Discrete Mathematics 310 (2010) 230–240

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

# Asymptotic enumeration of 2-covers and line graphs

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ARTICLE INFO

*Article history:*
Received 4 July 2007
Accepted 2 September 2008
Available online 20 November 2008

*Keywords:*
Asymptotic enumeration
Line graphs
Set partitions

ABSTRACT

A *2-cover* is a multiset of subsets of $[n] := \{1, 2, \dots, n\}$ such that each element of $[n]$ lies in exactly two of the subsets. A 2-cover is called *proper* if all of the subsets are distinct, and is called *restricted* if any two of them intersect in at most one element.

In this paper we find asymptotic enumerations for the number of line graphs on $n$ labelled vertices and for 2-covers.

We find that the number $s_n$ of 2-covers and the number $t_n$ of proper 2-covers both have asymptotic growth

$$s_n \sim t_n \sim B_{2n} 2^{-n} \exp\left(-\frac{1}{2}\log(2n/\log n)\right),$$

where $B_{2n}$ is the 2$n$th Bell number. Moreover, the numbers $u_n$ of restricted 2-covers on $[n]$ and $v_n$ of restricted, proper 2-covers on $[n]$ and $l_n$ of line graphs all have growth

$$u_n \sim v_n \sim l_n \sim B_{2n} 2^{-n} n^{-1/2} \exp\left(-\left[\frac{1}{2}\log(2n/\log n)\right]^2\right).$$



## 1. Introduction

A 2-cover of $[n] := \{1, 2, \dots n\}$ is a multiset of subsets $\{S_1, S_2, \dots, S_m\}$, $S_i \subseteq [n]$, (possibly with $S_i = S_j$ for some $i \neq j$), such that for each $d \in [n]$ the number of $j$such that $d \in S_j$ is exactly 2. A 2-cover is called *proper* if $S_i \neq S_j$ whenever $i \neq j$. A 2-cover is called *restricted* if the intersection of any 2 of the $S_i$ contains at most one element. These definitions have been taken from [4]. Note that a proper 2-cover $\{S_1, \dots, S_m\}$ is a set.

The *line graph* $L(G)$ of a simple graph $G$ is the graph whose vertex set is the edge set of $G$ and such that two vertices are adjacent in $L(G)$ if and only if the corresponding edges of $G$ have a common vertex.

Let $s_n$ be the number of 2-covers of $[n]$; let $t_n$ be the number of proper 2-covers of $[n]$; let $u_n$ be the number of restricted 2-covers of $[n]$; let $v_n$ be the number of restricted, proper 2-covers of $[n]$; and let $l_n$ be the number of line graphs on $n$ labelled vertices. Let $B_n$ be the $n$th Bell number. Given sequences $a_n$ and $b_n$, we write $a_n \sim b_n$ to mean $\lim_{n\to\infty} a_n/b_n = 1$.

**Theorem 1.** *The number of 2-covers and the number of proper 2-covers have asymptotic growth*

$$s_n \sim t_n \sim B_{2n} 2^{-n} \exp\left(-\frac{1}{2}\log(2n/\log n)\right) = B_{2n} 2^{-n}\sqrt{\frac{\log n}{2n}} \tag{1}$$

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0012-365X/$ – see front matter 
doi:10.1016/j.disc.2008.09.008

*while the number of restricted 2-covers, restricted, proper 2-covers and line graphs all have asymptotic growth*

$$u_n \sim v_n \sim l_n \sim B_{2n} 2^{-n} n^{-1/2} \exp\left(-\left[\frac{1}{2}\log(2n/\log n)\right]^2\right). \tag{2}$$

We note that the expression in the exponential in (2) is the square of the expression in the exponential in (1). A heuristic explanation was suggested by an anonymous referee. In counting 2-covers, the problem is nearly equivalent to that of counting partitions of $[2n]$ such that $i$ and $n+i$ are in different parts for each $i \in [n]$. For a uniformly chosen random partition, the probability that $i$ and $n+i$ are in the same part is $B_{2n-1}/B_{2n} \simeq \log(2n/\log n)/2n$. (This is shown rigorously in the proof of Lemma 3.) One might expect, therefore, that the number of $i$'s in a uniform random partition such that $i$ and $n+i$ are in the same part is asymptotically a Poisson distributed random variable with mean $\mu = \log(2n/\log n)/2$ and that the probability that there are no such $i$ is about $e^{-\mu} = \sqrt{\log n/(2n)}$. For the class of problems where multiple edges are not allowed, one can translate the question again into one about random partitions of $[2n]$: what is the probability that $i$ and $j$ are in the same part, and $n+i$ and $n+j$ are in the same part: this is $B_{2n-2}/B_{2n} \simeq \mu^2/n^2$.

One expects the number of pairs $(i, j)$ with either (a) $i$ and $j$ in the same part, and $n+i$ and $n+j$ in the same part, or (b) $i$ and $n+j$ in the same part, and $n+i$ and $j$ in the same part, should be asymptotically Poisson with mean $\mu^2$. The probability that there are no such pairs should therefore be about $e^{-\mu^2}$ and indeed we see an $e^{-\mu^2}$ term in (2).

The main term $B_{2n}2^{-n}$ in (1) and (2) can be roughly explained as follows. Take $2n$ half edges $\{1, 2, \ldots, 2n\}$, partition them into blocks, and form $n$ edges $\{j, j+n\}$ for $j \in \{1, 2, \ldots, n\}$, making sure $j$ and $j+n$ go into different blocks for all $j$ to avoid loops.

We make some initial observations regarding 2-covers, special graphs and orbits in Section 2. In particular, Section 2.3 connects our work to orbits of oligomorphic permutation groups and is not used in the rest of the paper. We use a probabilistic method to prove (1) in Section 3. A pair of technical lemmas are proven in Section 3.1, (1) is proven for $s_n$ in Section 3.2 and it is proven for $t_n$ in Section 3.3. We prove (2) in Section 4.

In both probabilistic and generating function proofs we will make use of Lambert's $W$-function $W(t)$, which is a solution to

$$W(t)e^{W(t)} = t \tag{3}$$

and which has asymptotics (see (3.10) of [7])

$$W(t) = \log t - \log\log t + \frac{\log\log t}{\log t} + o\left(\frac{1}{\log t}\right) \quad \text{as } t \to \infty. \tag{4}$$

For each 2-cover $\{S_1, \ldots, S_m\}$ of $[n]$ we define an associated $m \times n$ incidence matrix $M$ with entries given by

$$M_{i,j} = \begin{cases} 1 & \text{if } j \in S_i; \\ 0 & \text{if } j \notin S_i. \end{cases}$$

Note that $M$ has exactly 2 ones in each column and that the ordering of the rows is arbitrary. A 2-cover is proper if and only if $M$ has no repeated rows. A 2-cover is restricted if and only if $M$ has no repeated columns. Therefore, Theorem 1 is equivalent to the asymptotic enumeration of certain 0-1 matrices. The general methods of this paper were used for the asymptotic enumeration of other 0-1 matrices called incidence matrices in [2,3].

## 2. 2-covers, line graphs and orbits

In this section we establish correspondences between 2-covers, line graphs and orbits of certain permutation groups.

### 2.1. 2-covers and graphs

We define a *special multigraph* to be a multigraph with no isolated vertices or loops. Our first result is:

**Proposition 1.** *There is a bijection between 2-covers on $[n]$ and special multigraphs having unlabelled vertices and $n$ labelled edges, such that*

- *proper 2-covers correspond to multigraphs having no connected component of size 2;*
- *restricted 2-covers correspond to simple graphs.*

**Proof.** Let $\{S_1, \ldots, S_m\}$ be a 2-cover of $[n]$. Construct a graph $G$ as follows:

- the vertex set is $[m]$;
- for each $i \in [n]$, there is an edge $e_i$ joining vertices $j$ and $k$, where $S_j$ and $S_k$ are the two sets of the 2-cover containing $i$.

The graph $G$ is a multigraph (that is, repeated edges are permitted), but it has no isolated vertices and no loops.

Conversely, given a multigraph without isolated vertices or loops, we can recover a 2-cover: number the edges $e_1, \ldots, e_n$, and let $S_i$ be the set of indices $j$ for which the $i$th vertex lies on edge $e_j$. Thus we have the first part of the proposition.

The second part comes from observing that a "repeated set" in a 2-cover corresponds to a pair of vertices lying on the same edges, while a pair of elements lying in two different sets correspond to a pair of edges incident to the same two vertices. ■

### 2.2. Generating function identities for 2-covers

Recall that $s_n$, $t_n$, $u_n$ and $v_n$ denote the numbers of 2-covers, proper 2-covers, restricted 2-covers, and restricted proper 2-covers respectively. Using Proposition 1 in this subsection we will find relationships between these quantities and derive corresponding generating function identities.

**Proposition 2.** *Let $S(n,k)$ denote the Stirling numbers of the second kind, that is, the number of set partitions of $[n]$ into exactly $k$ non-empty subsets. Then,*

$$s_n = \sum_{k=1}^{n} S(n,k)u_k$$
$$t_n = \sum_{k=1}^{n} S(n,k)v_k$$
$$u_n = \sum_{k=0}^{n} \binom{n}{k} v_k.$$

**Proof.** We prove these for the corresponding special multigraphs.

Any special multigraph with edges $e_1, \ldots, e_n$ can be described by giving a partition of $[n]$ into, say, $k$ parts, together with a special simple graph with $k$ labelled edges; simply replace the $i$th edge of the simple graph by the $i$th set of edges of the partition (where the edges are ordered lexicographically, say). This is clearly a bijection. Moreover, the simple graph has no connected components of size 2 if and only if the same holds for the multigraph. This proves the first two equations.

Given a special simple graph, there is a distinguished subset of $[n]$ (of size $n-k$, say) consisting of isolated edges; the remaining graph has no components of size 2. Again, the correspondence is bijective. So the third equation holds. ■

Proposition 2 can be reformulated in terms of exponential generating functions. Let $S(x) = \sum_{n\geq 0} s_n x^n/n!$, with similar definitions for the others. The proof of Proposition 3 is omitted.

**Proposition 3.**

$$S(x) = U(e^x - 1)$$
$$T(x) = V(e^x - 1)$$
$$U(x) = V(x)e^x.$$

It follows from Proposition 3 that $S(x) = T(x)B(x)$, where $B(x) = e^{e^x-1}$ is the exponential generating function for the Bell numbers. This is easily proved directly.

### 2.3. Unrestricted 2-covers and orbits

A permutation group $G$ acting on an infinite set $\Omega$ is *oligomorphic* if it has only finitely many orbits on the set of $n$-tuples of distinct elements of $\Omega$ (equivalently, on the set of all $n$-tuples). We denote the numbers of these orbits by $F_n(G)$ and $F_n^*(G)$ respectively.

By [6], if $G$ is oligomorphic and *primitive* (that is, preserves no non-trivial equivalence relation on $\Omega$), then $F_n(G) = c^n n!/p(n)$, where $c > 1$ is an absolute constant and $p(n)$ is a polynomial. There is some interest in groups $G$ with the growth of $F_n(G)$ close to this bound. One example is the permutation group $S_\infty^{\{2\}}$ induced by the infinite symmetric group on the set of 2-element subsets of its domain.

**Proposition 4.** $F_n(S_\infty^{\{2\}}) = u_n$ *and* $F_n^*(S_\infty^{\{2\}}) = s_n$.

**Proof.** Simply observe that an $n$-tuple of distinct 2-sets is the edge set of a special simple graph with $n$ labelled edges, while an arbitrary $n$-tuple of 2-sets is the edge set of a special multigraph with $n$ labelled edges. ■

We note that the relation

$$F_n^*(G) = \sum_{k=1}^{n} S(n,k)F_k(G)$$

gives an alternative proof of the first equation in Proposition 2. We do not know of a similar interpretation of the other two parameters.

### 2.4. Generating function identities for line graphs

The relationship between line graphs and restricted 2-covers is essentially contained in Proposition 1 by considering the simple graph corresponding to a restricted 2-cover and letting labelled edges map to the labelled vertices of a line graph. This map to line graphs is one-to-one excepting cases contained in results of Whitney and Sabidussi.

Let $L(x) = \sum_{n\geq 0} l_n x^n/n!$. We now prove:

**Proposition 5.**

$$L(x) = e^{-x^3/3!-6x^4/4!-15x^5/5!-15x^6/6!}U(x).$$

**Proof.** According to Whitney's Theorem [5], an isomorphism between line graphs $L(G_1)$ and $L(G_2)$ of connected graphs is induced by an isomorphism from $G_1$ to $G_2$, except in one case: the line graphs of the triangle $K_3$ and the star $K_{1,3}$ are isomorphic. Moreover, Sabidussi [10] has shown that if $G$ is a connected graph with at least three vertices, then the automorphism groups of $G$ and $L(G)$ are isomorphic if $G$ is not $K_4$, $K_4$ with an edge deleted or $K_4$ with two adjacent edges deleted, which we shall denote by $K_4'$ and $K_4''$, respectively.

Now the connected components of line graphs which are triangles contribute a factor $e^{x^3/3!}$ to the exponential generating function $L(x)$ for line graphs on $[n]$; that is, $L(x) = e^{x^3/3!}W'(x)$, where $W'(x)$ is the e.g.f. for line graphs with no such components. Similarly, components which are triangles or stars contribute a factor $(e^{x^3/3!})^2$ to the e.g.f. for special simple graphs with $n$ edges, leading to an overall multiplication by a factor of $e^{-x^3/3!}$

Next, while $K_4$ has $S_4$ as an automorphism group and therefore admits $6!/4! = 30$ different edge labellings, the order of the automorphism group of $L(K_4)$ is $2 \cdot 4!$ and therefore $L(K_4)$ admits 15 different vertex labellings. Similarly to the above, this leads to a correction by a factor of $e^{-15x^6/6!}$.

Similar arguments hold for $K_4'$ and $K_4''$, leading to factors $e^{-15x^5/5!}$ and $e^{-6x^4/4!}$, correspondingly.

Proposition 5 now follows by Whitney's Theorem, Sabidussi's result, and Proposition 3. ■

## 3. Unrestricted 2-covers: A probabilistic approach

In this section we prove (1) of Theorem 1 by using a probabilistic construction.

### 3.1. Technical results

We proceed with the following definitions and lemma. Let $T_n$ be the set of proper 2-covers on $[n]$. Let $\mathcal{S}_n$ be the set of set partitions of $[2n]$. Let $E_{1,n} \subset \mathcal{S}_n$ be the subset of set partitions of $[2n]$ such that $j$ and $j+n$ are contained in different blocks for each $j \in [n]$. Define the function $\psi$ from a subset $\tilde{S}$ of $[2n]$ to a subset of $[n]$ by $\psi(\tilde{S}) = \{j : j \in \tilde{S} \text{ or } j+n \in \tilde{S}\}$. Let $E_{2,n} \subset \mathcal{S}_n$ be the subset of set partitions of $[2n]$ with blocks $\{\tilde{S_1}, \dots, \tilde{S_m}\}$ such that $\psi(\tilde{S}_{i_1}) \neq \psi(\tilde{S}_{i_2})$ for each $i_1 \neq i_2$. Let $C_n = E_{1,n} \cap E_{2,n}$. Let $\phi$ be the function on $\mathcal{S}_n$ given by

$$\phi(\{\tilde{S_1}, \dots, \tilde{S_m}\}) = \{\psi(\tilde{S_1}), \dots, \psi(\tilde{S_m})\}.$$

**Lemma 1.** *$\phi$ maps $C_n$ onto $T_n$ and $|\phi^{-1}(\mathbf{a})| = 2^n$ for all $\mathbf{a} \in T_n$.*

**Proof.** Fix $\{\tilde{S_1}, \dots, \tilde{S_m}\} \in C_n$. Each $j \in [n]$ appears in exactly two blocks of $\phi(\{\tilde{S_1}, \dots, \tilde{S_m}\})$ because of the definition of $E_{1,n}$ and the blocks of $\{\tilde{S_1}, \dots, \tilde{S_m}\}$ are unique because of the definition of $E_{2,n}$ so $\phi(\{\tilde{S_1}, \dots, \tilde{S_m}\}) \in T_n$.

Let $\mathbf{a} = \{S_1, \dots, S_m\} \in T_n$. For each $j \in [n]$ there are two ways of assigning $j, j+n$ to the appearances of $j$ in $\mathbf{a}$ (think of a fixed ordering of the blocks of $\mathbf{a}$ to see this). The choices made for every $j \in [n]$ determine an *assignment*. Clearly, every element of $\phi^{-1}(\mathbf{a})$ must be of the form $\chi(\mathbf{a})$ for some assignment $\chi$. There are $2^n$ assignments. We also write $\chi(S_i)$ for the block $\tilde{S_i}$ corresponding to $S_i$ in $\chi(\mathbf{a})$.

We claim that each assignment $\chi(\mathbf{a})$ gives a unique element of $C_n$. To see this, first note that $j$ and $j+n$ are clearly in different blocks of $\chi(\mathbf{a})$, so $\chi(\mathbf{a}) \in E_{1,n}$. Secondly, $\phi \circ \chi$ is the identity map on $T_n$. Therefore, $\chi(\mathbf{a}) \in E_{2,n}$ because $\mathbf{a}$ is a proper 2-cover. Moreover, $\chi_1(\mathbf{a}_1) \neq \chi_2(\mathbf{a}_2)$ for all $\mathbf{a}_1, \mathbf{a}_2 \in T_n$ such that $\mathbf{a}_1 \neq \mathbf{a}_2$ and for all assignments $\chi_1$ and $\chi_2$, which gives $\phi^{-1}(\mathbf{a}_1) \cap \phi^{-1}(\mathbf{a}_2) = \emptyset$.

We next prove that if $\chi_1$ and $\chi_2$ are two assignments such that $\chi_1(\mathbf{a}) = \chi_2(\mathbf{a})$, then $\chi_1 = \chi_2$. To see this, let

$$\mathcal{U} = \{j \in [n] : \chi_1 \text{ and } \chi_2 \text{ differ for } j\}.$$

Without loss of generality, assume that $j \in S_1$ and $j \in S_2$. Then, either $j \in \chi_1(S_1)$ and $j \in \chi_2(S_2)$ or $j+n \in \chi_1(S_1)$ and $j+n \in \chi_2(S_2)$. It follows that $\chi_1(S_1) = \chi_2(S_2)$. Therefore, $\phi \circ \chi_1(S_1) = \phi \circ \chi_2(S_2)$ or $S_1 = S_2$ violating the assumption that $\mathbf{a}$ is proper. We conclude that $\mathcal{U} = \emptyset$ and that $\chi_1 = \chi_2$. This implies that $|\phi^{-1}(\mathbf{a})| = 2^n$. ■

Next we generalize Lemma 1 to (possibly) improper covers. Let $U_n$ denote the set of 2-covers of $[n]$.

**Lemma 2.** *$\phi$ maps $E_{1,n}$ onto $U_n$. Let $\mathbf{a} = \{S_1, S_2, \dots, S_m\}$ be a 2-cover of $[n]$. Let $\mathcal{M}$ be the set of $i \in [m]$ such that there does not exist any $j \in [m] \setminus \{i\}$, $S_j = S_i$. Let*

$$\rho = \frac{m - |\mathcal{M}|}{2}$$

*be the number of pairs $\{i, j\}$ such that $S_i = S_j$. Then*

$$|\phi^{-1}(\mathbf{a})| = 2^{n-\rho}.$$

**Proof.** Clearly $\phi$ maps $E_{1,n}$ onto $U_n$. Let $\mathcal{N} = [n] \setminus \{\cup_{i \in \mathcal{M}} S_i\}$. Then $\{S_i : i \in \mathcal{M}\}$ is a proper cover of $\mathcal{N}$ and Lemma 1 implies that

$$|\phi^{-1}(\{S_i : i \in \mathcal{N}\})| = 2^{|\mathcal{N}|}.$$

For each pair $S_{i_1}, S_{i_2}$ such that $i_1 \neq i_2$ and $S_{i_1} = S_{i_2}$, it must be true that $\phi^{-1}(S_i)$ consists of two sets $\tilde{S}_1$ and $\tilde{S}_2$ such that for each $j \in S_{i_1}$ either $j \in \tilde{S}_{i_1}$ and $j + n \in \tilde{S}_{i_2}$ or $j + n \in \tilde{S}_{i_1}$ and $j \in \tilde{S}_{i_2}$. The number of unordered sets $\tilde{S}_{i_1}, \tilde{S}_{i_2}$ is $2^{|S_{i_1}|-1}$. Therefore,

$$|\phi^{-1}(\mathbf{a})| = 2^{|\mathcal{N}|} \prod 2^{|S_{i_1}|-1} = 2^{n-\rho},$$

where the product is over pairs $i_1, i_2$ such that $i_1 \neq i_2$ and $S_{i_1} = S_{i_2}$. ■

### *3.2. Asymptotic enumeration of proper 2-covers*

From Lemma 1 we conclude that $|C_n| = 2^n t_n$ so

$$t_n = 2^{-n}|C_n| = 2^{-n}\frac{|C_n|}{B_{2n}}B_{2n} \tag{5}$$

where $B_{2n}$ is the $2n$th Bell number.

We will now prove:

**Lemma 3.**

$$\frac{|E_{1,n}|}{B_{2n}} \sim \sqrt{\frac{\log n}{2n}} \tag{6}$$

*and*

$$\frac{|E_{2,n}|}{B_{2n}} = 1 - O\left(\frac{\log^2 n}{n}\right). \tag{7}$$

**Proof.** To prove (6), choose an element of $\mathcal{S}_n$ uniformly at random and let $X$ be the number of $j \in [n]$ for which $j$ and $j + n$ are in the same block. We have

$$\mathbb{P}(X = 0) = \frac{|E_{1,n}|}{B_{2n}}. \tag{8}$$

We have $X = \sum_{j=1}^{n} I_j$ where $I_j$ is the indicator random variable so that $j$ and $j + n$ are in the same block. The $r$th falling moment of $X$ is

$$\begin{aligned}\mathbb{E}(X)_r &= \mathbb{E}X(X-1)\cdots(X-r+1)\\ &= \sum \mathbb{E}(I_{j_1} I_{j_2} \cdots I_{j_r})\end{aligned}$$

where the sum is over $(j_1, \dots, j_r)$ with no repetitions. To find $\mathbb{E}(I_{j_1} I_{j_2} \cdots I_{j_r})$ we take $[2n] \setminus \{j_1, j_2, \dots, j_r\}$ and form a set partition. We then add $j_k$ to the block containing $j_k + n$ for each $k \in [r]$. This process is uniquely reversible. Therefore,

$$\mathbb{E}(X)_r = \frac{(n)_r B_{2n-r}}{B_{2n}}.$$

We apply the formula in Corollary 13, page 18, of [1] to obtain

$$\mathbb{P}(X = 0) = \sum_{r=0}^{\infty} (-1)^r \frac{\mathbb{E}(X)_r}{r!} = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \frac{(n)_r B_{2n-r}}{B_{2n}}. \tag{9}$$

To analyze (9) we use the expansion of the Bell numbers [7,9]

$$\log B_n = e^w(w^2 - w + 1) - \frac{1}{2}\log(1+w) - 1 - \frac{w(2w^2+7w+10)}{24(1+w)^3}e^{-w} - \frac{w(2w^4+12w^3+29w^2+40w+36)}{48(1+w)^6}e^{-2w} + O(e^{-3w}),$$

where $w = W(n)$ is given by (3) and (4), from which we obtain (using Maple)

$$\log B_{n-r} - \log B_n = -rw + \frac{rw}{2n}\left(\frac{r}{w+1} + \frac{1}{(w+1)^2}\right) + O\left(\frac{r^3 w}{n^2}\right).$$

In particular,

$$\frac{B_{n-1}}{B_n} \sim \frac{\log n}{n}$$

so there exists a constant $C > 0$ such that

$$\frac{B_{n-r}}{B_n} \leq \frac{(C\log n)^r}{(n)_r}. \tag{10}$$

Moreover,

$$\begin{aligned}\log B_{2n-r} - \log B_{2n} &= -rv + \frac{rv}{4n}\left(\frac{r}{v+1} + \frac{1}{(v+1)^2}\right) + O\left(\frac{r^3 v}{n^2}\right)\\ &= -r\log n + rc_n + r^2 d_n + O\left(\frac{r^3\log n}{n^2}\right),\end{aligned}$$

where $v = W(2n)$ has the expansion

$$v = \log n - \log\log n + \log 2 + \frac{\log\log n}{\log n} - \frac{\log 2}{\log n} + o\left(\frac{1}{\log n}\right),$$

where

$$\begin{aligned}c_n &= \log n - v - \frac{rv}{4n(v+1)^2}\\ &= \log\log n - \log 2 - \frac{\log\log n}{\log n} + \frac{\log 2}{\log n} + o\left(\frac{1}{\log n}\right)\end{aligned}$$

and where

$$d_n = O\left(\frac{1}{n}\right).$$

Using (10) we estimate

$$\begin{aligned}\left|\sum_{r>\log^{3/2} n}(-1)^r\frac{\mathbb{E}(X)_r}{r!}\right| &\leq \sum_{r>\log^{3/2} n}\frac{(n)_r B_{2n-r}}{r!B_{2n}}\\ &\leq \sum_{r>\log^{3/2} n}\frac{(C\log 2n)^r}{r!}\\ &= (2n)^C\sum_{r>\log^{3/2} n}e^{-C\log 2n}\frac{(C\log 2n)^r}{r!}\\ &= o(1).\end{aligned} \tag{11}$$

For $r \leq \log^{3/2} n$, we have

$$\frac{B_{2n-r}}{B_{2n}} = n^{-r}\exp\left(rc_n + r^2 d_n + O\left(\frac{\log^9 n}{n^2}\right)\right)$$

and

$$(n)_r = n^r\exp\left(O\left(\frac{r^2}{n}\right)\right),$$

hence

$$\mathbb{E}(X)_r = \exp\left(rc_n + r^2 d_n + O\left(\frac{\log^9 n}{n^2}\right)\right).$$

Therefore,

$$\begin{aligned}
\sum_{0\le r\le \log^{3/2} n} (-1)^r \frac{\mathbb{E}(X)_r}{r!} &= \sum_{0\le r\le \log^{3/2} n} \frac{(-1)^r}{r!} e^{rc_n + r^2 d_n}\left(1 + O\left(\frac{\log^9 n}{n^2}\right)\right) \\
&= \sum_{0\le r\le \log^{3/2} n} \frac{(-1)^r}{r!} e^{rc_n}\left(1 + d_n r^2 + O\left(\frac{\log^9 n}{n^2}\right)\right) \\
&= \sum_{0\le r\le \log^{3/2} n} \frac{(-1)^r}{r!} e^{rc_n} + d_n \sum_{0\le r\le \log^{3/2} n} \frac{(-1)^r r^2}{r!} e^{rc_n} \\
&\quad + O\left(\frac{\log^9 n}{n^2}\right) \sum_{0\le r\le \log^{3/2} n} \frac{e^{rc_n}}{r!}. \qquad (12)
\end{aligned}$$

We proceed to approximate the terms in (12). First, we find that

$$\begin{aligned}
\sum_{0\le r\le \log^{3/2} n} \frac{(-1)^r}{r!} e^{rc_n} &= \exp\left(-e^{c_n}\right) + O\left(\sum_{\log^{3/2} n\le r\le n} \frac{e^{rc_n}}{r!}\right) \\
&= \exp\left(-\frac{\log n}{2}\left[1 - \frac{\log\log n}{\log n} + \frac{\log 2}{\log n} + o\left(\frac{1}{\log n}\right)\right]\right) + o(n^{-1/2}) \\
&\sim \sqrt{\frac{\log n}{2n}}. \qquad (13)
\end{aligned}$$

We estimate

$$\begin{aligned}
d_n \left|\sum_{0\le r\le \log^{3/2} n} \frac{(-1)^r}{r!} r^2 e^{rc_n}\right| &= d_n \left|\sum_{2\le r\le \log^{3/2} n} \frac{(-1)^r}{(r-2)!} e^{rc_n} + \sum_{1\le r\le \log^{3/2} n} \frac{(-1)^r}{(r-1)!} e^{rc_n}\right| \\
&= d_n \left|e^{2c_n} \sum_{2\le r\le \log^{3/2} n} \frac{(-1)^r}{(r-2)!} e^{(r-2)c_n} + e^{c_n} \sum_{1\le r\le \log^{3/2} n} \frac{(-1)^r}{(r-1)!} e^{(r-1)c_n}\right| \\
&= d_n \left(\exp\left(-e^{c_n} + 2c_n\right) + \exp\left(-e^{c_n} + c_n\right) + O\left(e^{2c_n} \sum_{\log^{3/2} n\le r\le n} \frac{e^{rc_n}}{r!}\right)\right) \\
&= o(n^{-1/2}). \qquad (14)
\end{aligned}$$

Finally, we have

$$\begin{aligned}
O\left(\frac{\log^9 n}{n^2}\right) \sum_{0\le r\le \log^{3/2} n} \frac{e^{rc_n}}{r!} &\le O\left(\frac{\log^9 n}{n^2}\right) e^{e^{c_n}} \\
&= o(n^{-1/2}). \qquad (15)
\end{aligned}$$

Together, (8), (9) and (11)–(15) prove (6).

To show (7), let $Y$ be the number of pairs $S_i, S_j$ in an partition in $\mathcal{S}_n$ chosen uniformly at random for which $\psi(S_i) = \psi(S_j)$. For such $S_i, S_j$ of size $|S_i| = |S_j| = k$, the probability that they are present in the random partition is $B(2n-2k)/B(2n)$. The total number of pairs $S_i, S_j$ of size $k$ is bounded by $\binom{n}{k} 2^k$ (the number of ways of choosing a subset $J$ of size $k$ from $[n]$ times a bound on the number of ways of choosing two subsets $S_1, S_2$ of $[2n]$ of size $k$ such that either $j \in S_1$ and $j + n \in S_2$ or $j + n \in S_1$ and $j \in S_2$ for all $j \in J$). Therefore, using (10) we get

$$\begin{aligned}
1 - \frac{|E_{2,n}|}{B_{2n}} &= \mathbb{P}(Y > 0) \\
&\le \mathbb{E}Y
\end{aligned}$$

$$\leq \sum_{k=1}^{n} \binom{n}{k} 2^k \frac{B_{2n-2k}}{B_{2n}}$$
$$\leq \sum_{k=1}^{n} \binom{n}{k} 2^k \frac{(C\log 2n)^{2k}}{(2n)_{2k}}$$
$$\leq \sum_{k=1}^{n} \frac{(n)_k (2C^2 \log^2 2n)^k}{(2n)_{2k} k!}$$
$$= O\left(\frac{\log^2 n}{n}\right). \quad \blacksquare$$

Lemma 3 and (5) along with

$$\frac{|C_n|}{B_{2n}} \leq \frac{|E_{1,n}|}{B_{2n}}$$

and

$$\frac{|C_n|}{B_{2n}} \geq \frac{|E_{1,n}| - (B_{2n} - |E_{2,n}|)}{B_{2n}}$$

prove (1) for $t_n$.

### 3.3. *Asymptotic enumeration of 2-covers*

In this subsection we prove (1) for $s_n$. Recall that $U_n$ denotes the set of 2-covers of $[n]$. Each element of $E_{1,n}$ is mapped to a unique $\mathbf{a} \in U_n$ by $\phi$. Given $\omega = \{\tilde{S}_1, \tilde{S}_2, \ldots, \tilde{S}_m\} \in \mathcal{S}_n$, let $Z(\omega)$ be the number of pairs $\{i_1, i_2\}$ such that $\psi(\tilde{S}_{i_1}) = \psi(\tilde{S}_{i_2})$. Note that in the case $\omega \in E_{1,n}$ we have $Z(\omega) = \rho$ with $\rho$ defined with respect to $\mathbf{a} = \phi(\omega)$ in the statement of Lemma 2.

Define $D_{\rho,n}$ for $\rho \in \{0, 1, \ldots, n\}$ to be

$$D_{\rho,n} = \{\omega \in E_{1,n} : Z(\omega) = \rho\}.$$

Note that $D_{0,n} = C_n$. By Lemma 2,

$$u_n = \sum_{\rho=0}^{n} |D_{\rho,n}| 2^{-n+\rho}$$
$$= |C_n| 2^{-n} + \sum_{\rho=1}^{n} |D_{\rho,n}| 2^{\rho}$$
$$= B_{2n} 2^{-n} \left( \frac{|C_n|}{B_{2n}} + \sum_{\rho=1}^{n} \frac{|D_{\rho,n}|}{B_{2n}} 2^{\rho} \right).$$

We have shown in the previous section that $C_n/B_{2n} \sim \sqrt{\log n/2n}$. Observe that $\sum_{\rho=1}^{n} |D_{\rho,n}| 2^{\rho}/B_{2n} \leq \sum_{\rho=1}^{n} \mathbb{P}(Z = \rho) 2^{\rho}$, where $Z$ was defined in the last paragraph and $\omega$ is chosen uniformly at random from $\mathcal{S}_n$. In light of these observations, to prove (1) for $s_n$ it suffices to prove that

$$\sum_{\rho=1}^{n} \mathbb{P}(Z = \rho) 2^{\rho} = o\left(\sqrt{\frac{\log n}{2n}}\right). \tag{16}$$

The quantity $\mathbb{P}(Z \geq \rho)$ is equal to the probability that the randomly chosen element of $\mathcal{S}_n$ contains at least $\rho$ disjoint pairs of equal sets, therefore,

$$\mathbb{P}(Z \geq \rho) \leq \sum_{s_1=1}^{n} \sum_{s_2=1}^{n} \cdots \sum_{s_\rho=1}^{n} \binom{n}{s_1, s_2, \ldots, s_\rho, n - \sum s_i} \frac{B_{2n-2\sum s_i}}{B_{2n}}.$$

Let $\sigma$ be defined by $\sigma = \sum_{i=1}^{\rho} s_i$. We can assume $\sigma \leq n$. From (10) we have

$$\mathbb{P}(Z \geq \rho) \leq \sum_{s_1=1}^{n} \sum_{s_2=1}^{n} \cdots \sum_{s_\rho=1}^{n} \binom{n}{s_1, s_2, \ldots, s_\rho, n - \sigma} \frac{(C\log n)^{2\sigma}}{(2n)_{2\sigma}}$$
$$= \sum_{s_1=1}^{n} \sum_{s_2=1}^{n} \cdots \sum_{s_\rho=1}^{n} \frac{(n)_\sigma}{\prod_i s_i!} \frac{(C\log n)^{2\sigma}}{(2n)_{2\sigma}}.$$

Observing that

$$\frac{(n)_\sigma}{(2n)_{2\sigma}} = \frac{(n)_\sigma}{(2n)_\sigma(2n-\sigma)_\sigma} \le \frac{1}{(2n)_\sigma} \le n^{-\sigma},$$

we have

$$\begin{aligned}\mathbb{P}(Z \ge \rho) &\le \sum_{\sigma=\rho}^{n} \sum_{\substack{s_1,\dots,s_\rho:\\ \sum_i s_i=\sigma}} \frac{1}{\prod_i s_i!}\left(\frac{C^2\log^2 n}{n}\right)^\sigma \\ &= \sum_{\sigma=\rho}^{n} \frac{\rho^\sigma}{\sigma!}\left(\frac{C^2\log^2 n}{n}\right)^\sigma.\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{\rho=1}^{n}\mathbb{P}(Z=\rho)2^\rho &\le \sum_{\rho=1}^{n}\mathbb{P}(Z\ge\rho)2^\rho \\ &\le \sum_{\rho=1}^{n}\sum_{\sigma=\rho}^{n}\frac{2^\rho\rho^\sigma}{\sigma!}\left(\frac{C^2\log^2 n}{n}\right)^\sigma \\ &= \sum_{\sigma=1}^{n}\sum_{\rho=1}^{\sigma}\frac{2^\rho\rho^\sigma}{\sigma!}\left(\frac{C^2\log^2 n}{n}\right)^\sigma \\ &\le \sum_{\sigma=1}^{n}\sum_{\rho=1}^{\sigma}\frac{\rho^\sigma}{\sigma!}\left(\frac{2C^2\log^2 n}{n}\right)^\sigma \\ &\le \sum_{\sigma=1}^{n}\frac{(\sigma+1)^\sigma}{\sigma!}\left(\frac{2C^2\log^2 n}{n}\right)^\sigma \\ &= O\left(\frac{\log^2 n}{n}\right) \\ &= o\left(\sqrt{\frac{\log n}{2n}}\right).\end{aligned}$$

The last estimate proves (16). ■

## 4. Restricted 2-covers and line graphs: An analytic approach

Our proof of (2) will use generating function analysis. Let $a_{n,m}$ be the number of restricted, proper 2-covers on $[n]$ with $m$ blocks. The generating function for restricted, proper 2-covers

$$A(x,y) = \sum_{n=0}^{\infty}\sum_{m=1}^{2n}\frac{a_{n,m}}{n!}x^n y^m$$

equals

$$A(x,y) = \exp\left(-y-\frac{xy^2}{2}\right)\sum_{m\ge 0}\frac{y^m}{m!}(1+x)^{\binom{m}{2}}; \tag{17}$$

see page 203 of [4]. A brief proof of (17) is that $(1+x)^{\binom{m}{2}}$ is the generating function for labelled graphs on $m$ vertices and so $\sum_{m\ge 0}\frac{y^m}{m!}(1+x)^{\binom{m}{2}}$ is the exponential generating function of labelled graphs. Now, the factor $\exp\left(-y-xy^2/2\right)$ forbids isolated vertices and isolated edges.

Therefore,

$$V(x) = A(x,1) = e^{-1}\sum_{m=0}^{\infty}\frac{1}{m!}(1+x)^{\binom{m}{2}}e^{-x/2} \tag{18}$$

and

$$v_n = n!e^{-1}\sum_{m=0}^{\infty}\frac{1}{m!}\sum_{k=0}^{n}\frac{1}{k!}\left(-\frac{1}{2}\right)^k\binom{\binom{m}{2}}{n-k}$$
$$= n!e^{-1}\sum_{m=2}^{\infty}\frac{m^{2n}}{m!}\sum_{k=0}^{n}\frac{1}{k!}\left(-\frac{1}{2}\right)^k m^{-2n}\binom{\binom{m}{2}}{n-k} + o(1). \tag{19}$$

Note that for $m \geq 2$,

$$\left|\sum_{k=0}^{n}\frac{n!}{k!}\left(-\frac{1}{2}\right)^k m^{-2n}\binom{\binom{m}{2}}{n-k}\right| \leq \sum_{k=0}^{n}\binom{n}{k}\left(\frac{1}{2}\right)^k m^{-2n}\binom{m}{2}^{n-k}$$
$$\leq 2^{-n}\sum_{k=0}^{n}\binom{n}{k}m^{-2k}$$
$$= 2^{-n}\left(\frac{1+m^{-2}}{2}\right)^n = O(2^{-n}). \tag{20}$$

We will make use of the asymptotic analysis of the Bell numbers in Example 5.4 of [8], which uses the identity

$$B_n = e^{-1}\sum_{m=0}^{\infty}\frac{m^n}{m!}.$$

Let $m_0$ be the nearest integer to $\frac{2n}{W(2n)}$, where $W$ is defined by (3). (Here the choice of $m_0$ is slightly different from that in [8], but the analysis giving (21) and (22) below remains valid.) In [8] it is proved that

$$\sum_{\substack{1\leq m\leq n\\ |m-m_0|>\sqrt{n}\log n}}\frac{m^{2n}}{m!} = O\left(\frac{m_0^{2n}}{m_0!}\sqrt{n}\exp\left(-(\log n)^3\right)\right) \tag{21}$$

and that

$$\sum_{\substack{1\leq m\leq n\\ |m-m_0|\leq\sqrt{n}\log n}}\frac{m^{2n}}{m!} = \frac{m_0^{2n+1}}{m_0!}\sqrt{\frac{2\pi}{2n+m_0}}\left(1+O\left((\log n)^6 n^{-1/2}\right)\right) \tag{22}$$
$$\sim eB_{2n}. \tag{23}$$

It follows from (20) and (21) that

$$\sum_{\substack{1\leq m\leq n\\ |m-m_0|>\sqrt{n}\log n}}\frac{m^{2n}}{m!}\sum_{k=0}^{n}\frac{n!}{k!}\left(-\frac{1}{2}\right)^k m^{-2n}\binom{\binom{m}{2}}{n-k} = O\left(\frac{m_0^{2n}}{m_0!}\sqrt{n}2^{-n}\exp\left(-(\log n)^3\right)\right)$$
$$= O\left(B_{2n}2^{-n}\exp\left(-\frac{(\log n)^3}{2}\right)\right). \tag{24}$$

We have

$$\sum_{\substack{1\leq m\leq n\\ |m-m_0|\leq\sqrt{n}\log n}}\frac{m^{2n}}{m!}\sum_{k=0}^{n}\frac{n!}{k!}\left(-\frac{1}{2}\right)^k m^{-2n}\binom{\binom{m}{2}}{n-k} = \sum_{\substack{1\leq m\leq n\\ |m-m_0|\leq\sqrt{n}\log n}}\frac{m^{2n}}{m!}m^{-2n}n!\binom{\binom{m}{2}}{n} + \Delta, \tag{25}$$

where

$$\Delta := \sum_{\substack{1\leq m\leq n\\ |m-m_0|\leq\sqrt{n}\log n}}\frac{m^{2n}}{m!}\sum_{k=1}^{n}\frac{n!}{k!}\left(-\frac{1}{2}\right)^k m^{-2n}\binom{\binom{m}{2}}{n-k}$$

is bounded by

$$|\Delta| \leq \sum_{\substack{1\leq m\leq n\\ |m-m_0|\leq\sqrt{n}\log n}}\frac{m^{2n}}{m!}\sum_{k=1}^{n}\frac{n!}{k!}m^{-2n}\binom{\binom{m}{2}}{n}\left(\frac{n}{\binom{m}{2}-n}\right)^k$$
$$= O\left(\frac{\log^2 n}{n}\right)\sum_{\substack{1\leq m\leq n\\ |m-m_0|\leq\sqrt{n}\log n}}\frac{m^{2n}}{m!}m^{-2n}n!\binom{\binom{m}{2}}{n}.$$

One may show that uniformly for $m$ in the range $|m - m_0| \le \sqrt{n}\log n$

$$m^{-2n}\binom{\binom{m}{2}}{n} n! = 2^{-n}\exp\left(-\frac{n}{m_0} - \frac{n^2}{m_0^2}\right)\left(1 + O\left(n^{-1/2}\log^6 n\right)\right),$$

hence,

$$|\Delta| = O\left(\frac{\log^2 n}{n}\right) 2^{-n}\exp\left(-\frac{n}{m_0} - \frac{n^2}{m_0^2}\right) B_{2n}. \qquad (26)$$

The main term of (25) is

$$\sum_{\substack{1\le m\le n\\ |m-m_0|\le\sqrt{n}\log n}} \frac{m^{2n}}{m!} m^{-2n} n! \binom{\binom{m}{2}}{n} = 2^{-n}\exp\left(-\frac{n}{m_0} - \frac{n^2}{m_0^2}\right)(1+o(1)) \sum_{\substack{1\le m\le n\\ |m-m_0|\le\sqrt{n}\log n}} \frac{m^{2n}}{m!}$$

$$= eB_{2n}2^{-n}\exp\left(-\frac{n}{m_0} - \frac{n^2}{m_0^2}\right)(1+o(1))$$

$$= eB_{2n}\frac{1}{2^n\sqrt{n}} e^{-\left(\frac{1}{2}\log(2n/\log n)\right)^2}(1+o(1)) \qquad (27)$$

where we have used the asymptotic expansion (4) and the definition of $m_0$ at the last step. Now (19), (24), (26) and (27) prove (2) for $v_n$.

In the previous argument the result would have been the same if the $e^{-x/2}$ in (18) were replaced by 1 because in the Taylor expansion of $e^{-x/2}$ the constant term 1 corresponds to the main term of (25) and the higher order terms contribute to $\Delta$, which is negligible. The arguments for restricted partitions and line graphs are similar, starting from the identities obtained from Proposition 5 and (18)

$$U(x) = e^{-1}\sum_{m=0}^{\infty}\frac{1}{m!}(1+x)^{\binom{m}{2}} e^{x/2}$$

and

$$L(x) = e^{-1}\sum_{m=0}^{\infty}\frac{1}{m!}(1+x)^{\binom{m}{2}} e^{x/2-x^3/6-x^4/4-x^5/8-x^6/48}.$$

In each case only the contribution of the constant term of the Taylor expansion of the exponential is 1 and the remaining terms contribute to a quantity like $\Delta$ which is asymptotically insignificant. ■

## Acknowledgement

We thank a referee for an observation that led to the correction of Proposition 5.

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