Asymptotic enumeration of incidence matrices

Peter Cameron, Thomas Prellberg and Dudley Stark

School of Mathematical Sciences, Queen Mary, University of London, Mile End Road, London E1 4NS, United Kingdom

E-mail: p.j.cameron, t.prellberg, d.stark@qmul.ac.uk

Abstract. We discuss the problem of counting *incidence matrices*, i.e. zero-one matrices with no zero rows or columns. Using different approaches we give three different proofs for the leading asymptotics for the number of matrices with n ones as $n \to \infty$. We also give refined results for the asymptotic number of $i \times j$ incidence matrices with n ones.

1. Introduction

We call an *incidence matrix* a zero-one matrix with no zero rows and columns and denote by $F(n)$ the number of incidence matrices with exactly n ones, where $n \in \mathbb{N}$. For example, the four incidence matrices with $n = 2$ are

$$\begin{pmatrix} 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The first few terms of the sequence $F(n)$ for $n \in \mathbb{N}$ are

$$1, 4, 24, 196, 2016, 24976, 361792, 5997872, 111969552, \ldots$$

taken from the *On-Line Encyclopedia of Integer Sequences* [8], where this appears as sequence A101370. For convenience, we further define $F(0) = 1$.

If one imposes additional symmetries or constraints, such as allowing or prohibiting repeated rows or columns, or considering equivalence classes under row or column permutations, one is led to many different enumeration problems, as discussed in [5].

The counting problem can be interpreted in a surprisingly rich variety of different ways, leading to rather different mathematical approaches.

- **Counting hypergraphs by weight**
  Given a hypergraph on the vertex set $\{x_1, \ldots, x_r\}$, with edges $E_1, \ldots, E_s$ (each a non-empty set of vertices), the *incidence matrix* $A = (a_{ij})$ is the matrix with (i, j) entry 1 if $x_i \in E_j$, and 0 otherwise. The *weight* of the hypergraph is the sum of the cardinalities of the edges. Thus $F(n)$ is the number of vertex- and edge-labelled hypergraphs of weight n with no isolated vertices, up to isomorphism.
- **Counting bipartite graphs by edges**
  Given a zero-one matrix $A = (A_{ij})$, there is a (simple) bipartite graph whose vertices are indexed by the rows and columns of A, with an edge from r_i to c_j if $A_{ij} = 1$. The graph has a distinguished bipartite block (consisting of the rows). Thus, $F(n)$ counts labelled bipartite graphs with n edges and a distinguished bipartite block.

- **Counting pairs of partitions, or binary block designs**
  A *block design* is a set of *plots* carrying two partitions, the *treatment partition* and the *block partition*. It is said to be *binary* if no two distinct points lie in the same part of both partitions; that is, if the meet of the two partitions is the partition into singletons. Thus, $F(n)$ is the number of binary block designs with n plots and labelled treatments and blocks.
- **Counting orbits of certain permutation groups**
  A permutation group G on a set X is *oligomorphic* if the number $F_n^*(G)$ of orbits of G on n-tuples of elements of X is finite for all n. Equivalently, the number $F_n(G)$ of orbits on ordered n-tuples of distinct elements of X is finite, and the number $f_n(G)$ of orbits on n-element subsets of X is finite, for all n. These numbers satisfy various conditions, including the following:
  - $F_n^*(G) = \sum_{k=1}^n S(n,k)F_k(G)$ and its inverse $F_n(G) = \sum_{k=1}^n s(n,k)F_k^*(G)$, with $s(n,k)$ and $S(n,k)$ Stirling numbers of the first and second kind, respectively;
  - $f_n(G) \le F_n(G) \le n!f_n(G)$, where the right-hand bound is attained if and only if the group induced on a finite set by its setwise stabiliser is trivial.

  For example, let A be the group of all order-preserving permutations of the rational numbers. Then $f_n(A) = 1$ and $F_n(A) = n!$.
  Now if H and K are permutation groups on sets X and Y, then the direct product $H \times K$ acts coordinatewise on the Cartesian product $X \times Y$. It is easy to see that $F_n^*(H \times K) = F_n^*(H)F_n^*(K)$. Let (x_1, y_1), …, (x_n, y_n) be n distinct elements of $X \times Y$. If both X and Y are ordered, then the set of n pairs can be described by a matrix with n ones in these positions, where the rows and columns of the matrix are indexed by the sets $\{x_1, \dots, x_n\}$ and $\{y_1, \dots, y_n\}$ respectively (in the appropriate order). Thus
  $$F(n) = f_n(A \times A).$$
  Discussion of this "product action" can be found in [4].

For an extended discussion of these interpretations see [5]. For instance, when considering hypergraphs it is more natural to consider the unlabelled problem, which leads to identification of incidence matrices which are equivalent under permutation of rows or columns. Also, forbidding repeated rows corresponds to counting simple hypergraphs with no repeated edges.

2. The asymptotics of $F(n)$

It is possible to compute $F(n)$ explicitly. For fixed n, let $m_{ij}(n)$ be the number of $i \times j$ matrices with n ones (and no zero rows or columns). We set $m_{00}(0) = 1$ and $F(0) = 1$. Then

$$\sum_{i\le k}\sum_{j\le l}\binom{k}{i}\binom{l}{j}m_{ij}(n) = \binom{kl}{n}, \tag{2.1}$$

so by Möbius inversion,

$$m_{kl}(n) = \sum_{i\le k}\sum_{j\le l}(-1)^{k+l-i-j}\binom{k}{i}\binom{l}{j}\binom{ij}{n}, \tag{2.2}$$

and then

$$F(n) = \sum_{i\le n}\sum_{j\le n} m_{ij}(n). \tag{2.3}$$

For sequence a_n, b_n, we use the notation $a_n \sim b_n$ to mean $\lim_{n\to\infty} a_n/b_n = 1$. It is clear from the argument above that

$$F(n) \le \binom{n^2}{n} \sim \frac{1}{\sqrt{2\pi n}}(n\mathrm{e})^n,$$

and of course considering permutation matrices shows that

$$F(n) \geq n! \sim \sqrt{2\pi n}\left(\frac{n}{\mathrm{e}}\right)^n .$$

Theorem 1.

$$F(n) \sim \frac{n!}{4} e^{-\frac{1}{2}(\log 2)^2} \frac{1}{(\log 2)^{2n+2}} .$$

We remark that for $n = 10$, the asymptotic expression is about 2.5% less than the actual value of 2324081728.

As announced in [5], we have three different proofs of Theorem 1. The first proof employs pairs of random preorders and a probabilistic argument, the second proof uses counting of orbits of products of permutation groups, and the third proof employs a surprisingly simple identity.

First Proof: This proof uses a procedure which, when successful, generates an incidence matrix uniformly at random from all incidence matrices. The probability of success can be estimated and the asymptotic formula for $F(n)$ results.

Let R be a binary relation on a set X. We say R is *reflexive* if $(x, x) \in R$ for all $x \in X$. We say R is *transitive* if $(x, y) \in R$ and $(y, z) \in R$ implies $(x, z) \in R$. A *partial preorder* is a relation R on X which is reflexive and transitive. A relation R is said to satisfy *trichotomy* if, for any $x, y \in X$, one of the cases $(x, y) \in R$, $x = y$, or $(y, x) \in R$ holds. We say that R is a *preorder* if it is a partial preorder that satisfies trichotomy. The members of X are said to be the *elements* of the preorder.

A relation R is *antisymmetric* if, whenever $(x, y) \in R$ and $(y, x) \in R$ both hold, then $x = y$. A relation R on X is a *partial order* if it is reflexive, transitive, and antisymmetric. A relation is a *total order*, if it is a partial order which satisfies trichotomy. Given a partial preorder R on X, define a new relation S on X by the rule that $(x, y) \in S$ if and only if both (x, y) and (y, x) belong to R. Then S is an equivalence relation. Moreover, R induces a partial order $\overline{R}$ on the set of equivalence classes of S in a natural way: if $(x, y) \in R$, then $(\overline{x}, \overline{y}) \in \overline{R}$, where $\overline{x}$ is the S-equivalence class containing x and similarly for y. We will call an S-equivalence class a *block*. If R is a preorder, then the relation $\overline{R}$ on the equivalence classes of S is a total order. See Section 3.8 and question 19 of Section 3.13 in [3] for more on the above definitions and results. Random preorders are considered in [6].

Given a preorder on elements $[n] := \{1, 2, \ldots, n\}$ with K blocks, let $B_1, B_2, \ldots, B_K$ denote the blocks of the preorder. Generate two preorders uniformly at random, $B_1, B_2, \ldots, B_K$ and $B'_1, B'_2, \ldots, B'_L$. For each $1 \leq i < j \leq n$, define the event $D_{i,j}$ to be

$$D_{i,j} = \{\text{for each of the two preorders } i \text{ and } j \text{ are in the same block}\}.$$

Furthermore, define

$$W = \sum_{1 \leq i < j \leq n} I_{D_{i,j}},$$

where the indicator random variables are defined by

$$I_{D_{i,j}} = \begin{cases} 1 & \text{if } D_{i,j} \text{ occurs;} \\ 0 & \text{otherwise.} \end{cases}$$

If $W = 0$, then the procedure is successful, in which case $B_k \cap B'_l$ consists of either 0 or 1 elements for each $1 \leq k \leq K$ and $1 \leq l \leq L$. If the procedure is successful, then we define the corresponding $K \times L$ incidence matrix A by

$$A_{k,l} = \begin{cases} 1 & \text{if } B_k \cap B'_l \neq \emptyset; \\ 0 & \text{if } B_k \cap B'_l = \emptyset. \end{cases}$$

It is easy to check that the above definition of A in fact produces an incidence matrix and that each incidence matrix occurs in $n!$ different ways by the construction. It follows that

$$F(n) = \frac{P(n)^2 \mathbb{P}(W=0)}{n!},$$

where $P(n)$ is the number of preorders on n elements if $n \geq 1$ and $P(0) = 1$.

It is known (see [1], for example) that the exponential generating function of $P(n)$ is

$$\sum_{n=0}^{\infty} \frac{P(n)}{n!} z^n = \frac{1}{2 - e^z}. \tag{2.4}$$

The preceding equality implies that $P(n)$ has asymptotics given by

$$P(n) \sim \frac{n!}{2} \left(\frac{1}{\log 2} \right)^{n+1}. \tag{2.5}$$

It remains to find the asymptotics of $\mathbb{P}(W = 0)$.

The rth falling moment of W is

$$\begin{aligned} \mathbb{E}(W)_r &= \mathbb{E}W(W-1)\cdots(W-r+1) \\ &= \mathbb{E}\left(\sum_{\text{pairs } (i_s, j_s) \text{ different}} I_{i_1,j_1} \cdots I_{i_r,j_r} \right) && (2.6) \\ &= \mathbb{E}\left(\sum_{\text{all } i_s \text{ and } j_s \text{ different}} I_{i_1,j_1} \cdots I_{i_r,j_r} \right) + \mathbb{E}\left(\sum^{*} I_{i_1,j_1} \cdots I_{i_r,j_r} \right), && (2.7) \end{aligned}$$

with $\sum^{*}$ defined to be the sum with all pairs (i_s, j_s) different, but not all i_s, j_s different.

First we find the asymptotics of the first term in (2.7). For given sequences $i_1, i_2, \ldots, i_r, j_1, j_2, \ldots, j_r$, the expectation $\mathbb{E}(I_{i_1,j_1} \cdots I_{i_r,j_r})$ is the number of ways of forming two preorders on the set of elements $[n] \backslash \{j_1, j_2, \ldots, j_r\}$ and then for each s adding the element j_s to the block containing i_s in both preorders (which ensures that D_{i_s,j_s} occurs for each s) and dividing the result by $P(n)^2$. Since the number of ways of choosing $i_1, i_2, \ldots, i_r, j_1, j_2, \ldots, j_r$ equals $\frac{n!}{2^r(n-2r)!}$, This gives

$$\begin{aligned} \mathbb{E}\left(\sum_{\text{all } i_s \text{ and } j_s \text{ different}} I_{i_1,j_1} \cdots I_{i_r,j_r} \right) &= \frac{n!}{2^r(n-2r)!} \frac{P(n-r)^2}{P(n)^2} \\ &\sim \left(\frac{(\log 2)^2}{2} \right)^r, \end{aligned}$$

where we have used (2.5).

The second term is bounded in the following way. For each sequence $(i_1, j_1), (i_2, j_2), \ldots, (i_s, j_s)$ in the second term we form the graph G on vertices $\bigcup_{s=1}^{r} \{i_s, j_s\}$ with edges $\bigcup_{s=1}^{r} \{\{i_s, j_s\}\}$. Consider the unlabelled graph G' corresponding to G consisting of v vertices and c components. The number of ways of labelling G' to form G is bounded by n^v. The number of preorders corresponding to this labelling is $P(n - v + c)$ because we form a preorder on $n - v + c$ vertices after which the vertices in the connected component of G containing a particular vertex get added to that block. Therefore, we have

$$\begin{aligned} \mathbb{E}\left(\sum^{*} I_{i_1,j_1} \cdots I_{i_r,j_r} \right) &\leq \sum_{G'} n^v \frac{P(n-v+c)^2}{P(n)^2} \\ &= \sum_{G'} O\left(n^{2c-v}\right), \end{aligned}$$

where the constant in $O\left(n^{2c-v}\right)$ is uniform over all G' because $v \le 2r$. Since at least one vertex is adjacent to more than one edge, the graph G is not a perfect matching. Furthermore, each component of G contains at least two vertices. It follows that $2c < v$ and, as a result,

$$\mathbb{E}\left(\sum^{*} I_{i_1,j_1} \cdots I_{i_r,j_r}\right) = O\left(n^{-1}\right).$$

The preceding analysis shows that

$$\mathbb{E}(W)_r \sim \left(\frac{(\log 2)^2}{2}\right)^r$$

for each $r \ge 0$. The method of moments implies that the distribution converges weakly to the distribution of a Poisson$((\log 2)^2/2)$ distributed random variable and therefore

$$\mathbb{P}(W = 0) \sim \exp\left(-\frac{(\log 2)^2}{2}\right). \tag{2.8}$$

□

Second Proof: We now give a proof using product actions of groups, as discussed in the introduction. First of all, this approach leads to a different and simpler expression than (2.3) for $F(n)$ as a sum of terms of alternating sign.

Proposition 2.

$$F(n) = \frac{1}{n!}\sum_{k=1}^{n} s(n,k)P(k)^2,$$

where

$$P(n) = \sum_{k=1}^{n} S(n,k)k!$$

is the number of (total) preorders of $\{1,\ldots,n\}$*, and* $s(n,k)$ *and* $S(n,k)$ *are Stirling numbers of the first and second kind respectively.*

This is proved in [4], but can be seen as follows. Using the group A of all order-preserving permutations acting on $\mathbb{Q}$, we consider the direct product $A \times A$ acting on $\mathbb{Q} \times \mathbb{Q}$. We have $F_n(A) = n!$, whence it follows that $F^*_n(A) = \sum_{k=1}^{n} S(n,k)k! = P(n)$. Thus

$$F^*_n(A \times A) = P(n)^2 = \sum_{k=1}^{n} S(n,k)F_k(A \times A),$$

and so the inverse relation between the two kinds of Stirling numbers gives

$$F_n(A \times A) = \sum_{k=1}^{n} s(n,k)P(k)^2.$$

Finally, the group $A \times A$ has the property that the setwise stabiliser of a finite set fixes it pointwise, and so $f_n(A \times A) = F_n(A \times A)/n!$. □

The reader may find an alternative proof of Proposition 2 at the end of the paper.

We now replace $P(k)$ by the asymptotic form (2.5) given earlier. For $k \ge n/2$, the difference is exponentially small; and we will show below that the contribution of the terms with $k < n/2$ is negligible, so it suffices to note that the error we make is smaller than the approximated term.

So let

$$F'(n) = \frac{1}{4} \cdot \frac{1}{n!} \sum_{k=1}^{n} s(n,k)(k!)^2 c^{k+1},$$

where $c = 1/(\log 2)^2$ is as in the statement of the theorem. As we have argued, $F(n) \sim F'(n)$.

Now $(-1)^{n-k} s(n,k)$ is the number of permutations in the symmetric group S_n which have k cycles. So we can write the formula for $F'(n)$ as a sum over S_n, where the term corresponding to a permutation with k cycles is $(-1)^{n-k}(k!)^2 c^{k+1}$. In particular, the identity permutation gives us a contribution

$$g(n) = \frac{1}{4}\, n!\, c^{n+1},$$

and we have to show that $F'(n) \sim Cg(n)$ as $n \to \infty$, where $C = \exp(-(\log 2)^2/2)$.

To prove this, we write $F'(n) = F'_1(n) + F'_2(n) + F'_3(n)$, where the three terms are sums over the following permutations:

F'_1: all involutions (permutations with $\sigma^2 = 1$);

F'_2: the remaining permutations with $k \geq \lceil n/2 \rceil$;

F'_3: the rest of S_n.

We argue that $F'_1(n) \sim Cg(n)$, while $F'_2(n), F'_3(n) = o(g(n))$.

Case F'_1: Let $l = n - k$. Now an involution with k cycles has l cycles of length 2 and $n - 2l$ fixed points; so $l \leq n/2$. The number of such permutations is

$$\binom{n}{2l} \frac{(2l)!}{2^l\, l!} = \frac{n(n-1)\cdots(n-2l+1)}{2^l\, l!}.$$

So

$$\begin{aligned} \frac{F'_1(n)}{g(n)} &= \sum_{l=0}^{\lfloor n/2 \rfloor} \frac{n(n-1)\cdots(n-2l+1)}{2^l\, l!} (-1)^l \frac{((n-l)!)^2}{(n!)^2} c^{-l} \\ &= \sum_{l=0}^{\lfloor n/2 \rfloor} \frac{1}{l!} \left(\frac{-1}{2c}\right)^l \frac{(n-l)\cdots(n-2l+1)}{n\cdots(n-l+1)}. \end{aligned}$$

Now

$$\sum_{l=0}^{\infty} \frac{1}{l!} \left(\frac{-1}{2c}\right)^l = \exp\left(\frac{-1}{2c}\right) = C,$$

so we have to show that the factor involving n makes no difference to the limit. Now this factor is always less than 1, so the series is absolutely convergent (and uniformly in n); so we can choose r large enough that the sum of r terms of each sequence is close to its limit. Then, since the factors tend to 1 as $n \to \infty$, for n large each of these r terms is close to its limit. So the assertion is true: that is, $F'_1(n) \sim Cg(n)$.

Case F'_2: A permutation which has $k = n - l$ cycles and is not an involution has at least $n - 2l + 1$ fixed points, and there are at most

$$\binom{n}{2l-1}(2l-1)! = n(n-1)\cdots(n-2l+2)$$

such permutations. So, ignoring signs,

$$\begin{aligned}\frac{F_2'(n)}{g(n)} &\le \sum_{l\ge 0} \frac{(n-l)(n-l-1)\cdots(n-2l+2)}{n(n-1)\cdots(n-l+1)}\, c^{-l} \\ &< \frac{1}{n}\cdot\frac{1}{1-c^{-1}},\end{aligned}$$

which is $O(1/n)$.

Case F_3': We simply observe that there are at most $n!$ such permutations, so

$$\frac{F_3'(n)}{g(n)} \le n! \sum_{k=1}^{\lfloor n/2\rfloor} \left(\frac{k!}{n!}\right)^2 .$$

Now $n!/(k!)^2 \ge \binom{n}{\lfloor n/2\rfloor} \ge (2-\epsilon)^n$ for large n, so this sum is $O((2-\epsilon)^{-n}n/2) = o(1)$ as $n\to\infty$. □

Third Proof: If one is interested in asymptotic enumeration of $F(n)$, formula (2.2), being a double sum over terms of alternating sign, is on first sight rather unsuitable for an asymptotic analysis. The expression in Proposition 2 is also an alternating sum. We present a derivation of the asymptotic form of $F(n)$ based on the following elegant and elementary identity, which gives $F(n)$ as a sum of positive terms. This identity and equation (2.2) were also derived in [7].

Proposition 3.

$$F(n) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{1}{2^{k+l+2}}\binom{kl}{n} . \tag{2.9}$$

Proof. Insert

$$1 = \sum_{k=i}^{\infty} \frac{1}{2^{k+1}}\binom{k}{i} = \sum_{l=j}^{\infty} \frac{1}{2^{l+1}}\binom{l}{j} \tag{2.10}$$

into (2.3) and resum using (2.1). □

We start the asymptotic analysis by rewriting (2.9) as

$$n!F(n) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{k^n}{2^{k+1}}\,\frac{l^n}{2^{l+1}}\,\frac{(kl)_n}{(kl)^n} , \tag{2.11}$$

where $(x)_n = x(x-1)\dots(x-n+1)$ is the falling factorial. Given the identity

$$P(n) = \sum_{k=0}^{\infty} \frac{k^n}{2^{k+1}} , \tag{2.12}$$

which follows from expanding (2.4), (2.11) is bounded above by $P(n)^2$, as the factor $(kl)_n/(kl)^n$ takes values in $[0,1]$.

For $n \le kl$, a straightforward expansion of the factor gives

$$\frac{(kl)_n}{(kl)^n} = \exp\left[-\sum_{j=1}^{\infty} \frac{B_{j+1}(n)-B_{j+1}(0)}{j(j+1)(kl)^j}\right] . \tag{2.13}$$

Here, we have used that $\sum_{k=0}^{n-1} k^j = (B_{j+1}(n) - B_{j+1}(0))/(j+1)$ where $B_j(x)$ is a Bernoulli polynomial. It follows that

$$\frac{(kl)_n}{(kl)^n} = e^{-\frac{n^2}{2kl}} \left(1 + O(n/kl) + O(n^3/(kl)^2)\right) \ . \tag{2.14}$$

(This argument will be presented more thoroughly for $(z)_n$ with complex-valued z in the next section.) The sum (2.11) is dominated by terms around $k = l = n/\log 2$, so that we expect the correction to give $e^{-(\log 2)^2/2}$, which in turn would imply $n!F(n) \sim P(n)^2 e^{-(\log 2)^2/2}$. The difference is given by

$$n!F(n) - P(n)^2 e^{-(\log 2)^2/2} = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{k^n}{2^{k+1}} \frac{l^n}{2^{l+1}} \left(\frac{(kl)_n}{(kl)^n} - e^{-(\log 2)^2/2}\right) \ . \tag{2.15}$$

To proceed we choose $m_0 < n/\log 2 < m_1$ and split the summation. We obtain

$$\begin{aligned} \left|n!F(n) - P(n)^2 e^{-(\log 2)^2/2}\right| \ &\leq \ \sum_{k=m_0}^{m_1}\sum_{l=m_0}^{m_1} \frac{k^n}{2^{k+1}} \frac{l^n}{2^{l+1}} \left|\frac{(kl)_n}{(kl)^n} - e^{-(\log 2)^2/2}\right| \\ &\quad + 2\sum_{k=0}^{\infty} \frac{k^n}{2^{k+1}} \left(\sum_{k=0}^{m_0-1} \frac{k^n}{2^{k+1}} + \sum_{k=m_1+1}^{\infty} \frac{k^n}{2^{k+1}}\right) \ . \end{aligned} \tag{2.16}$$

Specifying $m_0 = n/\log 2 - cn^\delta$ and $m_1 = n/\log 2 + cn^\delta$ for $1/2 < \delta < 1$ and $c > 0$, we use (2.14) to estimate

$$\begin{aligned} \frac{(kl)_n}{(kl)^n} - e^{-(\log 2)^2/2} \ &= \ e^{-\frac{n^2}{2kl}} \left(1 + O(n/kl) + O(n^3/(kl)^2)\right) - e^{-(\log 2)^2/2} \\ &= \ e^{-(\log 2)^2/2} \left(1 + O(n^{\delta-1})\right) \left(1 + O(n^{-1})\right) - e^{-(\log 2)^2/2} \\ &= \ O(n^{\delta-1}) \end{aligned}$$

for $m_0 \leq k, l \leq m_1$. This allows us to bound the first term in (2.16) by $P(n)^2 O(n^{\delta-1})$. To get a bound on the second term, we utilize the following lemma.

Lemma 4. *(a) For $K, n \in \mathbb{N}$ and $K < n/\log 2$,*

$$\sum_{k=0}^{K} \frac{k^n}{2^{k+1}} \leq \frac{K^n}{2^K} \frac{e^{n/K}}{2} \frac{1}{e^{n/K} - 2} \ . \tag{2.17}$$

(b) For $K, n \in \mathbb{N}$ and $K > n/\log 2$,

$$\sum_{k=K+1}^{\infty} \frac{k^n}{2^{k+1}} \leq \frac{K^n}{2^K} \frac{e^{n/K}}{2} \frac{1}{2 - e^{n/K}} \ . \tag{2.18}$$

Proof. Part (a) follows from the estimate

$$\sum_{k=0}^{K} \frac{k^n}{2^{k+1}} \ = \ \frac{K^n}{2^{K+1}} \sum_{k=0}^{K} \frac{(k/K)^n}{2^{k-K}} \leq \frac{K^n}{2^{K+1}} \sum_{k=0}^{K} 2^k (1 - k/K)^n \leq \frac{K^n}{2^{K+1}} \sum_{k=0}^{\infty} \left(2e^{-n/K}\right)^k$$

and part (b) similarly from

$$\sum_{k=K+1}^{\infty} \frac{k^n}{2^{k+1}} \ = \ \frac{K^n}{2^{K+1}} \sum_{k=K+1}^{\infty} \frac{(k/K)^n}{2^{k-K}} \leq \frac{K^n}{2^{K+1}} \sum_{k=1}^{\infty} \frac{(1 + k/K)^n}{2^k} \leq \frac{K^n}{2^{K+1}} \sum_{k=1}^{\infty} \left(\frac{e^{n/K}}{2}\right)^k \ .$$

□

For $K = n/\log 2 \mp cn^\delta$, we find

$$\frac{K^n}{2^K}\frac{e^{n/K}}{2}\frac{1}{|e^{n/K}-2|} = \frac{n^n e^{-n}}{(\log 2)^n} e^{-\alpha n^{2\delta-1}} O(n^{1-\delta}) = P(n) O\left(e^{-\alpha n^{2\delta-1}}\right)$$

where $\alpha = c^2(\log 2)^2/2$. Using Lemma 4, we therefore bound the second term in (2.16) by $P(n)^2 O\left(\exp(-\alpha n^{2\delta-1})\right)$. Altogether we find

$$n!F(n) - P(n)^2 e^{-(\log 2)^2/2} = P(n)^2\left(O(n^{\delta-1}) + O\left(\exp(-\alpha n^{2\delta-1})\right)\right)$$

and as $1/2 < \delta < 1$, we have

$$\lim_{n\to\infty}\frac{n!F(n)}{P(n)^2} = e^{-(\log 2)^2/2}$$

which completes the proof. □

3. The asymptotics of $m_{kl}(n)$

In this section we present results on the number of incidence matrices with specified numbers of rows and columns. To obtain the desired asymptotic form of $m_{kl}(n)$ from eqn. (2.2), we need to deal with the challenge that summing over large terms with alternating signs can lead to enormous cancellations. Fortunately, there is a standard trick using the calculus of residues.

Proposition 5.

$$m_{kl}(n) = \frac{k!l!}{n!}\mathrm{Res}\left(\frac{(st)_n}{(s)_{k+1}(t)_{l+1}};\ s=\infty,\ t=\infty\right). \tag{3.1}$$

Proof. Using the fact that

$$\mathrm{Res}(\Gamma(s),\ s=-m) = \frac{(-1)^m}{m!},$$

we write

$$\begin{aligned} m_{kl}(n) &= \frac{k!l!}{n!}(-1)^{k+l}\sum_{i=0}^{k}\sum_{j=0}^{l}\frac{(-1)^i}{i!}\frac{(-1)^j}{j!}\frac{(ij)_n}{(k-i)!(l-j)!} \\ &= \frac{k!l!}{n!}\frac{(-1)^{k+l}}{(2\pi i)^2}\int_{\mathcal{C}_{[-k,0]}} ds \int_{\mathcal{C}_{[-l,0]}} dt\ \frac{(st)_n}{(s+k)_{k+1}(t+l)_{l+1}} \\ &= \frac{k!l!}{n!}\frac{1}{(2\pi i)^2}\int_{\mathcal{C}_{[0,k]}} ds \int_{\mathcal{C}_{[0,l]}} dt\ \frac{(st)_n}{(s)_{k+1}(t)_{l+1}}. \end{aligned}$$

Here, the contours $\mathcal{C}_{[a,b]}$ encircle the (real) interval $[a,b]$ counterclockwise. As the integrand is a rational function, the contour integrals can be expressed as residues at infinity. □

This formulation allows us to do the asymptotic analysis via saddle point analysis of a contour integral. We consider the scaling behaviour of $m_{k,l}(n)$ as $n\to\infty$ with $k=\kappa n$ and $l=\lambda n$ for fixed λ, κ. As a preparation, we state the following Lemma.

Lemma 6. *Let $n\in\mathbb{N}$ and $z\in\mathbb{C}$ with $|z|>n$. Then*

$$(z)_n = z^n\exp\left[-\sum_{j=1}^{\infty}\frac{B_{j+1}(n)-B_{j+1}(0)}{j(j+1)z^j}\right]. \tag{3.2}$$

Moreover, we have the asymptotic expansion

$$\log(z)_n \sim (z+1/2)\log z - (z-n+1/2)\log(z-n) - n + \sum_{k=1}^{\infty} \frac{B_{2k}}{2k(2k-1)} \left(\frac{1}{z^{2k-1}} - \frac{1}{(z-n)^{2k-1}} \right) . \tag{3.3}$$

as $|z-n|$ and $|z|$ tend to infinity. Here $B_n(x)$ is the n-th Bernoulli polynomial and $B_n = B_n(0)$ the n-th Bernoulli number.

Proof. We write

$$(z)_n = z^n \prod_{k=0}^{n-1} \left(1 - \frac{k}{z}\right) .$$

For $|z| > n$ we take logarithms and expand $\log(1-k/z)$ in k/z. Exchanging the order of summation and using that

$$\sum_{k=0}^{n-1} k^j = (B_{j+1}(n) - B_{j+1}(0))/(j+1)$$

gives (3.2). One can obtain (3.3) by substituting $B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k}$ and exchanging the order of summation again. As the double sum here is not absolutely convergent, the resulting series (3.3) cannot be expected to be convergent. Instead of labouring to prove that one still arrives at an asymptotic expansion, we point out that the result is just the difference between the Stirling series for $\log(z!)$ and $\log((z-n)!)$ and argue that standard arguments used in the derivation of the Stirling series also lead to (3.3). In contrast with the Stirling series, which is valid for $|\arg(z)| < \pi$, the validity of this expansion is not restricted to a sector of the complex plane. □

From (3.2) we obtain that

$$(z)_n = z^n e^{-\frac{n^2}{2z}} \left(1 + O(n/|z|) + O(n^3/|z|^2)\right) , \tag{3.4}$$

whereas from (3.3) we obtain that

$$(z)_n = (z-n)^{n-z-1/2} z^{z+1/2} e^{-n} \left(1 + O(1/|z|) + O(1/|z-n|)\right) . \tag{3.5}$$

We now state the main theorem of this section.

Theorem 7. *For fixed $\sigma, \tau > 0$,*

$$m_{kl}(n) \sim \frac{n^{2n}}{n!} e^{nw(\sigma)} v(\sigma) e^{nw(\tau)} v(\tau) e^{-\frac{1}{2\sigma\tau}} \tag{3.6}$$

with

$$w(x) = x(1-e^{-1/x})\log(1-e^{-1/x}) + \log x - e^{-1/x} ,$$
$$v(x) = \sqrt{\frac{x(1-e^{-1/x})}{x(1-e^{-1/x}) - e^{-1/x}}}$$

and $k = n\sigma(1-e^{-1/\sigma})$, $l = n\tau(1-e^{-1/\tau})$.

Proof. In order to evaluate the integral

$$\frac{n!}{k!l!}m_{kl}(n) = \frac{1}{(2\pi i)^2}\int_{\mathcal{C}_{[0,k]}} ds \int_{\mathcal{C}_{[0,l]}} dt \, \frac{(st)_n}{(s)_{k+1}(t)_{l+1}}$$

asymptotically, we approximate the integrand uniformly using (3.4) and (3.5) on contours satisfying $|s| = R_s > k$ and $|t| = R_t > l$. We find

$$\frac{n!}{k!l!}m_{kl}(n) = \frac{e^{k+l}}{(2\pi i)^2}\int_{|s|=R_s} ds \int_{|t|=R_t} dt \, \frac{(s-k)^{s-k-1/2}}{s^{s-n+1/2}}\frac{(t-l)^{t-l-1/2}}{t^{t-n+1/2}}e^{-\frac{n^2}{2st}}(1+R)$$

where

$$R = O(n/R_sR_t) + O(n^3/(R_sR_t)^2) + O(1/R_s) + O(1/R_t) + O(1/|R_s-k|) + O(1/|R_t-l|) \ .$$

Substituting $s = n\sigma$, $t = n\tau$, $k = n\kappa$, $l = n\lambda$, this simplifies to $R = O(1/n)$ and we arrive at

$$m_{kl}(n) \sim \kappa^{n\kappa}\lambda^{n\lambda}n^n e^n \frac{\sqrt{2\pi\kappa\lambda n}}{(2\pi i)^2}\int_{|\sigma|=\rho_\sigma} d\sigma \int_{|\tau|=\rho_\tau} d\tau \; e^{nf(\sigma,\kappa)}g(\sigma,\kappa)e^{nf(\tau,\lambda)}g(\tau,\lambda)e^{-\frac{1}{2\sigma\tau}}$$

with $R_s = n\rho_\sigma$ and $R_t = n\rho_\tau$ and

$$f(x,y) = (x-y)\log(x-y) - (x-1)\log x \quad \text{and} \quad g(x,y) = ((x-y)x)^{-1/2} \ .$$

As n tends to infinity, each integration is dominated by saddle points σ_s and τ_s on the positive real axis. The saddle point equation is

$$0 = \partial_1 f(\sigma_s,\kappa) = \log\left(1 - \frac{\kappa}{\sigma_s}\right) + \frac{1}{\sigma_s} \ ,$$

(here, ∂_1 denotes taking the derivative with respect to the first argument) and an identical expression for τ_s. There exists a unique positive solution $\sigma_s(\kappa)$, and a standard saddle-point evaluation (see e.g. [2]) gives

$$m_{kl}(n) \sim \kappa^{n\kappa}\lambda^{n\lambda}n^n e^n \frac{\sqrt{2\pi\kappa\lambda n}}{(2\pi)^2}e^{nf(\sigma_s,\kappa)}g(\sigma_s,\kappa)e^{nf(\tau_s,\lambda)}g(\tau_s,\lambda)e^{-\frac{1}{2\sigma_s\tau_s}}\frac{2\pi}{\sqrt{\partial_1^2 f(\sigma_s,\kappa)\partial_1^2 f(\tau_s,\lambda)}}$$

which simplifies to the desired result. □

Theorem 7 can be used for a fourth proof of Theorem 1 via an asymptotic evaluation of the sum over $m_{kl}(n)$. The sum is dominated by terms near $\sigma_s = \tau_s = 1/\log 2$ from whence it follows that the distribution has a peak about $k_s = l_s = n/(2\log 2)$.

We conclude this paper with giving an identity for $m_{kl}(n)$ which is a refinement of Proposition 2.

Proposition 8.

$$m_{kl}(n) = \frac{k!l!}{n!}\sum_{r=1}^{n} s(n,r)S(r,k)S(r,l) \tag{3.7}$$

Proof. Using $(ij)_n = \sum_{r=1}^n s(n,r)(ij)^r$, we write (2.2) as

$$m_{kl}(n) = \frac{1}{n!}\sum_{r=1}^{n} s(n,r)\sum_{i\le k}\sum_{j\le l}(-1)^{k+l-i-j}\binom{k}{i}\binom{l}{j}(ij)^r$$

and resum using $k!S(r,k) = \sum_{i\le k}(-1)^{k-i}\binom{k}{i}i^r$. □

Inversion of (3.7) gives

$$k!S(n,k)l!S(n,l) = \sum_{r=1}^{n} r!S(n,r)m_{kl}(r) \tag{3.8}$$

which has a straightforward combinatorial interpretation, as $r!S(n,r)$ is the number of preorders of n elements into r blocks. The left hand side of (3.8) is just the number of ways of choosing two preorders of an n-set into k and l blocks, respectively. The right hand side of (3.8) corresponds to counting the number of ways in which elements of an n-set can be distributed into r cells of a $k \times l$-array, where the cells are given by $k \times l$-incidence matrices with r ones, for arbitrary r.

Summing (3.7) over k and l provides another proof of Proposition 2. Proposition 8 could also be used as a basis for Theorem 7. We leave this as an exercise for the reader.

References

[1] Barthelemy J P 1980 An asymptotic equivalent for the number of total preorders on a finite set *Discrete Math.* **29** 311–3
[2] Bleistein N and Handelsman R A 1986 *Asymptotic Expansions of Integrals* (New York: Dover)
[3] Cameron P J 1994 *Combinatorics: Topics, Techniques, Algorithms* (Cambridge: Cambridge University Press)
[4] Cameron P J, Gewurz D A and Merola F 2004 Product action *Discrete Math.* (in press)
[5] Cameron P J, Prellberg T and Stark D 2005 Asymptotics for incidence matrix classes *Preprint*
[6] Cameron P J and Stark D 2005 Random Preorders *Preprint*
[7] Maia M and Mendez M 2005 *Preprint* arXiv:math.CO/0503436
[8] Sloane N J A (ed.) *The On-Line Encyclopedia of Integer Sequences* `http://www.research.att.com/~njas/sequences/`