> APPENDIX 2.0 – FOUR TIMESCLAE

> restart;

> deqn := diff(x(t), t, t) + x(t) + epsilon·(x(t)^2 - 1)·diff(x(t), t);

$$deqn := \frac{d^2}{dt^2}\, x(t) + x(t) + \varepsilon\, (x(t)^2 - 1)\, \left(\frac{d}{dt}\, x(t)\right) \tag{1}$$

> dsolve(deqn, x(t));

$$x(t) = \_a \ \&\text{where} \left[\left\{\left(\frac{d}{d\_a}\, \_b(\_a)\right) \_b(\_a) + \_a + \varepsilon\, \_b(\_a)\, \_a^2 - \varepsilon\, \_b(\_a) = 0\right\}, \left\{\_a = x(t),\ \_b(\_a) = \frac{d}{dt}\, x(t)\right\}, \left\{t = \int \frac{1}{\_b(\_a)}\, d\_a + \_C1,\ x(t) = \_a\right\}\right] \tag{2}$$

> xansatz := t → X[0](t, epsilon * t, epsilon^2 * t, epsilon^3 * t) + epsilon * X[1](t, epsilon * t, epsilon^2 * t, epsilon^3 * t) + epsilon^2 * X[2](t, epsilon * t, epsilon^2 * t, epsilon^3 * t) + ε^3 ·X[3](t, epsilon·t, ε^2·t, ε^3·t);

$$xansatz := t \to X_0(t, \varepsilon\, t, \varepsilon^2\, t, \varepsilon^3\, t) + \varepsilon\, X_1(t, \varepsilon\, t, \varepsilon^2\, t, \varepsilon^3\, t) + \varepsilon^2\, X_2(t, \varepsilon\, t, \varepsilon^2\, t, \varepsilon^3\, t) + \varepsilon^3\, X_3(t, \varepsilon\, t, \varepsilon^2\, t, \varepsilon^3\, t) \tag{3}$$

> deqnseries := series(subs(epsilon^3 * t = upsilon, subs(epsilon^2 * t = sigma, subs(epsilon * t = tau, eval(subs(x = xansatz, deqn))))), epsilon, 4);

$$\begin{aligned} deqnseries := {} & D_{1,1}(X_0)(t, \tau, \sigma, \upsilon) + X_0(t, \tau, \sigma, \upsilon) + \Big(X_1(t, \tau, \sigma, \upsilon) + 2\, D_{1,2}(X_0)(t, \tau, \sigma, \upsilon) \\ & + \left(X_0(t, \tau, \sigma, \upsilon)^2 - 1\right) D_1(X_0)(t, \tau, \sigma, \upsilon) + D_{1,1}(X_1)(t, \tau, \sigma, \upsilon)\Big)\, \varepsilon + \Big(X_2(t, \tau, \sigma, \upsilon) \\ & + \left(X_0(t, \tau, \sigma, \upsilon)^2 - 1\right) \left(D_2(X_0)(t, \tau, \sigma, \upsilon) + D_1(X_1)(t, \tau, \sigma, \upsilon)\right) + 2\, X_0(t, \tau, \sigma, \\ & \upsilon)\, X_1(t, \tau, \sigma, \upsilon)\, D_1(X_0)(t, \tau, \sigma, \upsilon) + 2\, D_{1,3}(X_0)(t, \tau, \sigma, \upsilon) + D_{2,2}(X_0)(t, \tau, \sigma, \upsilon) \\ & + 2\, D_{1,2}(X_1)(t, \tau, \sigma, \upsilon) + D_{1,1}(X_2)(t, \tau, \sigma, \upsilon)\Big)\, \varepsilon^2 + \Big(2\, D_{1,4}(X_0)(t, \tau, \sigma, \upsilon) + X_3(t, \tau, \\ & \sigma, \upsilon) + D_{1,1}(X_3)(t, \tau, \sigma, \upsilon) + 2\, D_{2,3}(X_0)(t, \tau, \sigma, \upsilon) + 2\, D_{1,3}(X_1)(t, \tau, \sigma, \upsilon) \\ & + D_{2,2}(X_1)(t, \tau, \sigma, \upsilon) + \left(X_0(t, \tau, \sigma, \upsilon)^2 - 1\right) \left(D_3(X_0)(t, \tau, \sigma, \upsilon) + D_2(X_1)(t, \tau, \sigma, \upsilon)\right. \\ & \left.+ D_1(X_2)(t, \tau, \sigma, \upsilon)\right) + \left(2\, X_0(t, \tau, \sigma, \upsilon)\, X_2(t, \tau, \sigma, \upsilon) + X_1(t, \tau, \sigma, \upsilon)^2\right) D_1(X_0)(t, \tau, \sigma, \\ & \upsilon) + 2\, X_0(t, \tau, \sigma, \upsilon)\, X_1(t, \tau, \sigma, \upsilon)\, \left(D_2(X_0)(t, \tau, \sigma, \upsilon) + D_1(X_1)(t, \tau, \sigma, \upsilon)\right) \\ & + 2\, D_{1,2}(X_2)(t, \tau, \sigma, \upsilon)\Big)\, \varepsilon^3 + O(\varepsilon^4) \end{aligned} \tag{4}$$

> deqn0 := coeff(deqnseries, epsilon, 0);

$$deqn0 := D_{1,1}(X_0)(t, \tau, \sigma, \upsilon) + X_0(t, \tau, \sigma, \upsilon) \tag{5}$$

> deqn1 := coeff(deqnseries, epsilon, 1);

$$deqn1 := X_1(t, \tau, \sigma, \upsilon) + 2\, D_{1,2}(X_0)(t, \tau, \sigma, \upsilon) + \left(X_0(t, \tau, \sigma, \upsilon)^2 - 1\right) D_1(X_0)(t, \tau, \sigma, \upsilon) + D_{1,1}(X_1)(t, \tau, \sigma, \upsilon) \tag{6}$$

> deqn2 := coeff(deqnseries, epsilon, 2);

$$deqn2 := X_2(t,\tau,\sigma,\upsilon) + \left(X_0(t,\tau,\sigma,\upsilon)^2 - 1\right)\left(D_2(X_0)(t,\tau,\sigma,\upsilon) + D_1(X_1)(t,\tau,\sigma,\upsilon)\right) + 2\,X_0(t,\tau,\sigma,\upsilon)\,X_1(t,\tau,\sigma,\upsilon)\,D_1(X_0)(t,\tau,\sigma,\upsilon) + 2\,D_{1,3}(X_0)(t,\tau,\sigma,\upsilon) + D_{2,2}(X_0)(t,\tau,\sigma,\upsilon) + 2\,D_{1,2}(X_1)(t,\tau,\sigma,\upsilon) + D_{1,1}(X_2)(t,\tau,\sigma,\upsilon) \quad \textbf{(7)}$$

> *deqn3* := *coeff*(*deqnseries*, epsilon, 3);

$$deqn3 := 2\,D_{1,4}(X_0)(t,\tau,\sigma,\upsilon) + X_3(t,\tau,\sigma,\upsilon) + D_{1,1}(X_3)(t,\tau,\sigma,\upsilon) + 2\,D_{2,3}(X_0)(t,\tau,\sigma,\upsilon) + 2\,D_{1,3}(X_1)(t,\tau,\sigma,\upsilon) + D_{2,2}(X_1)(t,\tau,\sigma,\upsilon) + \left(X_0(t,\tau,\sigma,\upsilon)^2 - 1\right)\left(D_3(X_0)(t,\tau,\sigma,\upsilon) + D_2(X_1)(t,\tau,\sigma,\upsilon) + D_1(X_2)(t,\tau,\sigma,\upsilon)\right) + \left(2\,X_0(t,\tau,\sigma,\upsilon)\,X_2(t,\tau,\sigma,\upsilon) + X_1(t,\tau,\sigma,\upsilon)^2\right) D_1(X_0)(t,\tau,\sigma,\upsilon) + 2\,X_0(t,\tau,\sigma,\upsilon)\,X_1(t,\tau,\sigma,\upsilon)\left(D_2(X_0)(t,\tau,\sigma,\upsilon) + D_1(X_1)(t,\tau,\sigma,\upsilon)\right) + 2\,D_{1,2}(X_2)(t,\tau,\sigma,\upsilon) \quad \textbf{(8)}$$

> *deqn0*;

$$D_{1,1}(X_0)(t,\tau,\sigma,\upsilon) + X_0(t,\tau,\sigma,\upsilon) \quad \textbf{(9)}$$

> *dsolve*(*deqn0*, *X*[0](*t*, tau, sigma, upsilon));

$$X_0(t,\tau,\sigma,\upsilon) = \_F1(\tau,\sigma,\upsilon)\sin(t) + \_F2(\tau,\sigma,\upsilon)\cos(t) \quad \textbf{(10)}$$

> *X*[0] := (*t*, tau, sigma, upsilon) → *A*(tau, sigma, upsilon) * cos(*t*-Phi(tau, sigma, upsilon));

$$X_0 := (t,\tau,\sigma,\upsilon) \rightarrow A(\tau,\sigma,\upsilon)\cos(t - \Phi(\tau,\sigma,\upsilon)) \quad \textbf{(11)}$$

> *deqn0*;

$$0 \quad \textbf{(12)}$$

> *deqn1temp* := *collect*(*combine*(*deqn1*, *trig*), [sin, cos]);

$$deqn1temp := \left(-2\,D_1(A)(\tau,\sigma,\upsilon) - \frac{1}{4}A(\tau,\sigma,\upsilon)^3 + A(\tau,\sigma,\upsilon)\right)\sin(t - \Phi(\tau,\sigma,\upsilon)) + X_1(t,\tau,\sigma,\upsilon) + 2\,A(\tau,\sigma,\upsilon)\cos(t-\Phi(\tau,\sigma,\upsilon))\,D_1(\Phi)(\tau,\sigma,\upsilon) - \frac{1}{4}A(\tau,\sigma,\upsilon)^3\sin(3\,t - 3\,\Phi(\tau,\sigma,\upsilon)) + D_{1,1}(X_1)(t,\tau,\sigma,\upsilon) \quad \textbf{(13)}$$

> *secular_terms* := {*coeff*(*deqn1temp*, sin(*t*-Phi(tau, sigma, upsilon))), *coeff*(*deqn1temp*, cos(*t* -Phi(tau, sigma, upsilon)))}

$$secular\_terms := \left\{2\,A(\tau,\sigma,\upsilon)\,D_1(\Phi)(\tau,\sigma,\upsilon),\ -2\,D_1(A)(\tau,\sigma,\upsilon) - \frac{1}{4}A(\tau,\sigma,\upsilon)^3 + A(\tau,\sigma,\upsilon)\right\} \quad \textbf{(14)}$$

> *dsolve*(*secular_terms*, {Phi(tau, sigma, upsilon), *A*(tau, sigma, upsilon)});

$$\{A(\tau,\sigma,\upsilon) = 0, \Phi(\tau,\sigma,\upsilon) = \Phi(\tau,\sigma,\upsilon)\}, \left\{A(\tau,\sigma,\upsilon) = -\frac{2}{\sqrt{1 + 4\,e^{-\tau}\,\_F1(\sigma,\upsilon)}}, \Phi(\tau,\sigma,\upsilon) = \_F2(\sigma,\upsilon)\right\} \quad \textbf{(15)}$$

> *X*[0] := (*t*, tau, sigma, upsilon) →2·cos(*t*-Phi(sigma, upsilon));

$$X_0 := (t,\tau,\sigma,\upsilon) \rightarrow 2\cos(t - \Phi(\sigma,\upsilon)) \quad \textbf{(16)}$$

> $deqn1temp := map(simplify, collect(combine(deqn1, trig), [\sin, \cos]));$

$$deqn1temp := X_1(t, \tau, \sigma, \upsilon) - 2\sin(3t - 3\Phi(\sigma, \upsilon)) + D_{1,1}(X_1)(t, \tau, \sigma, \upsilon) \quad \textbf{(17)}$$

> $dsolve(deqn1temp, X[1](t, \mathrm{tau}, \mathrm{sigma}, \mathrm{upsilon}));$

$$X_1(t, \tau, \sigma, \upsilon) = \sin(t)\ \_F2(\tau, \sigma, \upsilon) + \cos(t)\ \_F1(\tau, \sigma, \upsilon) - \frac{1}{4}\sin(3t - 3\Phi(\sigma, \upsilon)) \quad \textbf{(18)}$$

> $X_1(t, \tau, \sigma, \upsilon) = -\frac{1}{4}\sin(3t - 3\Phi(\sigma, \upsilon));$

$$X_1(t, \tau, \sigma, \upsilon) = -\frac{1}{4}\sin(3t - 3\Phi(\sigma, \upsilon)) \quad \textbf{(19)}$$

> $X[1] := (t, \mathrm{tau}, \mathrm{sigma}, \mathrm{upsilon}) \rightarrow -\frac{1}{4}\sin(3t - 3\Phi(\sigma, \upsilon));$

$$X_1 := (t, \tau, \sigma, \upsilon) \rightarrow -\frac{1}{4}\sin(3t - 3\Phi(\sigma, \upsilon)) \quad \textbf{(20)}$$

> $deqn2temp := map(simplify, collect(combine(deqn2, trig), [\sin, \cos]));$

$$deqn2temp := \frac{1}{4}\left(16\, D_1(\Phi)(\sigma, \upsilon) - 1\right)\cos(t - \Phi(\sigma, \upsilon)) + X_2(t, \tau, \sigma, \upsilon) + D_{1,1}(X_2)(t, \tau, \sigma, \upsilon) - \frac{5}{4}\cos(5t - 5\Phi(\sigma, \upsilon)) - \frac{3}{4}\cos(3t - 3\Phi(\sigma, \upsilon)) \quad \textbf{(21)}$$

> $secular\_terms := \{coeff(deqn2temp, \sin(t-\mathrm{Phi}(\mathrm{sigma}, \mathrm{upsilon}))), coeff(deqn2temp, \cos(t-\mathrm{Phi}(\mathrm{sigma}, \mathrm{upsilon})))\};$

$$secular\_terms := \left\{0, 4\, D_1(\Phi)(\sigma, \upsilon) - \frac{1}{4}\right\} \quad \textbf{(22)}$$

> $deqn2htemp := map(simplify, collect(combine(deqn2h, trig), [\sin, \cos]));$

$$deqn2htemp := deqn2h \quad \textbf{(23)}$$

> $dsolve(secular\_terms, \{\mathrm{Phi}(\mathrm{sigma}, \mathrm{upsilon})\});$

$$\left\{\Phi(\sigma, \upsilon) = \frac{1}{16}\sigma + \_F1(\upsilon)\right\} \quad \textbf{(24)}$$

> $\Phi(\sigma, \upsilon) = \frac{1}{16}\sigma + \_F1(\upsilon)$

$$\Phi(\sigma, \upsilon) = \frac{1}{16}\sigma + \_F1(\upsilon) \quad \textbf{(25)}$$

> $\Phi := (\sigma, \upsilon) \rightarrow \frac{1}{16}\sigma + \_F1(\upsilon);$

$$\Phi := (\sigma, \upsilon) \rightarrow \frac{1}{16}\sigma + \_F1(\upsilon) \quad \textbf{(26)}$$

> $X[0](t, \mathrm{tau}, \mathrm{sigma}, \mathrm{upsilon});$

$$2\cos\left(t - \frac{1}{16}\sigma - \_F1(\upsilon)\right) \quad \textbf{(27)}$$

> $X[0] := (t, \mathrm{tau}, \mathrm{sigma}, \mathrm{upsilon}) \rightarrow 2\cos\left(t - \frac{1}{16}\sigma - \_F1(\upsilon)\right);$

$$X_0 := (t, \tau, \sigma, \upsilon) \rightarrow 2\cos\left(t - \frac{1}{16}\sigma - \_F1(\upsilon)\right) \quad \textbf{(28)}$$

> $X[1](t, \mathrm{tau}, \mathrm{sigma}, \mathrm{upsilon});$

(29)

$$-\frac{1}{4}\sin\left(3t-\frac{3}{16}\sigma-3\_F1(\upsilon)\right) \quad \mathbf{(29)}$$

> $X[1] := (t, \text{tau}, \text{sigma}, \text{upsilon}) \rightarrow -\frac{1}{4}\sin\left(3t-\frac{3}{16}\sigma-3\_F1(\upsilon)\right);$

$$X_1 := (t, \tau, \sigma, \upsilon) \rightarrow -\frac{1}{4}\sin\left(3t-\frac{3}{16}\sigma-3\_F1(\upsilon)\right) \quad \mathbf{(30)}$$

> $deqn2temp := map(factor, map(simplify, collect(combine(deqn2, trig), [\sin, \cos])));$

$$deqn2temp := X_2(t, \tau, \sigma, \upsilon) - \frac{5}{4}\cos\left(5t-\frac{5}{16}\sigma-5\_F1(\upsilon)\right) - \frac{3}{4}\cos\left(3t-\frac{3}{16}\sigma-3\_F1(\upsilon)\right) + \mathrm{D}_{1,1}(X_2)(t, \tau, \sigma, \upsilon) \quad \mathbf{(31)}$$

> $dsolve(deqn2temp, X[2](t, \text{tau}, \text{sigma}, \text{upsilon}));$

$$X_2(t, \tau, \sigma, \upsilon) = \sin(t)\ \_F3(\tau, \sigma, \upsilon) + \_F2(\tau, \sigma, \upsilon)\cos(t) - \frac{5}{96}\cos\left(5t-\frac{5}{16}\sigma-5\_F1(\upsilon)\right) - \frac{3}{32}\cos\left(3t-\frac{3}{16}\sigma-3\_F1(\upsilon)\right) \quad \mathbf{(32)}$$

> $X[2] := (t, \text{tau}, \text{sigma}, \text{upsilon}) \rightarrow -\frac{5}{96}\cos\left(5t-\frac{5}{16}\sigma-5\_F1(\upsilon)\right) - \frac{3}{32}\cos\left(3t-\frac{3}{16}\sigma-3\_F1(\upsilon)\right);$

$$X_2 := (t, \tau, \sigma, \upsilon) \rightarrow -\frac{5}{96}\cos\left(5t-\frac{5}{16}\sigma-5\_F1(\upsilon)\right) - \frac{3}{32}\cos\left(3t-\frac{3}{16}\sigma-3\_F1(\upsilon)\right) \quad \mathbf{(33)}$$

> $deqn3temp := map(factor, map(simplify, collect(combine(deqn3, trig), [\sin, \cos])));$

$$deqn3temp := 4\cos\left(t-\frac{1}{16}\sigma-\_F1(\upsilon)\right)\mathrm{D}(\_F1)(\upsilon) + X_3(t, \tau, \sigma, \upsilon) + \mathrm{D}_{1,1}(X_3)(t, \tau, \sigma, \upsilon) + \frac{9}{32}\sin\left(3t-\frac{3}{16}\sigma-3\_F1(\upsilon)\right) + \frac{1}{32}\sin\left(t-\frac{1}{16}\sigma-\_F1(\upsilon)\right) + \frac{7}{12}\sin\left(7t-\frac{7}{16}\sigma-7\_F1(\upsilon)\right) + \frac{85}{96}\sin\left(5t-\frac{5}{16}\sigma-5\_F1(\upsilon)\right) \quad \mathbf{(34)}$$

> $secular\_terms := \{coeff(deqn3temp, \sin(t-\mathrm{Phi}(\text{sigma}, \text{upsilon}))), coeff(deqn3temp, \cos(t-\mathrm{Phi}(\text{sigma}, \text{upsilon})))\};$

$$secular\_terms := \left\{\frac{1}{32}, 4\,\mathrm{D}(\_F1)(\upsilon)\right\} \quad \mathbf{(35)}$$

> $dsolve(4\,\mathrm{D}(\_F1)(\upsilon));$

$$\_F1(\upsilon) = \_C1 \quad \mathbf{(36)}$$

> $\_F1 := (\upsilon) \rightarrow \_C1;$

$$\_F1 := \upsilon \rightarrow \_C1 \quad \mathbf{(37)}$$

> $deqn3temp := X_3(t, \tau, \sigma, \upsilon) + \mathrm{D}_{1,1}(X_3)(t, \tau, \sigma, \upsilon) - \frac{9}{32}\sin\left(-3t+\frac{3}{16}\sigma+3\_C1\right) - \frac{7}{12}\sin\left(-7t+\frac{7}{16}\sigma+7\_C1\right) - \frac{85}{96}\sin\left(-5t+\frac{5}{16}\sigma+5\_C1\right)$

$$deqn3temp := X_3(t, \tau, \sigma, \upsilon) + \mathrm{D}_{1,1}(X_3)(t, \tau, \sigma, \upsilon) - \frac{9}{32}\sin\left(\frac{3}{16}\sigma-3t+3\_C1\right) \quad \mathbf{(38)}$$

$$-\frac{7}{12}\sin\left(-7\,t+\frac{7}{16}\,\sigma+7\,\_C1\right)-\frac{85}{96}\sin\left(\frac{5}{16}\,\sigma-5\,t+5\,\_C1\right)$$

```
> dsolve(deqn3temp, X[3](t, tau, sigma, upsilon));
```

$$X_3(t,\tau,\sigma,\upsilon)=\sin(t)\,\_F3(\tau,\sigma,\upsilon)+\_F2(\tau,\sigma,\upsilon)\cos(t)-\frac{9}{256}\sin\left(\frac{3}{16}\,\sigma-3\,t+3\,\_C1\right)-\frac{7}{576}\sin\left(-7\,t+\frac{7}{16}\,\sigma+7\,\_C1\right)-\frac{85}{2304}\sin\left(\frac{5}{16}\,\sigma-5\,t+5\,\_C1\right) \quad (39)$$

```
> X[3] := (t, tau, sigma, upsilon) -> -9/256 sin(-3 t + 3/16 sigma + 3 _C1) - 7/576 sin(-7 t + 7/16 sigma + 7 _C1) - 85/2304 sin(-5 t + 5/16 sigma + 5 _C1);
```

$$X_3:=(t,\tau,\sigma,\upsilon)\rightarrow-\frac{9}{256}\sin\left(-3\,t+\frac{3}{16}\,\sigma+3\,\_C1\right)-\frac{7}{576}\sin\left(-7\,t+\frac{7}{16}\,\sigma+7\,\_C1\right)-\frac{85}{2304}\sin\left(-5\,t+\frac{5}{16}\,\sigma+5\,\_C1\right) \quad (40)$$

```
> xansatz(t);
```

$$2\cos\left(-t+\frac{1}{16}\,\varepsilon^2\,t+\_C1\right)+\frac{1}{4}\,\varepsilon\sin\left(-3\,t+\frac{3}{16}\,\varepsilon^2\,t+3\,\_C1\right)+\varepsilon^2\left(-\frac{5}{96}\cos\left(-5\,t+\frac{5}{16}\,\varepsilon^2\,t+5\,\_C1\right)-\frac{3}{32}\cos\left(-3\,t+\frac{3}{16}\,\varepsilon^2\,t+3\,\_C1\right)\right)+\varepsilon^3\left(-\frac{9}{256}\sin\left(-3\,t+\frac{3}{16}\,\varepsilon^2\,t+3\,\_C1\right)-\frac{7}{576}\sin\left(-7\,t+\frac{7}{16}\,\varepsilon^2\,t+7\,\_C1\right)-\frac{85}{2304}\sin\left(-5\,t+\frac{5}{16}\,\varepsilon^2\,t+5\,\_C1\right)\right) \quad (41)$$

```
> foo := dsolve([subs(epsilon = 1/2, deqn), x(0) = 2.0, D(x)(0) = 0], range = 0. ..100, numeric);
```

$$foo := \mathbf{proc}(x\_rkf45)\ \ldots\ \mathbf{end\ proc} \quad (42)$$

```
> _C1 := 0.2;
```

$$\_C1 := 0.2 \quad (43)$$

```
> xansatz(t);
```

$$2\cos\left(-t+\frac{1}{16}\,\varepsilon^2\,t+0.2\right)+\frac{1}{4}\,\varepsilon\sin\left(-3\,t+\frac{3}{16}\,\varepsilon^2\,t+0.6\right)+\varepsilon^2\left(-\frac{5}{96}\cos\left(-5\,t+\frac{5}{16}\,\varepsilon^2\,t+1.0\right)-\frac{3}{32}\cos\left(-3\,t+\frac{3}{16}\,\varepsilon^2\,t+0.6\right)\right)+\varepsilon^3\left(-\frac{9}{256}\sin\left(-3\,t+\frac{3}{16}\,\varepsilon^2\,t+0.6\right)-\frac{7}{576}\sin\left(-7\,t+\frac{7}{16}\,\varepsilon^2\,t+1.4\right)-\frac{85}{2304}\sin\left(-5\,t+\frac{5}{16}\,\varepsilon^2\,t+1.0\right)\right) \quad (44)$$

```
> #xapprox:=subs(epsilon=1/2, subs(tmp,xansatz(t)));
> xapprox := subs(epsilon = 1/2, xansatz(t));
```

$$xapprox := 2\cos\left(-\frac{63}{64}\,t+0.2\right)+\frac{247}{2048}\sin\left(-\frac{189}{64}\,t+0.6\right)-\frac{5}{384}\cos\left(-\frac{315}{64}\,t+1.0\right)-\frac{3}{128}\cos\left(-\frac{189}{64}\,t+0.6\right)-\frac{7}{4608}\sin\left(-\frac{441}{64}\,t+1.4\right)-\frac{85}{18432}\sin\left(-\frac{315}{64}\,t+1.0\right) \quad (45)$$

```
> with(plots):
```

```
> pnum:=odeplot(foo,t=0..50,refine=1,color=red):
> pexp:=plot(xapprox,t=0..50,color=green):
> display(pexp,pnum);
```

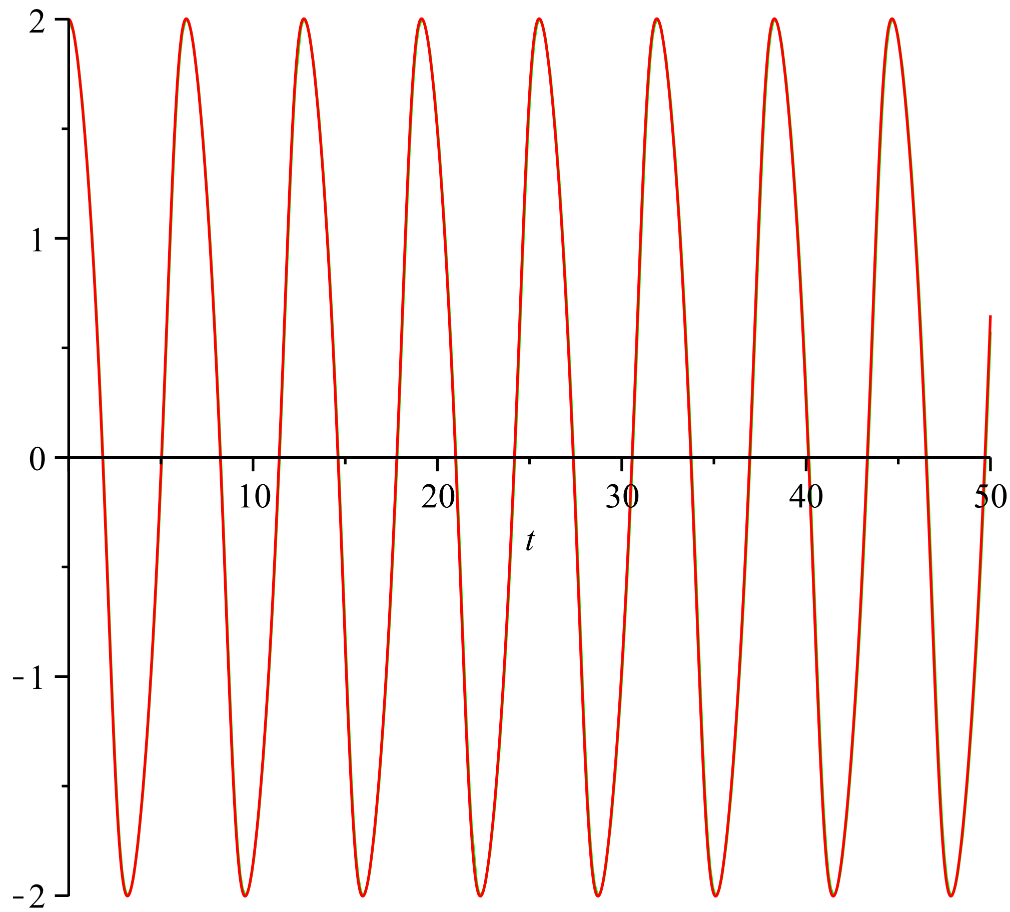

Figure 6.0(a) numerical integration of the van der Pol in red vs approximate solution in green by the 4-timing method for phi = 0.2 and epsilon = 0.5

```
> pnum2:=odeplot(foo,[x(t),D(x)(t)],t=0..10,refine=1,color=red):
> pexp2:=plot([xapprox,diff(xapprox,t),t=0..10],color=green):
> display(pnum2,pexp2);
```

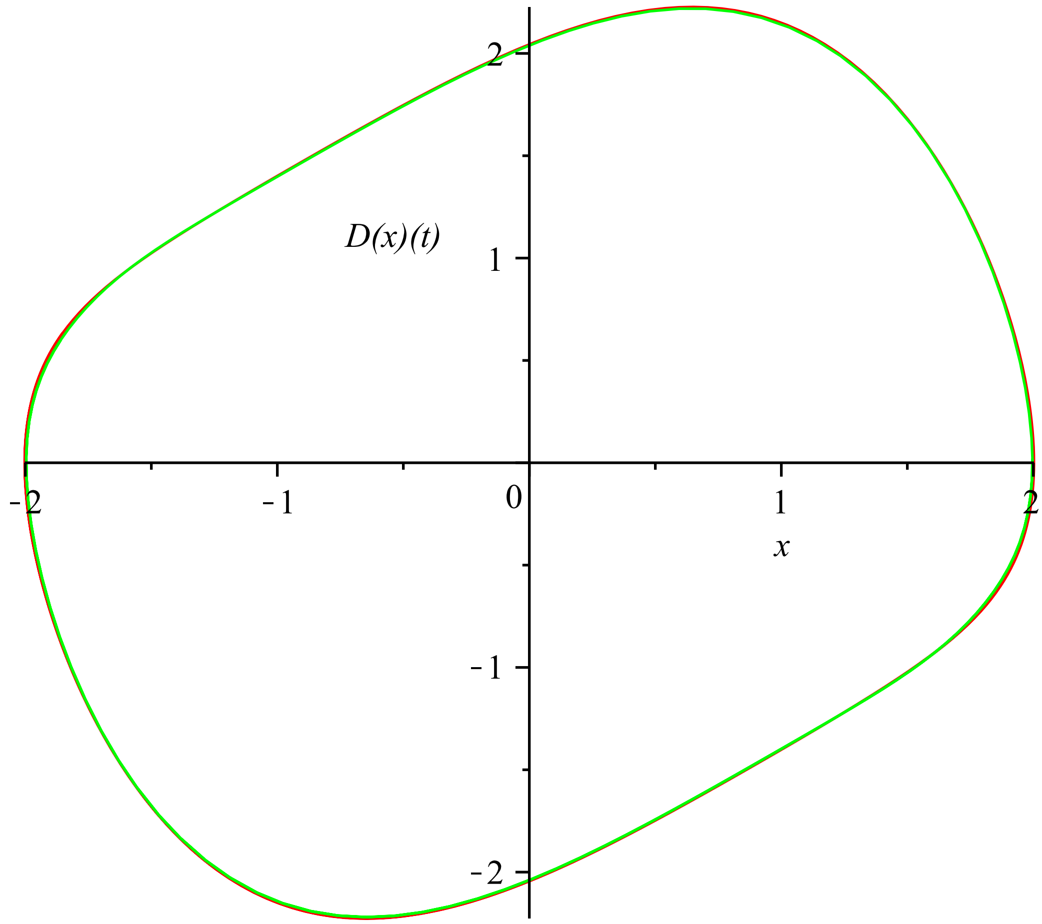

Figure 6.0(b) phase plot of the numerical integration of the van der Pol in red vs approximate solution in green by the 4-timing method for phi = 0.2 and epsilon = 0.5