An asymptotic expansion for the mode of the maximum of a set of Poisson random variables

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We deal with a set of independent Poisson random variables $\{X_1, X_2, \ldots, X_n\}$ with common mean λ, so that $\Pr[X_i = k] = e^{-\lambda}\lambda^k/k!$. The mean λ is considered to be a fixed constant throughout. We let $M_n = \max(X_i)$ and wish to describe the distribution of M_n. Our motivation was a problem in random graph theory, in which we were interested in the distribution of maximum degree in graphs with Poisson degree distribution. Our contribution in this note is a theorem giving a very precise asymptotic expansion for the most probable (modal) value of the maximum.

Theorem 1. *With the definitions stated above, we have that* $M_n \sim x_0 \equiv \log n/W(\log n/(e\lambda))$ *as* $n \to \infty$, *where* $W(\cdot)$ *is Lambert's W function. Furthermore, we have the even more accurate asymptotic expansion*

$$M_n \sim x_0 + \left(\log\lambda - \lambda - \log(2\pi)/2 - \frac{3}{2}\log(x_0)\right) / (\log(x_0) - \log\lambda)$$

.

To establish this, we start with the Poisson cumulative probability:

$$\Pr[X_i < k] = Q(k, \lambda) \equiv \Gamma(k, \lambda)/\Gamma(k)$$

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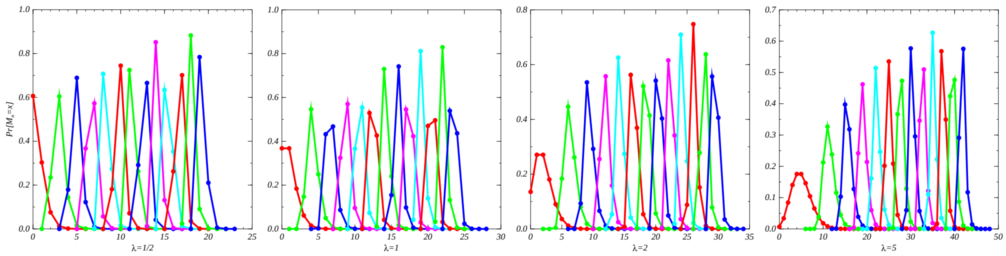

Figure 1: From left to right: the distribution of the maximum of Poisson variables for $\lambda = 1/2, 1, 2, 5$ (left to right) and $n = 10^0, 10^2, 10^4, \ldots, 10^{24}$. Note that there is an error in Fig 1 in Anderson et al. (1997), where the curves labelled $k = 6$ and $k = 8$ are incorrect.

where Q and $\Gamma(\cdot, \cdot)$ are incomplete Gamma functions; that is,

$$\Gamma(a, x) = \int_x^\infty t^{a-1} e^{-t} \mathrm{d}t.$$

The assumed independence of the Poisson variables implies that

$$\Pr[M_n \leqslant k] = \Pr[X_1 \leqslant k]^n = Q(k+1, \lambda)^n = \Gamma(k+1, \lambda)^n / \Gamma(k+1)^n.$$

Our aim is to approximate the distribution of M_n. We have

$$\begin{aligned} \Pr[M_n = k] &= \Pr[M_n \leqslant k] - \Pr[M_n \leqslant k-1] \\ &= Q(k+1, \lambda)^n - Q(k, \lambda)^n. \end{aligned}$$

It is useful now to make some precise numerical experiments to understand the difficulty of the problem; in particular we notice an extremely slow growth of M_n with n. Examples of these distributions are shown in Figure 1. These numerical results demonstrate the so-called *focussing* effect; the maxima M_n are concentrated on at most two adjacent integers for large n; we call them *modal values*. It is this focussing that allows us to characterize the distributions very precisely by a single asymptotic estimate.

In previous work on this problem, Anderson (1970) proved the existence of integers I_n such that $\Pr[M_n \in \{I_n, I_n+1\}] \to 1$ as n tends to infinity for

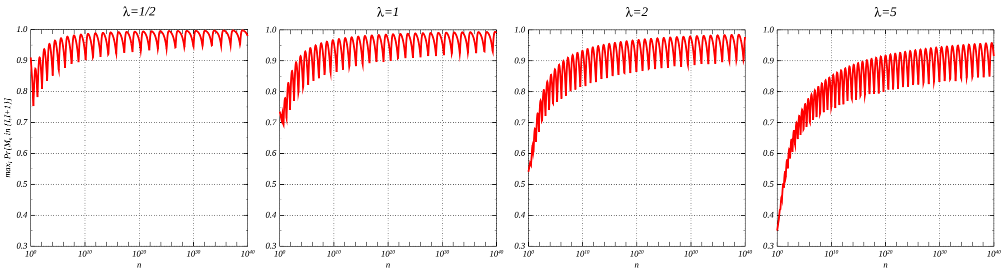

Figure 2: The maximal probability (with respect to I_n) that $M_n \in \{I_n, I_n{+}1\}$ for $\lambda = 1/2, 1, 2, 5$ (left to right) and $10^0 \leqslant n \leqslant 10^{40}$. The curves show the probability that M_n takes either of its two most frequently occurring values.

fixed $\lambda > 0$; and that $I_n \sim \beta_n$, where β_n is defined as the unique solution of $Q(\beta_n, \lambda) = 1/n$. We have that $\beta_n \to \infty$ as n tends to infinity.

Following this work, Kimber (1983) computed an asymptotic result; he showed $I_n \sim \log n / \log\log n$ and $P_n \sim (k/I_n)^{1+B_n}$ with B_n dense in $[-1/2, 1/2]$, and concluded that to the first order, the rate of growth of I_n is independent of the Poisson mean λ. He moreover showed that P_n, defined as $P_n = \Pr[M_n \in \{I_n, I_n{+}1\}]$, oscillates and the oscillation persists for arbitrarily large n. We illustrate in Figure 2 exactly how this probability oscillates as $n \to \infty$. Our numerical experiments show that $\log n / \log\log n$ estimates I_n very poorly. Our aim to improve this asymptotic formula, and especially to account accurately for the dependence on λ.

Our method is a refinement of that of Kimber; that is, we consider a continuous distribution g which interpolates the Poisson maximum distribution, and we solve $g(x) = 1/n$. Consider $g_\lambda(x) \equiv 1 - \Gamma(x{+}1, \lambda)/\Gamma(x{+}1)$ for fixed $\lambda \in \mathbb{R}^+$, which is a strictly decreasing function on $(0, \infty)$. If $\epsilon = 1/n$ is a small positive real, then $g_\lambda(x)$ has a unique root $x(\epsilon)$ which increases as $\epsilon \to 0^+$. We will develop an asymptotic expansion (as $\epsilon \to 0$) of this root $x(\epsilon)$.

We have, from the definition of the Γ function:

$$g_\lambda(x) = \exp(-\lambda)\,\lambda^x \sum_{i=1}^{\infty} \frac{\lambda^i}{\Gamma(x{+}i{+}1)}.$$

We make a Taylor expansion of $g_\lambda(x)$ in powers of x^{-1}, separating the $x\log(x)$ and $\log(x)$ terms, and thus work with

$$\log(g_\lambda(x)) = -x\log(x) + (1+\log\lambda)x - \frac{3}{2}\log(x) + \left(\log\lambda - \lambda - \frac{\log(2\pi)}{2}\right) + \frac{\lambda - 13/12}{x} + \mathcal{O}(x^{-2}). \quad (1)$$

To apply the method of Kimber, we need to approximate the solution of $\log(g_\lambda(x)) = -\log n$. Let $W(\cdot)$ be the principal branch of Lambert's W function; that is, the unique positive root of the mapping $x \mapsto x\exp x$ (Corless et al., 1996). Then a first approximation to the solution of $\log(g_\lambda(x)) = -\log n$ (with n large and positive) is given by keeping only the dominant first two terms in Equation (1):

$$M_n \sim x_0 \equiv \frac{\log n}{W\left(\frac{\log n}{\exp(1)\lambda}\right)}. \quad (2)$$

That this is already quite accurate can be seen from the dark blue curves in Figure 3. However, we would like to do better; ideally the error should be less than unity so that the mode (necessarily an integer) of the distribution is correctly identified. A refinement x_1 may be generated by making a single Newton correction step; that is, $x_1 = x_0 - (h(x_0) + \log n)/h'(x_0)$, where h is some approximation to $\log(g_\lambda)$. For example, by keeping all terms in $\log(g_\lambda(x))$ and $\log(g_\lambda(x))'$ which do not vanish as $n \to \infty$, we obtain

$$M_n \sim x_1 = x_0 + \frac{\log\lambda - \lambda - \log(2\pi)/2 - 3\log(x_0)/2}{\log(x_0) - \log\lambda}.$$

This establishes our theorem. The asymptotic estimate x_1 appears to have error less than unity for all values of n and λ considered in Figure 3, and so is probably sufficiently precise for all practical purposes. If further accuracy is needed, it may be obtained by additional Newton steps. In any case, both x_0 and x_1 are considerably more precise than Kimber's approximation. We emphasize that this claim is based on the very precise computations shown in our figures; these results which we label "exact" are not from simulations, but from numerical evaluation of the exact formulas with careful control of floating-point errors. Our numerical experience convinces us that it is unlikely that any formula as simple as that in Theorem 1 is more accurate.

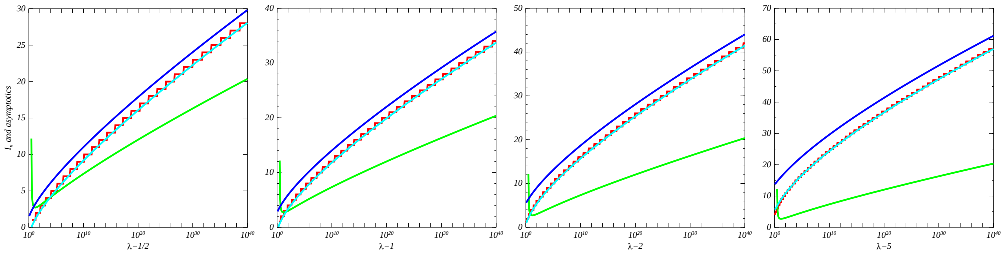

Figure 3: Exact values and asymptotics of I_n for $\lambda = 1/2, 1, 2, 5$ (left to right) and $n = 10^0, \ldots, 10^{40}$ (horizontal axis on each plot). The staircase red line (almost hidden by the cyan line) represents the *exact* mode I_n; the other lines represent asymptotic approximations: green for the result of Kimber (1983) (which is independent of λ), and dark blue and cyan for our new results x_0 and x_1 respectively. The cyan curve always sits between the steps of I_n, meaning that x_1 has error less than unity.

References

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