left(1 - \frac{1}{16}\, \varepsilon^2 + \frac{17}{3072}\, \varepsilon^4 + \frac{35}{884736}\, \varepsilon^6\right)\right)$$
$$+ \frac{24061}{28311552} \cos\left(3\, t\, \left(1 - \frac{1}{16}\, \varepsilon^2 + \frac{17}{3072}\, \varepsilon^4 + \frac{35}{884736}\, \varepsilon^6\right)\right)\Bigg)$$

> *foo* := *dsolve*([*subs*(epsilon = 0.5, *deqn*), *x*(0) = 2.0, D(*x*)(0) = 0], *range* = 0. ..100, *numeric*);

$$foo := \mathbf{proc}(x\_rkf45)\ \ldots\ \mathbf{end\ proc} \tag{1.58}$$

> *xapprox* := *subs*(epsilon = 0.5 , *xansatz*(*T*));

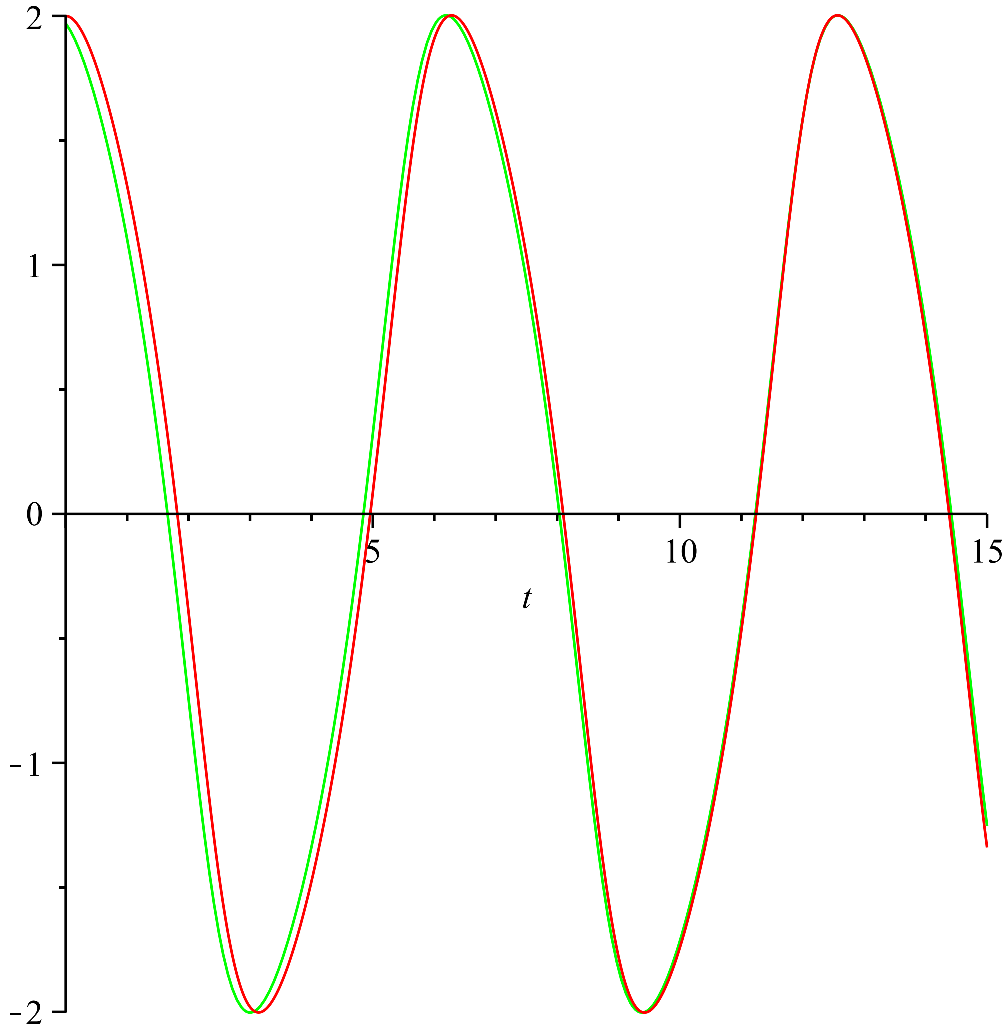
$$\begin{aligned} xapprox := {} & 2.003872256 \cos(0.9847214840\, t) - 0.1213585271 \sin(2.954164452\, t) \\ & - 0.02291050829 \cos(2.954164452\, t) - 0.01198701131 \cos(4.923607420\, t) \\ & + 0.004420928013 \sin(4.923607420\, t) + 0.001283699789 \sin(6.893050388\, t) \\ & + 0.0007257422352 \cos(6.893050388\, t) + 0.0001392849552 \cos(8.862493356\, t) \\ & - 0.00002345191108 \sin(10.83193632\, t) - 0.0001244990031 \sin(8.862493356\, t) \\ & - 0.00001930288621 \cos(10.83193632\, t) - 0.000003007551560 \cos(12.80137929\, t) \end{aligned} \tag{1.59}$$

> *with*(*plots*) :

> *pnum* := *odeplot*(*foo*, *t* = 0 ..15, *refine* = 1, *color* = *red*) :

> *pexp* := *plot*(*xapprox*, *t* = 0 ..15, *color* = *green*) :

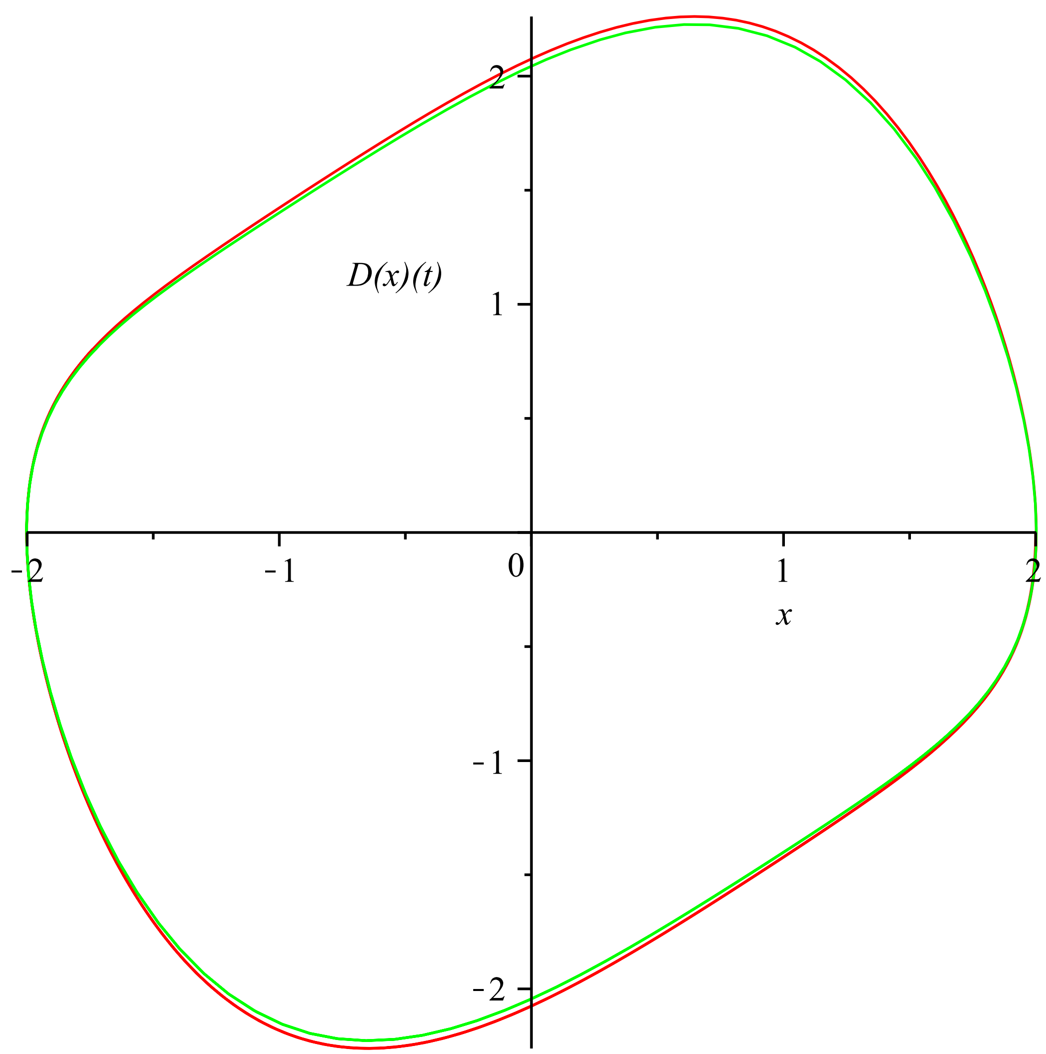
> *display*(*pexp*, *pnum*);



> $pnum2 := odeplot(foo, [x(t), D(x)(t)], t = 0..10, refine = 1, color = red)$:

> $pexp2 := plot([xapprox, diff(xapprox, t), t = 0..10], color = green)$:

> $display(pnum2, pexp2)$;



D(x)(t)
x
2
1
-1
-2
-2
-1
0
1
2