Multifractals, Thermodynamic Formalism and Intermittency

Revised Honours Thesis

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Chapter 1

Introduction

We first present an overview of the basic concept of multifractals and the various methods used for analysing such entities. The $f(\alpha)$ and the D_q spectra are presented as the most natural ways of describing the distributions of scaling exponents. These are shown to be equivalent in the sense that they are related by the well known Legendre Transform. The Thermodynamic Formalism is next introduced and is shown to also contain the same information about a multifractal as the $f(\alpha)$ and the D_q spectra.

The notion of a Transfer Operator is developed for the analysis of point sets arranged on trees as per the work of Feigenbaum et al [3]. An equivalence between the Thermodynamic Potentials of the previous section and the leading eigenvalue of the operator is established. The multifractal nature of a set can then be investigated through an eigenvalue calculation.

We then use the above theory to examine simple maps that possess intermittent behaviour when iterated. The notion of invariant density is discussed and shown to be inadequate for the description of the long term statistical behaviour of intermittent maps which are characterised by an 'indifferent fixed point'. A fixed point is said to be indifferent if iterates of local points move away 'extremely slowly'; this slowness is brought about by the fact that such maps have a slope of 1 at the fixed point where it can therefore be neither expanding nor contracting. The Thermodynamic Formalism of the previous section is seen to be a generalisation of the invariant density approach and is then applied to the Farey map and its piecewise linear version.

Another important quality of a map is 'how long' the long term behaviour takes to set in. This can be determined by observation of the structure of the eigenvalue spectrum of the transfer operator. The speed of convergence will naturally depend on the dominance of this operator's leading eigenvalue. Note that the existence of a spectral gap has been proven by Prellberg [6] but there are no analytic results with regards to its size. In the language of statistical mechanics, this is an investigation into the decay of correlation lengths.

It turns out that the indifferent fixed point and hence intermittency implies the existence of a phase transition in the thermodynamic potential. The multifractal involved is produced by iteration of the inverse of the map. A Modified Transfer Operator is also introduced in order to further simplify the problem of the indifferent fixed point as per Prellberg [6].

Finally, it should be indicated that rather than relying on rigorous mathematics, this paper will hopefully present ideas and arguments in a more intuitive way.

Chapter 2

The Concept of Multifractals

2.1 Fractals

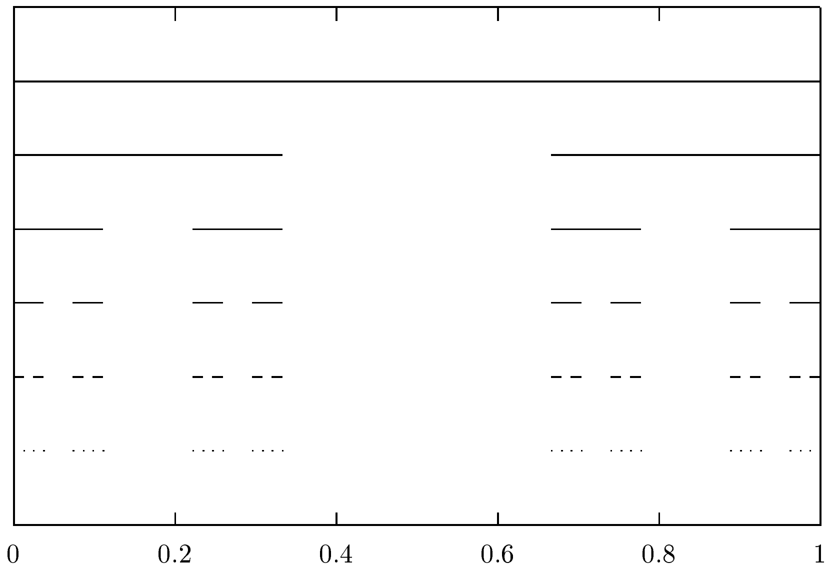
First of all, there is no hard mathematical definition of the word fractal. This is not a problem, as the objects that are generally recognised to be fractal are inherently hard to categorize. Indeed, Falconer [2] analogises the difficulty of rigorously defining a fractal to that of a biologist trying to pin down a definition of life. Certainly, any set definition would most probably exclude some example of a fractal or 'life'.

In lieu of a formal definition, we present the following which is a selection of features commonly found in fractal objects. It is by no means a complete list, nor should a fractal have all of those mentioned but it does provide a general overview of fractal characteristics. These are based on criteria suggested by Falconer [2]:

1. The existence of detail at any scale as well as the replication of similar patterns at all levels of magnification.
2. The failure of traditional geometry to adequately describe the object. The existence of complexity beyond that which can be achieved by a collection of simple shapes.
3. The existence of a very simple method of construction. This is often a recursive process.
4. The various fractal dimensions which are mentioned below being greater than the

topological dimension. (The *topological dimension* of a set is always an integer and is 0 if it is totally disconnected, 1 if the intersection of the set and the boundary of arbitrarily small neighbourhoods around any point, where it is not the null set, is of dimension 0 etc. [2].)

Figure 2.1: Construction of the Cantor set



A simple example is the ubiquitous Cantor set, see 2.1. This is constructed from the unit interval by removing the middle third of each interval at each stage. The number of elements grows like 2^n while their lengths vanish like 3^{-n}. The limit of this process is thus a set of uncountably many points, a kind of dust. Note that the topological dimension of the Cantor set would be 0 while, as will be shown below, other measures of dimension give non-zero values. It is self similar: every interval that appears in the construction may be blown up to the size of the unit interval where it will be identical to the original set.

The recent discovery by Mandelbrot [5] of fractal objects, formerly disregarded as 'pathological' or noise, and the fact that they have quantifiable characteristics has led to an enormous effort in the identification and measurement of fractal properties. Fractals, in comparison with circles and lines and squares which are the building blocks of traditional geometry, are the patterns of nature as we observe it. Coastlines, lungs, trees, the path of a falling leaf and household sponges are all fractal objects when considered within their relevant length scales.

2.1.1 Dimensions

One of the simplest measurements used to characterise a fractal is the box counting dimension (BCD). This is often loosely referred to as the fractal dimension. In our tackling of multifractals we will use the principle of box counting, so it is instructive to include a description at this point. Consider an object, which has a physical length L , that may be embedded in an N-dimensional Euclidean space, where N is the lowest such dimension of embedding. Now imagine the object to be in a grid of N-d hypercubes of side $l < L$. We calculate the number of boxes that intersect with the object, $N(l, L)$. Naturally, $N(l, L)$ will vary as l or L is varied. However, it clearly cannot depend on the absolute value of either but rather the quantity $\varepsilon = l/L$. Indeed, we expect $N(l, L) \equiv N(\varepsilon) \sim \varepsilon^{-D_0}$ [9].

In comparison to the BCD, there is the Hausdorff dimension (HD). This is really a more general approach to the idea of covering a set with cells and counting the intersecting ones. To calculate the HD, consider all possible collections of sets that cover an object (i.e. the intersection of all the sets and the object is the object itself). All sets in a collection must have a diameter $< \delta$. Again, $N(\delta)$ is calculated as δ is taken to zero. Clearly, the BCD is greater than or equal to the HD and in general, we find that the BCD and the HD are exactly equivalent, except in some more bizarre cases. Certainly, the HD is a most natural measure of dimension as it allows for sets that fit the form of the fractal in a precise way. The BCD is really an easy way of implementing the concept of the HD, and really a much more natural method for computers.

An important observation is that for any object, the above methods of determining dimension merely give the highest dimension present. Consider a balloon on a string. The BCD and the HD will be 2 (the balloon is locally 2-dimensional) even though there is a 1-dimensional structure present. As we shall see, it is possible to mathematically 'detect' the string on the balloon using the techniques associated with multifractals.

This is a simple example of geometric multifractality but multifractality extends beyond purely geometric notions.

In order to demonstrate the calculation of the BCD we consider the Cantor set 2.1. Dividing the unit interval into 'boxes' of length $\varepsilon = 3^{-n}$, it is clear from the construction that $N(\varepsilon) = 2^n$ such boxes cover the set completely. Rewriting $N(\varepsilon)$ as $(3^{-n})^{-\log 2/\log 3} = \varepsilon^{-\log 2/\log 3}$ we can pick off D_0 as $log 2/\log 3$.

2.2 Multifractals

2.2.1 Measure Theory

The recently developed concept of 'multifractality' refers to the behaviour of appropriate measures on geometric objects. Such entities in fact possess a spectrum of scaling exponents rather than a single one like a simple fractal and this will be regarded as the essential feature of a multifractal.[1] In the following, we will generally follow the work of Tél [9].

The added component of a measure means that the analysis is more than just that of a geometrical nature. Indeed, the support may have no fractal structure at all but, with a particular measure, would be considered to be a multifractal. However, if the measure is proportional to a constant mass distribution, the analysis will yield information about the geometric properties of the support itself.

Two examples of measures that have been experimentally observed to produce multifractal behaviour are: growth probability for aggregation processes such as crystal formation and the voltage distribution on a random resistor network. [9].

Two constructions used to express these scaling exponents are now presented, followed by an example to clarify the discussion. The connection between multifractals and thermodynamic systems is then developed in the final section of this chapter.

[1] As for fractals, a precise definition of a multifractal does not really exist in the literature.

2.2.2 The $f(\alpha)$ Spectrum

We begin by examining the properties of the following quantity [9]:

$$\Gamma_q(l, L) \equiv \sum_i \mu_i^q \tag{2.1}$$

where μ_i denotes the measure of the i^{th} box for a uniform partition of the object into boxes of dimension $l \ll L$ where L is the overall size of the object. The measure of a box with no intersection with the object is defined to be zero. Hence, for $q = 0$ we are merely counting the number of boxes intesecting the object which is the $N(\varepsilon)$ defined for the BCD in the previous section, i.e. $\Gamma_0 \equiv N(\varepsilon) \sim \varepsilon^{-D_0}$. Consider now the behaviour for arbitrary q of $\Gamma_q(l, L)$ as the parameter $\varepsilon \equiv l/L$ goes to zero. Again, one expects that $\Gamma_q(l, L)$ only depends on a dimensionless quantity which must be ε itself $\Rightarrow \Gamma_q(l, L) \equiv \Gamma_q(\varepsilon)$. With normalization imposed, we also have that $\Gamma_1(\varepsilon) \equiv 1$.

Now, we have two important observations. Firstly, for a given q and as $\varepsilon \searrow 0$, there will be a particular subset S_q of the object whose covering boxes dominate the value of $\Gamma_q(\varepsilon)$. Moreover, for $\varepsilon \sim 0$, each of these boxes will contribute the same amount to $\Gamma_q(\varepsilon)$, which will be denoted by $\mu_q(\varepsilon)^q$; [9], [2]. Again, the number of contributing boxes will diverge just as for a simple object. As per the previous section, we can write

$$N_q(\varepsilon) \sim \varepsilon^{-f_q} \tag{2.2}$$

where f_q must be the BCD of S_q .

The way in which $\mu_q(\varepsilon)$ goes to 0 may be expressed as a power law:

$$\mu_q \sim \varepsilon^{\alpha_q} \tag{2.3}$$

(α_q is often referred to as the *crowding index*). Thus, we obtain a spectrum of α_q and f_q from observation of μ_q and $N_q(\varepsilon)$. These two scaling exponents yield information on the dimension of the subset and the way in which the measure vanishes at each point of the subset. The elimination of q gives the so-called $f(\alpha)$ spectrum.

The $f(\alpha)$ characteristic may also be considered without resort to the $\Gamma_q(\varepsilon)$ construction. We examine the object at each point p by centering a ball B_p of diameter ε at p and then observe $\mu(B_p)$ as $\varepsilon \searrow 0$. As before, we expect that as $\varepsilon \searrow 0$, $\mu(B_p) \sim \varepsilon^{\alpha}$.

Clearly, there may be a number of points on the object that have the same crowding index. In keeping with our fractal notion, we may also expect that for a given α the number of these points will grow like $\varepsilon^{-g(\alpha)}$. Note that 'balls around points dimension' is a variation on the BCD that defined f_q and hence $f(\alpha)$ in the above work but the two approaches are similar enough to lead to the identification $g(\alpha) \equiv f(\alpha)$.[2]

Tél [9] notes that $f(\alpha)$ is usually a single humped function and that the range of α values is a closed interval.

The $f(\alpha)$ spectrum is naturally obtained from α_q and the f_q spectra. We show now that it is in fact possible to go back the other way when given the $f(\alpha)$ spectrum alone. This indicates that no information is lost by reducing our focus to $f(\alpha)$ itself. As has been discussed, $\Gamma_q(\varepsilon)$ is dominated by a subset of the overall object for each q . In particular, using 2.2 and 2.3, it can be written as

$$\Gamma_q(\varepsilon) \approx \mu_q(\varepsilon)^q N_q(\varepsilon) = \varepsilon^{q\alpha_q}\varepsilon^{-f_q} \tag{2.4}$$

In the second derivation of $f(\alpha)$, we have considered the behaviour of μ at every point and in so doing have made the analysis continuous. As $\varepsilon \searrow 0$, we may approximate $\Gamma_q(\varepsilon)$ by an integral over all the measures of the infinitesimal balls at each point of the object. Note that we have a natural parameter in the crowding index α . So we now write for small ε :

$$\Gamma_q(\varepsilon) \sim \int (\varepsilon^\alpha)^q \varepsilon^{-f(\alpha)}\, d\alpha \tag{2.5}$$

Comparing equations 2.4 and 2.5, we obtain a method for calculating q from the $f(\alpha)$ spectrum. To do so, note that ε is infinitesimal and therefore the integral of equation 2.5 is dominated by the interval of α where $q\alpha - f(\alpha)$ is a minimum (really just Laplace's method for finding limits of integrals). Assuming differentiability of $f(\alpha)$, the α at which this minimum occurs, α' , is given implicitly by:

$$\begin{aligned} 0 &= \frac{d}{d\alpha}(q\alpha - f(\alpha))\bigg|_{\alpha'} \qquad (2.6) \\ &= q - \frac{df(\alpha)}{d\alpha}\bigg|_{\alpha'} \end{aligned}$$

[2] There may well exist cases where these definitions of dimension differ and in the tradition of great science the possibility will be ignored at the moment as having pathological value only. One would intuitively expect that for the cases where HD = BCD, that the 'balls around points' dimension should also be the same as the BCD.

$$\Rightarrow q = \left.\frac{df(\alpha)}{d\alpha}\right|_{\alpha'}$$

In fact, for a given q , α' is equal to α_q as equations 2.4 and 2.5 must match up. So we now have:

$$\left.\frac{df(\alpha)}{d\alpha}\right|_{\alpha_q} = q$$

and $f_q = f(\alpha_q)$.

Hence, α_q may be calculated from the gradient of the $f(\alpha)$ characteristic and f_q is then given simply by $f(\alpha)$ itself evaluated at α_q . We next introduce the D_q spectrum and show its connection with the $f(\alpha)$ spectrum.

2.2.3 The D_q Spectrum

We again look at $\Gamma_q(\varepsilon)$. By the arguments in the previous section, $\Gamma_q(\varepsilon)$ must scale as a power of ε as $\varepsilon \searrow 0$ (see equation 2.4). We write:

$$\Gamma_q(\varepsilon) \sim \varepsilon^{\tau(q)} \tag{2.7}$$

For $q = 1$, we have $\Gamma_1(\varepsilon) = 1$, so $\tau(0)$ must be 0 . Hence, we can write $\tau(q) \equiv (q-1)D_q$ to emphasize this fact. The D_q's are referred to as the generalised dimensions of q^{th} order. Comparing equations 2.5 and 2.7 and recalling the discussion of minimization of the exponent in the integral 2.5, we have:

$$\tau(q) = \min_{\alpha}(q\alpha - f(\alpha)) \tag{2.8}$$

To invert this equation:

$$\begin{aligned}
\tau(q) &= \min_{\alpha}(q\alpha - f(\alpha)) \\
&\leq (q\alpha - f(\alpha)) \\
\Rightarrow f(\alpha) &\leq (q\alpha - \tau(q)) \\
&\leq \min_{q}(q\alpha - \tau(q)) \\
&= \min_{q}\max_{\alpha'}(q(\alpha - \alpha') + f(\alpha')) \\
&= \begin{cases} f(\alpha) & \text{if } \alpha = \alpha' \\ -\infty & \text{otherwise} \end{cases} \quad \dagger
\end{aligned}$$

$$\leq \quad f(\alpha)$$

Note that the above will fail at the line denoted by † unless $f(\alpha)$ eventually falls away faster than any linear function of α . Of course if $f(\alpha)$ is only defined on an interval of α then the transformation actually holds true regardless of the shape of $f(\alpha)$. With this restriction on $f(\alpha)$ relationship 2.8 has the inversion:

$$f(\alpha) = \min_q (q\alpha - \tau(q)) \tag{2.9}$$

The $f(\alpha)$ and D_q's are seen to be Legendre transformation of each other and both are therefore equivalent representations of multifractal characteristics. The reason for the term 'generalised dimensions' is made more clear by the following: D_0 is the BCD of the object, D_1 is the information dimension and D_2 is the correlation dimension [2].

In the case of simple fractals like the Cantor set, the D_q's are all equal to the BCD and the $f(\alpha)$ spectrum will consist of a single point. Note how the inversion relation between the D_q's and $f(\alpha)$ still works nicely in this degenerate case. If the measure is constructed to reflect the mass of the object, i.e. it is a measure of constant density, then the multifractal characteristics will be purely geometrical.

2.2.4 Thermodynamic Formalism

If a multifractal can be constructed recursively, its characteristics may be obtained via the use of thermodynamic potentials [9]. While this is, on the surface, a nice connection between different areas of mathematics and physics, it also allows us to tap into the powerful techniques of thermodynamic formalism [7]. In actual fact, the formalism can be shown to provide a fuller description of a multifractal in that the $f(\alpha)$ spectrum is only a subset of the information potentially afforded [9]. For this work, it will not be necessary to go this far and only enough machinery needed to deal with a simple measure will be developed. We will also restrict our commentary to multifractals formed on the unit interval, though it will be clear that the notions apply for more general settings.

The important ingredient is recursive construction. The two-scale Cantor set is an example of this. In a similar fashion to the construction of the Cantor set, each stage is formed from the previous stage by taking each interval and replacing it with two new ones reduced in length by ratios of r_1 and r_2 ($r_1 = r_2 = 1/3$ for the Cantor set). Since there are 2^n intervals at the n^{th} level of construction, we may *encode* this process by labelling each interval with a binary number of n digits. We may think of this as happening on a tree, branching at each level. The rules for the tree's construction are referred to as the 'grammar'. For the two-scale Cantor set, the grammar is trivial in that all 'words' appear at each level. Of course, not all encryptions are that simple and, in general, we expect $N(n)$, the number of allowed words at the tree's n^{th} level, to grow like $\exp(K_0 n)$ [9], where K_0 is the topological entropy [8] .

There are, by definition, $N(n)$ words at the n^{th} level and these will be represented by $S_J^{(n)} = s_1^{(n)}\, s_2^{(n)}\, \ldots\, s_n^{(n)}$ where $J \in 1, \ldots, N(n)$. The 'letters' of the word on a k-nary tree, $s_i^{(n)}$, range from 0 to $k-1$. Now, each word has a length related to it, which may be denoted by $l_J^{(n)}$.[3] For example, at the second stage of the two-scale Cantor set, we have the codes {00},{01},{10} and {11}. These naturally correspond to the lengths r_1^2, $r_1 r_2$, $r_2 r_1$ and r_2^2, respectively. Two observations are that these lengths are a perfect fit for their part of the fractal and that degeneracy occurs. The former statement means that we are working with Hausdorff ideals rather than box counting concessions. For more complicated objects, the formalism does indeed provide an optimal coverage meaning that we are obtaining more exact results than would be obtained by the methods of the previous sections.

It is clear that these length scales all approach zero in an exponential fashion. Thus, it is reasonable to express the length scales as follows:

$$l_J^{(n)} = \exp(-nE_J^{(n)}) \quad \text{or} \quad E_J^{(n)} = -\frac{1}{n} \ln l_J^{(n)} \tag{2.10}$$

We draw the analogy between these words on trees to spin chains of n interacting spins. For a k-nary hierarchy[4], these are k-state spin chains. The $E_J^{(n)}$ can then be interpreted

[3]Note that since the support is the unit interval, there is no need to rescale $l_J^{(n)}$ as $L = 1$ but this should be kept in mind for other examples.

[4]This means k branches stem from every point on the tree. At each level, there is a k-fold increase in the number of

as energies of these states. Note that in the limit $n \to \infty$, we have uncountably many states with much degeneracy of energies. Also, since the length scales will be confined to the interval (r_{min}^n, r_{max}^n), where r_{min} and r_{max} are the smallest and largest ratios of reduction present at any level, the energies will be similarly confined to a positive interval (E_{min}, E_{max}). The number of states with a certain energy E, $N(E,n)$, grows exponentially:

$$N(E,n) \sim \exp(S(E)n) \tag{2.11}$$

If the $l_J^{(n)}$ are all the same for fixed n, there is only one energy scale and, from before, we have $S(E) = K_0$. This is the case for simple fractals like the Cantor set. More generally, $S(E)$ must be bound by K_0.

If we now consider the lengths raised to a power β as a measure on the fractal we may, in comparison to previous definitions [eg. 2.1], define a new type of 'partition function':

$$\Gamma(n,\beta) \equiv \sum_{J=1}^{N(n)} \left(l_J^{(n)}\right)^{\beta} \tag{2.12}$$

We expect this sum to follow a power law as $n \to \infty$, [9].

$$\Gamma(n,\beta) \sim \exp(-\beta F(\beta)n) \tag{2.13}$$

$\beta F(\beta)$ has been introduced in analogy with thermodynamics. $F(\beta)$ is called the free energy and $\beta F(\beta)$ the Pressure Function. For a given β, we expect one particular length or energy (E say) to totally dominate the sum 2.13. Combining 2.11 and 2.12 we have $e^{nS(E)}e^{-nE\beta} \approx e^{-n\beta F(\beta)}$ which can be expressed in the relationship:

$$\beta F(\beta) = \min_E(E\beta - S(E)) \tag{2.14}$$

and the inverse, which is obtained with arguments and conditions similar to those used to obtain 2.9

$$S(E) = \min_\beta(E\beta - \beta F(\beta)) \tag{2.15}$$

In the language of thermodynamics 2.11 and 2.12 represent the microcanonical ensemble and canonical ensemble formulations [4]. They must therefore be equivalent

possible words.

descriptions in the thermodynamic limit which is born out by the existence of the Legendre transformation derived above.

We now consider a general type of measure based on the partition. For each length element or word at the n^{th} level, the measure μ will be simply the length, $l_J^{(n)}$ raised to a power σ. Note that $\Gamma(n, \sigma)$ is therefore an unnormalised sum of all probabilities. The correctly normalised measure of a word at the n^{th} level is then given by:

$$\mu(S_J^{(n)}) = \frac{\left(l_J^{(n)}\right)^\sigma}{\sum_{J=1}^{N(n)} \left(l_J^{(n)}\right)^\sigma} = \frac{\left(l_J^{(n)}\right)^\sigma}{\Gamma(n, \sigma)} \tag{2.16}$$

Since the measure is a power of the length and we know how the lengths behave asymptotically, it is trivial to find the crowding index. Using relations 2.10 and 2.13 in 2.16 we find the asymptotic form of $\mu(S_J^n)$:

$$\mu(S_J^{(n)}) \sim \exp(\sigma F(\sigma) n) \exp(-\sigma E_J^{(n)} n) \tag{2.17}$$

Thus for a given value of E we have an expression for the crowding index (see equation 2.3):

$$\alpha = \sigma - \sigma F(\sigma)/E \tag{2.18}$$

To determine the corresponding $f(\alpha)$ value we note that the number of words of the same length $l_J^{(n)}$, $N(n, E)$, must behave simply as a power of $l_J^{(n)}$ itself:

$$e^{S(E)n} \sim \left(l_J^{(n)}\right)^{-f(\alpha)} = e^{-En-f(\alpha)} \tag{2.19}$$

which implies

$$f(\alpha) = S(E)/E \tag{2.20}$$

when $\alpha = \sigma - \sigma F(\sigma)/E$. Thus, for a fixed sigma, the $f(\alpha)$ spectrum may be generated for a measure which is a power of the length scales of the covering set by finding $\beta F(\beta)$ and then performing a Legendre transform 2.15, to obtain $S(E)$.

In particular, for a k-nary tree, we have from the definition of $\Gamma(n, \sigma)$ 2.12, that $\Gamma(n, 0) = N(n) = k^n$. So for $\sigma = 0$, we have from equation 2.13 that

$$-\sigma F(\sigma)|_{\sigma=0} = \ln k \tag{2.21}$$

The crowding index, which is given by equation 2.18, is then simply

$$\alpha = \frac{\ln k}{E} \tag{2.22}$$

and using equation 2.20, we also have

$$f(\alpha) = \frac{\alpha}{\ln k} S(\frac{\ln k}{\alpha}) \tag{2.23}$$

We will use this in chapter 5 when an equimeasure partition is considered, i.e. a $\sigma = 0$ measure.

Chapter 3

Intermittency and Multifractals

Intermittency is a well recognised route to chaos [8]. That is, it describes a typical phenomenon observed at the boundary between chaos and order. It is generally characterised by some measured signal that moves randomly between states of unpredictability and regularity. It is observed that with the tuning of some external parameter the occurence and duration of these chaotic bursts continuously changes. At one extreme of the parameter range, the signal will become totally chaotic and at the other, fully periodic.

The process of intermittency may be easily modelled by simple maps of the unit interval. One such example is the Farey map and its piecewise linearised version which will be investigated further in the final chapter, see 5.1 and 5.2 as well as figures 5.1 and 5.2. Apart from being simple to investigate, the behaviour of such maps may be qualitatively generalised to more complex systems possessing intermittent behaviour. Toy models have always proved helpful in understanding more complicated systems as in the example of Lorenz's model of the atmosphere. Some examples of intermittent signals are the 'current' of highway traffic, the electric potential of a nerve membrane, and many currents and voltages observed in electronic components and circuits such as a junction diode and the relative velocities of the eddies of turbulence [8].

The general class of intermittent maps that are of interest to us are maps $g(x)$ of the unit interval that have the following main properties (see class of functions $\mathcal{C}_r$ in [6] for more details)

1. g has a fixed point at $x = 0$ and its asymptotic form near the origin is given by $g(x) \sim x + cx^{1+r}$ where $r > 0$.

2. g maps the unit interval onto the unit interval

3. $\exists$ an $a \in (0, 1)$ such that $g[0, a) = [a, 1)$ and $g[a, 1] = [0, 1]$. The inverses for these two branches are also well defined.

4. $|g'(x)| > 1 \; \forall \; x > 0$.

While iterates tend to get trapped at $x = 0$ and hence move, at least locally, in a slow, predictable way, iterates do eventually undergo chaotic behaviour. As with any chaotic behaviour, things may look somewhat haphazard in a short time frame so it is instructive to look at some sort of 'average' long term behaviour of iterates of the map. One simple way of doing this is to look at the 'invariant measure' of the map. Consider a sequence of iterates $\{x_n\}_{n=0}^{\infty}$ where $x_{n+1} = g(x_n)$ and the 'seed' point is x_0. In the following, we use Thompson's definitions [10] and the first one is the fraction of 'time' iterates spend in a particular region of $[0, 1]$.

$$\rho(\mathcal{J}) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \chi_{\mathcal{J}}(x_n) \tag{3.1}$$

where $\mathcal{J}$ is some interval in $[0, 1]$ and

$$\chi_{\mathcal{J}}(x) = \begin{cases} 1 & \text{if } \; x \in \mathcal{J} \\ 0 & \text{otherwise} \end{cases} \tag{3.2}$$

This is a definition of density in time but can be rephrased into one of a density in space. By taking small divisions $\mathcal{J}_i$ of the unit interval and measuring at $\rho(\mathcal{J}_i)$, a type of histogram can be drawn up. In the limit of the intervals $\mathcal{J}_i$ shrinking to zero width, this histogram becomes a kind of probability density on the unit interval, denoted by $\rho(x)$. Note that we require $\int_0^1 \rho(x)\, dx = 1$ for normalisation.

In the case of a stable point or any type of q-cycle, the iterates will only contribute to the $\rho(\mathcal{J}_i)$ of only one or q of the $\mathcal{J}_i$'s respectively, providing x_0 is in the appropriate basin of attraction. Thus, $\rho(x)$ will be made up of Dirac delta functions. For example, in the case of a map with a globally stable two-cycle (i.e. all iterates approach this cycle), the density function would be $\rho(x) = \frac{1}{2}[\delta(x-p_1)+\delta(x-p_2)]$ where p_1 and p_2 are the stable points. However, if iterates behave chaotically, the density funcion $\rho(x)$ will be a smooth curve as iterates will tend to wander through all parts of the unit interval. An important condition on $\rho(x)$ is that it depends on the seed point x_0. Clearly, this statistical description will not be very 'statistical' if it depends strongly on the choice of x_0. We will therefore be only interested in maps that possess 'typical' orbits, which means that almost all seed points give rise to the same $\rho(x)$. Here, almost all means that the set of points that realise a different $\rho(x)$ are of measure 0 on the unit interval. It is noted in passing that the Farey map which we will be concerned with in the last chapter satisfies this criterion.

It turns out that $\rho(x)$ may be expressed as a solution to what is known as the Perron-Frobenius equation (PFE). To start with we clarify the notion of $\rho(x)$ being an invariant measure with respect the function g. Recalling equation 3.1, we write:

$$\begin{aligned}
\rho(\mathcal{J}) &= \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \chi_{\mathcal{J}}(x_n) \qquad (3.3)\\
&= \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \chi_{\mathcal{J}}(g(x_{n-1}))\\
&= \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \chi_{g^{-1}\mathcal{J}}(x_{n-1})\\
&= \rho(g^{-1}\mathcal{J})
\end{aligned}$$

where g^{-1} is the inverse of g. Since, by definition, $\int_{\mathcal{J}} \rho(x)\, dx = \rho(\mathcal{J})$ we also have from 3.4 that

$$\int_{\mathcal{J}} \rho(x)\, dx = \int_{g^{-1}(\mathcal{J})} \rho(x)\, dx \qquad (3.4)$$

With some further manipulation of the above, we obtain the integral version of the Perron-Frobenius equation:

$$\begin{aligned}
\int_{\mathcal{J}} \rho(x)\, dx &= \int_{g^{-1}(\mathcal{J})} \rho(x)\, dx \qquad (3.5)\\
&= \int_0^1 \chi_{g^{-1}(\mathcal{J})}(x)\rho(x)\, dx\\
&= \int_0^1 \chi_{\mathcal{J}}(g(x))\rho(x)\, dx\\
&= \int_{\mathcal{J}} \left[\int_0^1 \delta(y-g(x))\rho(x)\, dx\right] dy
\end{aligned}$$

where we have used a delta function representation of the characteristic function. We may now identify the intergrands as $\mathcal{J}$ is an arbitrary interval in $[0,1]$. Switching x and y and then using a standard delta function result we obtain the following (assuming g is differentiable):

$$\begin{aligned} \rho(x) &= \int_0^1 \delta(x - g(y))\rho(y)\,dy \\ &= \sum_{g(y)=x} \frac{\rho(y)}{|g'(y)|} \end{aligned} \tag{3.6}$$

This is the Perron-Frobenius equation and it will be shown in chapter 4 that this is a particular case of a more general operator equation based on the the thermodynamic formalism for multifractals. It is important at this point to raise the point of convergence. Equation 3.7 may be thought of as an eigenfunction problem for an operator [1] Λ_{ID} where the eigenvalue must be 1:

$$\Lambda_{ID} \circ \rho(x) \equiv \sum_{g(y)=x} \frac{\rho(y)}{|g'(y)|} = \rho(x) \tag{3.7}$$

Mathematically, we 'discover' $\rho(x)$ by repeatedly operating Λ_{ID} on the constant function $1(x)$. Providing 1 is the largest eigenvalue and $\rho(x)$ is not orthogonal to the constant function, we would obtain, as $n \to \infty$

$$(\Lambda_{ID})^n \circ 1(x) \to \rho(x) \tag{3.8}$$

This preparation brings us to an important question, how long does it take for the long-term statistical behaviour to become apparent? In other words, how dominant is the eigenvalue 1? The answers lie in structure of the rest of the spectrum of the operator Λ_{ID} and, in particular, the strength of the next largest eigenvalue. This structure is at present an analytical 'unknown' and it is hoped that the numerical results presented in chapter 5 may lead to a better understanding of what is going on here.

We now discuss the situation for the invariant density of functions modelling intermittency. The behaviour of $\rho(x)$ for intermittent maps gives rise to the notion of an indifferent fixed point. While $x = 0$ is not a stable fixed point in the normal sense in that nearby iterates always move away from it, these iterates also move extremely

[1] ID = Invariant Density

slowly giving rise to a delta function density. As mentioned in the introduction, the map is neither expanding nor contracting at the fixed point. However, the fixed point is not an attracting fixed point and the map is technically said to be 'expansive' which has the following definition,[6]:

A map f is *expansive* if and only if there exist a $\delta > 0$ such that

$$x \neq y \quad \Rightarrow \quad \exists\, n \in (1, 2, \ldots) : \ d(f^n x, f^n y) > \delta$$

In particular, for the Farey map for which $r = 1$, Feigenbaum et al [3] show that a solution to the Perron-Frobenius equation is $\rho(x) = 1/x$. [2] This is clearly problematic as any attempt to normalise this density ends up with a delta function at $x = 0$ and this is not very informative. Thus, the invariant density setting does not suit the expansive maps of intermittency and a new approach must be found.

In search of a more subtle method of detecting the long term behaviour of intermittent maps, we arrive at the concept of multifractals. As will be demonstrated, the thermodynamic setting is actually a generalisation of the invariant density approach. This connection is presented in the following chapter with the construction of the transfer operator and its modified version.

[2] This is actually a more universal result for all maps with $r = 2$. In fact, Prellberg's lemma 4.14 [6] shows more generally that $\rho(x) \sim x^{-r}$ as $x \to 0$ where $r > 0$.

Chapter 4

The Transfer Operator

4.1 Derivation of Eigenvalue Equation

The multifractal in question has the unit interval as its support (so it is not geometrically inherently a fractal) and a measure based on the behaviour of a given map of the unit interval. The scaling properties of the multifractal will be seen as a reflection of the long term behaviour of the map and in general the phenomenon of intermittency.

We investigate an equimeasure partition [3], where the partition is one that arises naturally via iteration of the inverse of the map f. More precisely, the partition is based on a point set of the unit interval which can be constructed iteratively on a complete tree by f^{-1} acting on a seed point p. By a complete tree, we mean that at the n^{th} level, there are k^n points, where $k \in \{2, 3, 4, \ldots\}$. Each length element of the partition has equal probability so, in the thermodynamic formalism of chapter 2, we would use $\sigma = 0$ in our calculation of $f(\alpha)$ which was shown to lead to the identification $\alpha = \ln k/E$, see relationships 2.20 and 2.22. In particular, the Farey map (which is defined at the start of the next chapter) has two branches and therefore gives rise to 2^n points at the n^{th} level of construction. Thus, the $f(\alpha)$ spectrum for this equimeasure partition of the unit interval is given by equation 2.23 with $k = 2$

$$f(\alpha) = \frac{\alpha}{\ln 2} S\left(\frac{\ln 2}{\alpha}\right) \tag{4.1}$$

Recall also that $S(E)$ is obtained via a Legendre transform of $\beta F(\beta)$, see equation 2.15.

We now present the construction of the partition in detail: each point on the tree may be represented by a word $(e_n, e_{n-1}, \ldots, e_1)$ or a k-nary number $t = \sum_{j=1}^{n} e_j k^j$ where $e_i \in \{0, \ldots, k-1\}$. At each iteration, the next level is assumed to be obtained from k functions F_e and ultimately from an initial point p. As per the notion of the invariant density, p is chosen to be one that yields typical orbits. The F_e are also assumed to be *contractive*: that is, for $x \neq y$, $|f(x) - f(y)| < |x - y|$, $\forall x, y \in [0,1]$. A point at the n^{th} level of construction can be represented by $x^{(n)}(t) = x^{(n)}(e_n, \ldots, e_1) = F_{e_n} \circ F_{e_{n-1}} \circ \cdots \circ F_{e_1}(p)$. Here, e_n is the most significant digit of t. By construction, words representing adjacent points will differ by 1 in the least significant digit which corresponds to the function F_{e_1}.

To analyse this set we would of course like to employ the notions of the previous sections. We construct a cover for the n^{th} level with intervals of length

$$l_n(e_n, \ldots, e_1) = |x_n(e_n, \ldots, e_1) - x_n(e_n, \ldots, e_1 - 1)|$$

where $e_2, \ldots, e_n \in \{0, \ldots, k-1\}$ and $e_1 \in \{1, \ldots, k-1\}$. Note that this partition covers all of the interval between $x_n(0, 0, \ldots, 0)$ and $x_n(k-1, k-1, \ldots, k-1)$ which is not a fractal object. We now have a partition function:

$$\Gamma(n, \beta) = \sum_{e_i} {l_n}^{\beta} \tag{4.2}$$

From before, $\Gamma(n, \beta) \sim \exp -\beta F(\beta)$. Now consider the quantity

$$S(n, \beta) = \sum_{e_i} \lambda(\beta)^{-n} {l_n}^{\beta} \tag{4.3}$$

It is then clear that $S(n, \beta) \sim 1$ iff $\lambda(\beta) = \exp(-\beta F(\beta)$. We now derive an eigenvalue equation based on the assumption of convergence for $S(n, \beta)$ keeping in mind $\lambda(\beta)$'s relationship to the free energy. Since each F_{e_i} is contractive we employ a Taylor expansion for n sufficiently large:

$$\begin{aligned} & F_{e_n} \circ F_{e_{n-1}} \circ \ldots \circ F_{e_2}(F_{e_1}(p)) \\ \sim \; & F_{e_n} \circ F_{e_{n-1}} \circ \ldots \circ F_{e_2}(F_{e_1}(p)) \\ & +(F_{e_n} \circ F_{e_{n-1}} \circ \ldots \circ F_{e_2})'(F_{e_1}(p))(F_{e_1 - 1}(p) - F_{e_1}(p)) \\ & + \bigcirc \left(F_{e_1 - 1}(p) - Fe_1(p)\right)^2 \end{aligned} \tag{4.4}$$

Thus we have the following asymptotic form for l_n:

$$l_n \sim \left|(F_{e_n} \circ F_{e_{n-1}} \circ \ldots \circ F_{e_2})'(F_{e_1}(p))\right| \tag{4.5}$$

Using 4.5 in 4.3 :

$$\begin{aligned}
S(n,\beta) \;&\sim\; \sum_{e_i} \lambda(\beta)^{-n} |F'_{e_n} \cdots F'_{e_2}|^{\beta} \\
&= \sum_{e_2} \frac{|F'_{e_2}(F_{e_1}(p))|^{\beta}}{\lambda(\beta)} \sum_{e_3} \frac{|F'_{e_3}(F_{e_2} \circ F_{e_1}(p))|^{\beta}}{\lambda(\beta)} \cdots \\
&\quad \sum_{e_j} \frac{\left|F'_{e_j}(F_{e_{j-1}} \circ F_{e_{j-2}} \circ \cdots)\right|^{\beta}}{\lambda(\beta)} \sum_{e_{j+1}} \frac{\left|F'_{e_{j+1}}(F_{e_j} \circ F_{e_{j-1}} \circ \cdots)\right|^{\beta}}{\lambda(\beta)} \cdots \\
&\quad \sum_{e_n} \frac{\left|F'_{e_n}(F_{e_{n-1}} \circ F_{e_{n-2}} \circ \cdots)\right|^{\beta}}{\lambda(\beta)}
\end{aligned}$$

We now examine the middle terms of the above product and note that a kind of transfer operator may be constructed as follows: we define

$$\Phi_j^{(\beta)}(F_{e_j}(P)) \equiv \sum_{e_{j+1}} \frac{\left|F'_{e_{j+1}}(F_{e_j}(P))\right|^{\beta}}{\lambda(\beta)} \cdots \sum_{e_n} \frac{\left|F'_{e_n}(F_{e_{n-1}} \circ \cdots \circ F_{e_j}(P)\right|^{\beta}}{\lambda(\beta)} \tag{4.6}$$

where $P = F_{e_{j-1}} \circ F_{e_{j-2}} \circ \cdots \circ F_{e_1}(p)$. Thus we also have the recursion relation:

$$\Phi_{j-1}^{(\beta)}(P) = \sum_{e_j} \frac{\left|F'_{e_j}(P)\right|^{\beta}}{\lambda(\beta)} \Phi_j^{(\beta)}(F_{e_j}(P)) \tag{4.7}$$

Now, if $\Gamma(n,\beta)$ converges, we must have $\Phi_{j-1}^{(\beta)}(x) \sim \Phi_j^{(\beta)}(x)$ for large $n-j$. Equation 4.7 becomes, upon some rearrangement, the equation of a linear transfer operator which will be denoted by $\mathcal{L}^{(\beta)}$.

$$\lambda(\beta)\Phi^{(\beta)}(P) = \sum_{e_j} \left|F'_{e_j}(P)\right|^{\beta} \Phi^{(\beta)}(F_{e_j}(P)) \equiv \mathcal{L}^{(\beta)}\Phi^{(\beta)}(P) \tag{4.8}$$

$\lambda(\beta)$ is thus the largest eigenvalue of $\mathcal{L}^{(\beta)}$. Recalling the initial interpretation of $\lambda(\beta)$ as being equal to $\exp(-\beta F(\beta))$, we have recast the problem of finding the free energy of a map into one of finding the largest eigenvalue of a well defined linear operator. There is of course much more that could be said and the interested reader is referred to Ruelle's tome [7]. Also, noting the relationship $F'_{e_j}(P) = 1/f'(F_{e_j}(P))$ and replacing

P with x, equation 4.8 may be rewritten as the following:

$$\lambda(\beta)\Phi^{(\beta)}(x) = \sum_{f(y)=x} \frac{\Phi^{(\beta)}(y)}{|f'(y)|^{\beta}} \tag{4.9}$$

This will be the form used in the rest of this thesis and matches that used in [6]. It is known as the Ruelle-Perron-Frobenius equation (RPFE).

It is also important to note the relationship between the transfer operator and the partition function. A close inspection of the relationship 4.6 shows that, for large n, $\Gamma(n,\beta)$ is just the n^{th} iteration of the transfer operator acting on the constant function $1(x)$ with the eigenvalue $\lambda(\beta)$ divided out with each iteration. Along the lines of the argument for the existence of a well defined invariant measure based on the notion of 'typical' orbits, a seed point from almost all of the unit interval will asymptotically give the same partition funcion.

$$\mathcal{L}^{(\beta)^{n-1}} \circ 1(x) \sim \lambda(\beta)^{(n-1)}\Gamma(n,\beta) \tag{4.10}$$

The connection between the invariant measure developed in chapter 2 and the transfer operator may now be stated. As can be seen from equation 4.9, the Perron-Frobenius equation 3.7 is the particular case of the RPFE when $\beta = 1$. Further inspection of the two reveals that we must also have $\lambda(1) = 1$ in all cases. For the Farey map, confirmation of this is provided by Feigenbaum [3] and indeed, by the results of chapter 4. So we now have have a finer technique for the analysis of intermittent maps.

Also, for reasons similar to those mentioned regarding the operator Λ_{ID} (see chapter 3), we are interested in the speed of convergence of the partition function. Thus we would like to examine the general spectrum of $\mathcal{L}^{\beta}$ rather than just the leading eigenvalue.

4.2 Modified Transfer Operator

In the previous section we arrived at the problem of determining the largest eigenvalue of an operator. Whilst this is already a vast improvement in terms of ease of calculation, there is a further transformation of the transfer operator that will be of help to us. The new operator is the so-called modified transfer operator (MTO). The main importance of the MTO is that it removes the problem of the singularity at $x = 0$ by shifting the focus to a more well behaved 'induced map' on a smaller interval of $[0, 1]$. The use of the MTO has been developed extensively in relation to the process of intermittency by Prellberg [6]. We will present this induced version of the transfer operator along with the connecting relations between the two operators.

An important notion is that of the induced map. Given a point x_0 in a subinterval J of $[0, 1]$, we consider the iterates $\{f^k(x_0)\}_{k=1}^{\infty}$. If there exists an n such that $f^n(x_0) \in J$ and this is the first such iterate that returns to J, we then define the function $n(x)$ to be equal to n at the point x_0. Since the maps considered possess typical orbits, there are at most countably many points J_∞ where $n(x)$ is not defined. We now define the induced or 'first return' map $g : J \setminus J_\infty \to J \setminus J_\infty$ by (see [6])

$$g(x) = f^{n(x)}(x) \tag{4.11}$$

In fact, $g(x)$ can be extended to the whole of J by arbitrarily defining it on J_∞ [6].

Importantly, we now have a better map to deal with. If J is chosen to be $[\frac{1}{2}, 1]$ for example, g has no indifferent fixed points. So we are inducing on an interval to get away from the indifferent fixed point. Whereas f was said to be *expansive*, the induced map g, is *expanding.* The term *expanding* may be defined as follows:

A map g is *expanding* if and only if there exists a $\delta > 0$ and a $\lambda_0 > 0$ such that

$$d(x, y) < \delta \quad \Rightarrow \quad d(g(x), g(y)) \geq \lambda_0 \, d(x, y)$$

where in our case, $d(x, y) = |x - y|$ as we are simply dealing with the unit interval.

The fact that g is expanding is shown by Prellberg [6]. Note that an expansive map is also sometimes said to be 'marginally expanding', indicating its weaker quality. Hopefully, the statistical behaviour of this induced map will somehow connect to that of the original map. A first guess might be to simply write down the RPFE for the induced map:

$$\lambda(\beta)\Phi^{(\beta)}(x) = \sum_{g(y)=x} \frac{\Phi^{(\beta)}(y)}{|g'(y)|^{\beta}} \tag{4.12}$$

Unfortunately, there is no link back to the same equation for f. However, it is found that the introduction of another variable z does provide the connection we seek without creating too many problems. The modified transfer operator is thus defined as follows:

$$\mathcal{M}_{\beta z}\phi(x) = \sum_{g(y)=x} z^{n(y)} \frac{\phi(y)}{|g'(y)|^{\beta}} \tag{4.13}$$

For the purpose of this work, the important relationship between the two transfer operators is how their eigenvalues connect. We provide some notation in order to explain: $\sigma_p(\mathcal{O})$ is the set of all eigenvalues of the operator $\mathcal{O}$ and $r(\mathcal{O})$ is the essential spectral radius of this set. The essential spectral radius $r(\mathcal{O})$ is defined to be the smallest value such that the set of all eigenvalues with norm greater than $r(\mathcal{O})$ is a disconnected set. We simply quote the salient part of theorem 2.7 from Prellberg's work [6] without proof:

Theorem 4.1 *Suppose* $0 \neq |z| < 1/r(\mathcal{L}^{(\beta)})$. *Then* $z^{-1} \in \sigma_p(\mathcal{L}^{(\beta)})$ *if and only if* $1 \in \sigma_p(\mathcal{M}_{\beta z})$.

This says that if we can calculate analytically or numerically for a given β what value(s) of z the operator $\mathcal{M}_{\beta z}$ has an eigenvalue equal to one, then we will also find all the leading eigenvalues of the unmodified transfer operator, $\mathcal{L}^{(\beta)}$. By leading, it is meant that the eigenvalues so obtained will be the ones outside of the essential spectral radius.

So, the importance of the MTO is that we have removed the effect of the indifferent fixed point, which has been shown to completely dominate the behaviour of the map, as exemplified by the invariant density being a δ function. Thus, we have, in essence, restricted our view of the map to a picture of its behaviour on one side of a 'phase transition'.

Chapter 5

The Farey Map — Numerical Analysis

5.1 The Farey Map and its Piecewise Linear version

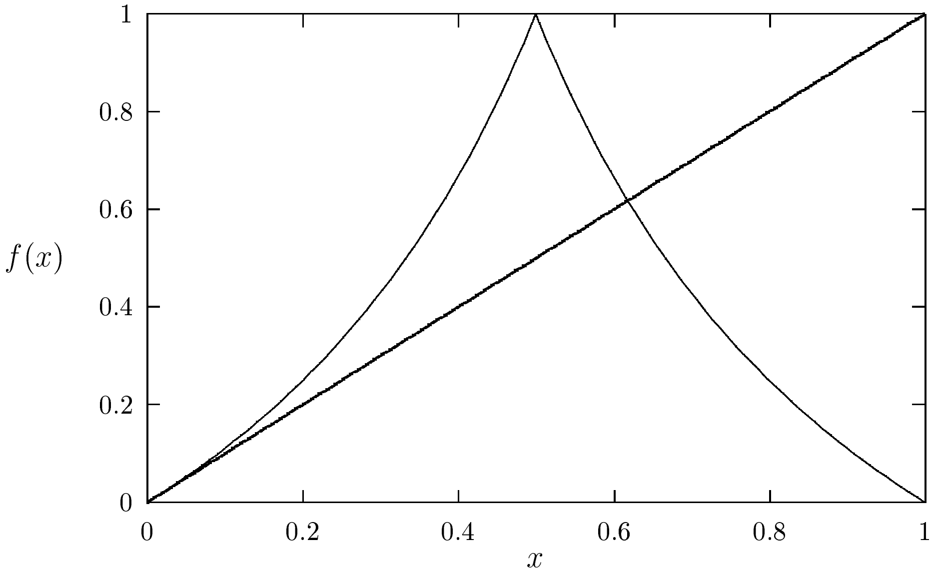
The Farey map f and its piecewise linear version $\tilde{f}$ (the LFM) are defined as follows (see figures 5.1 and 5.2)

$$f(x) = \begin{cases} \frac{x}{(1-x)}, & 0 \leq x < 1/2 \\ \frac{(1-x)}{x}, & 1/2 \leq x \leq 1 \end{cases} \tag{5.1}$$

$$\tilde{f}(x) = \begin{cases} \frac{n+2}{n}x - \frac{1}{n(n+1)}, & \frac{1}{n+2} \leq x < \frac{1}{n+1}, \quad n = 1, 2, \ldots \\ 2 - 2x, & 1/2 \leq x \leq 1 \end{cases} \tag{5.2}$$

As has already been noted, the Farey map is in the class $\mathcal{C}_1$, (see chapter 3), as for small x, $f(x) \sim x + x^2$. The importance of the LFM is that it closely approximates f and is a simpler map to analyse. In particular, its induced version is also piecewise linear, which simplifies the MTO as $\tilde{g}'$ can only be dependent on n. As will be seen, this allows for analytic solutions for all of the eigenvalues, which are in fact connected by a nice recursive property. The Farey map is understandably more complicated and the eigenvalues cannot, in general, be found explicitly. We are forced to go one step further and approximate the operator itself by a finite matrix. This will be discussed further in the following sections along with comments on the results of the work.

Figure 5.1: Farey map



5.2 Modified Transfer Operator for the Piecewise Linear Farey Map

We find that a closed form expression for the eigenvalues of the modified transfer operator may be obtained for the case of the piecewise linear version of the Farey Map. Let $\tilde{g}$ be the induced map on the interval $J = [\frac{1}{2}, 1] \subset I$ where I is the unit interval. We find the following expression for $\tilde{g}$:

$$\tilde{g}(y) = \begin{cases} -n(n+1)y + n^2 + \frac{n+1}{2} & \text{for } y \in \left(1 - \frac{1}{2n}, 1 - \frac{1}{2(n+1)}\right] \ , \ n = 1, 2, \ldots \\ 1 & \text{for } y = 1/2 \end{cases} \tag{5.3}$$

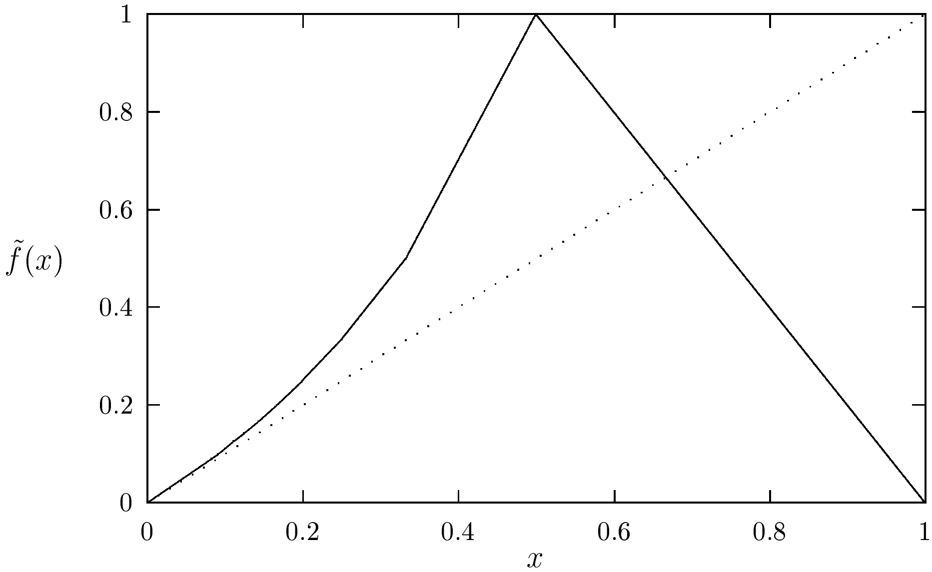
From Section 4.2 we recall the definition of the Modified Transfer Operator:

$$\mathcal{M}^{z\beta} \circ \phi(x) = \sum_{\substack{y \in J \\ \tilde{g}(y) = x}} z^{n(y)} \frac{\phi(y)}{|\tilde{g}'(y)|^{\beta}} \tag{5.4}$$

If we assume $\phi(y)$ can be written as a power series $\sum_{m=0}^{\infty} \phi_m y^m$ and using the inverse of $\tilde{g}(y)$ obtained by inverting the relationship 5.3 we may now rewrite operator equation 5.4 and procede to manipulate it as follows:

$$\mathcal{M}^{z\beta} \circ \phi(x) = \sum_{n=1}^{\infty} \frac{z^n}{(n(n+1))^{\beta}} \sum_{m=0}^{\infty} \phi_m \left(\frac{-x + n^2 + \frac{n+1}{2}}{n(n+1)} \right)^m$$

Figure 5.2: Piecewise linear Farey map



$$
\begin{aligned}
&= \sum_{n=1}^{\infty} \frac{z^n}{(n(n+1))^{\beta+m}} \sum_{m=0}^{\infty} \phi_m \sum_{M=0}^{m} \binom{m}{M} (-x)^M \left(n^2 + \frac{n+1}{2}\right)^{m-M} \\
&= \sum_{n=1}^{\infty} \sum_{M=0}^{\infty} \sum_{m=M}^{\infty} \frac{z^n}{(n(n+1))^{\beta+m}} \phi_m \binom{m}{M} (-1)^M x^M \left(n^2 + \frac{n+1}{2}\right)^{m-M} \\
&= \sum_{M=0}^{\infty} x^M \left[(-1)^M \sum_{n=1}^{\infty} \frac{z^n}{(n(n+1))^{\beta+m}} \phi_m \binom{m}{M} \left(n^2 + \frac{n+1}{2}\right)^{m-M} \right] \\
&= \sum_{M=0}^{\infty} x^M \left(\mathcal{M}^{z\beta} \circ \phi\right)_M
\end{aligned}
$$

where we have used a simple binomial expansion and an interchange of summation: $\sum_{m=0}^{\infty} \sum_{M=0}^{m} \equiv \sum_{M=0}^{\infty} \sum_{m=M}^{\infty}$ and made the following identification

$$
\left(\mathcal{M}^{z\beta} \circ \phi\right)_M = (-1)^M \sum_{n=1}^{\infty} \frac{z^n}{(n(n+1))^{\beta+m}} \phi_m \binom{m}{M} \left(n^2 + \frac{n+1}{2}\right)^{m-M} \tag{5.5}
$$

Consider a function $\phi(y)$ with $\phi_m = 0 \; \forall \; m > N$ and $\phi_N \neq 0$; i.e. $\phi(y)$ is just a polynomial in y of order m. On inspection of equation 5.5 it is seen that $\left(\mathcal{M}^{z\beta} \circ \phi\right)_M = 0 \; \forall \; M > N$. We are in fact left with a representation of the operator that is an upper triangular matrix of dimension $(N+1)$. In particular, we have

$$
\left(\mathcal{M}^{z\beta} \circ \phi\right)_N = (-1)^N \sum_{n=1}^{\infty} \frac{z^n}{(n(n+1))^{\beta+N}} \phi_N \tag{5.6}
$$

Since the matrix is upper triangular, we may perform a Gaussian elimination to find the appropriate ϕ_m for $m < N$ so that

$$\left(\mathcal{M}^{z\beta} \circ \phi\right)_m = (-1)^N \sum_{n=1}^{\infty} \frac{z^n}{(n(n+1))^{\beta+N}} \phi_m$$

holds for each $m < N$. Thus, we can construct an eigenfunction which is a polynomial of order N for each N . The set of all such eigenfunctions spans the space we are interested in, so we must have all of the eigenfunctions. Most importantly we have found the eigenvalues in closed form:

$$\lambda_N(z, \beta) = (-1)^N \sum_{n=1}^{\infty} \frac{z^n}{(n(n+1))^{\beta+N}} \tag{5.7}$$

Note also the recursion relation:

$$\lambda_N(z, \beta) = (-1)^N \lambda_0(z, \beta + N) \tag{5.8}$$

To find the free energy we must determine when $\lambda_N(z, \beta)$ is equal to one for each N. To do this, we need only look at the solutions to the equations $\lambda_0(z, \beta) = 1$ and $\lambda_1(z, \beta) = 1$ which differ because of the negative sign. All other solutions follow easily using relationship 5.8.

To obtain such solutions, the manipulation of the expression 5.7 into a sum involving the Lerch transcendent functions, which are defined in Appendix A (also, see [1]), is most desirable as such entities converge well and there are existing routines for their evaluation. This is done in the appendices where a definition of the functions used is given as well.

Using the above expansion, a search routine (see program *transfer* in appendix B) was developed to find for a range of β the values of z where $\lambda_N(z, \beta) = 1$, $N = 0, 1$. (Note that $\lambda_N(\beta, z)$ is only defined for $z \in [-1, 1]$ as the sum diverges outside of this range.) We were restricted to $\beta > -7$ because of convergence difficulties for $-z$. As mentioned above, it was only necessary to do this for $N = 0$ and 1 because of the recursion relation 5.8 . Included in what follows are some plots of $\lambda_0(\beta, z)$ as a function of z for various fixed β ranging from -4.2 to $+0.9$, together with the lines 1 and -1 to demonstrate the movement of the solutions to $\lambda_{0,1}(z, \beta) = 1$ which are themselves plotted in the following section. Note that the plots for $\lambda_0(z, \beta)$ for $N > 0$

are the same except they should be relabelled with $\beta \to \beta + N$ and multiplied by a factor $(-1)^N$. Hence the plots of $z(\beta)$ for $N > 0$ and even ($N > 1$ and odd) will be the same as the plots of $z(\beta)$ for $N = 0$ ($N = 1$) translated to the left by $\beta = N$ ($\beta = N - 1$) respectively. For example, the plot of $\lambda_0(-3, z) = 1$ 5.6 is also the plot for $-\lambda_1(-4, z) = -1$ and this is why both 1 and -1 appear on the plots.

Also important is the existence of a positive z solution. Recalling the theorem from the previous chapter, $1/z$ gives the corresponding eigenvalue for the transfer operator $\mathcal{L}_\beta$. It is observed that for at least the region $\beta > -5$ the positive solution is the smallest in magnitude and therefore the largest eigenvalue of the transfer operator. This is good in that the logarithm of this is then equated to the pressure function, $\beta F(\beta)$, so we would expect the positive solution to be the dominant one.

The algorithm was checked for $\beta = 0$ where it can be easily shown that $\lambda_0(z, 0) = z/(1 - z)$, see figure 5.9. The output from the program was seen to match up with a relative error of at most 10^{-10} even out at $\beta = -7$.

Figure 5.3: Eigenvalue of LFM vs z for $\beta = -4.2$

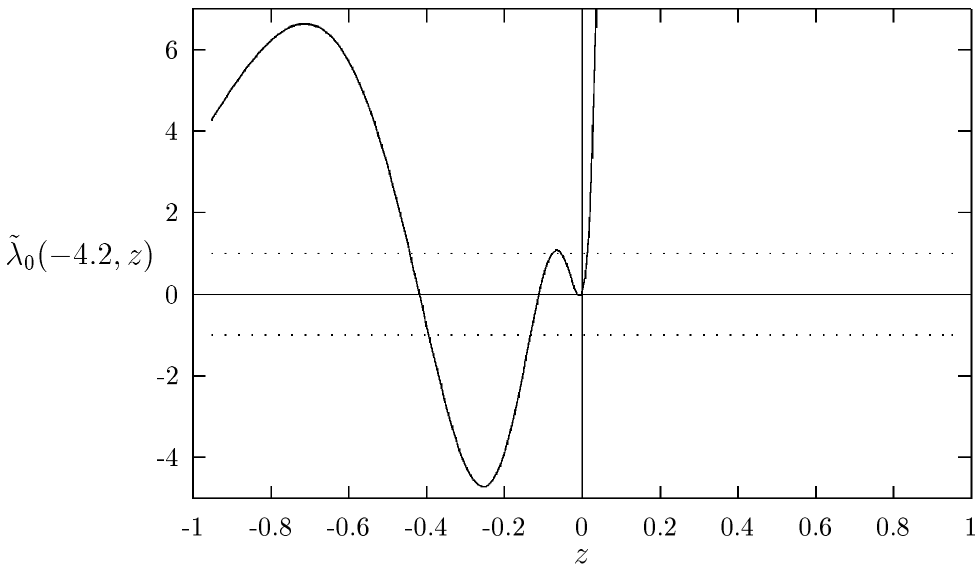


Figure 5.4: Eigenvalue of LFM vs z for $\beta = -4.1$

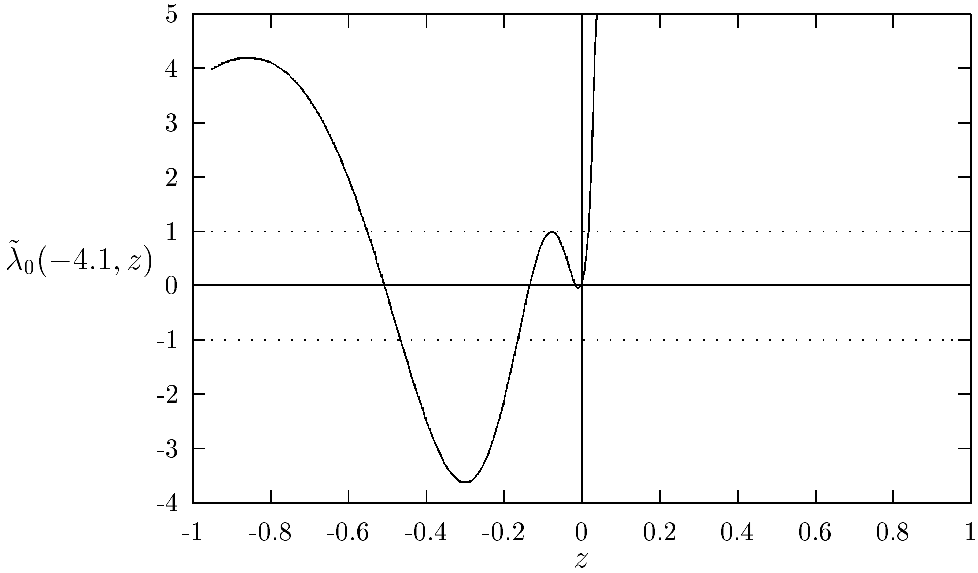


Figure 5.5: Eigenvalue of LFM vs z for $\beta = -3.7$

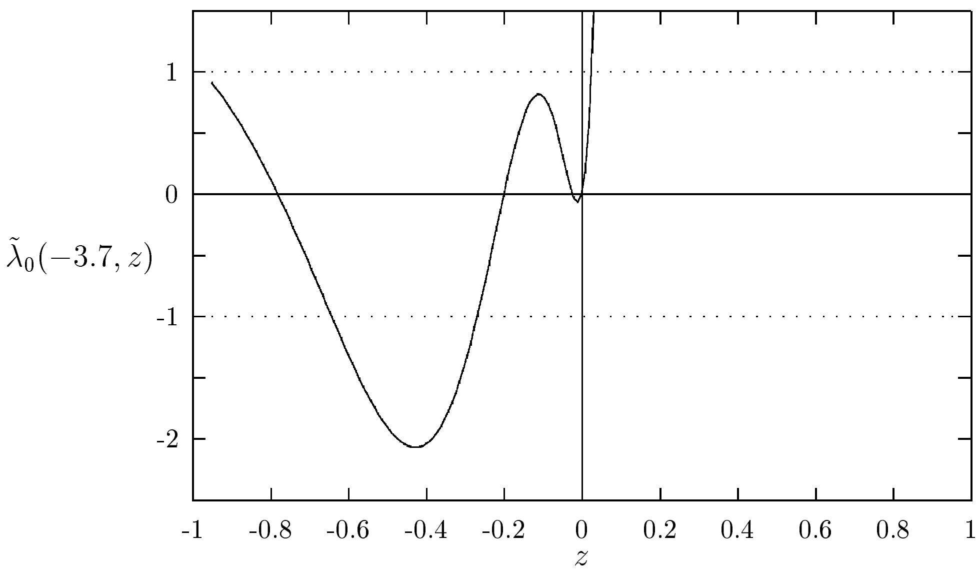


Figure 5.6: Eigenvalue of LFM vs z for $\beta = -3$

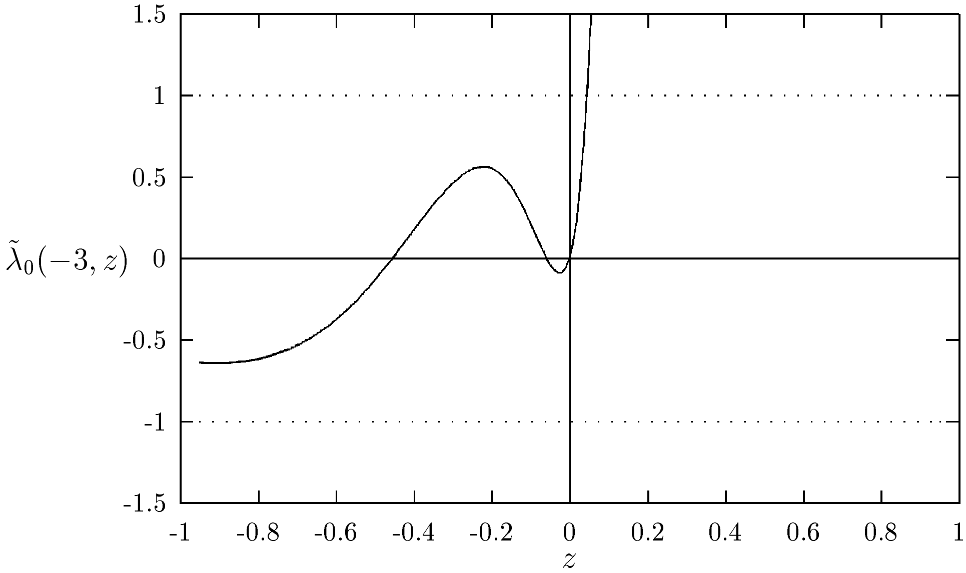


Figure 5.7: Eigenvalue of LFM vs z for $\beta = -2$

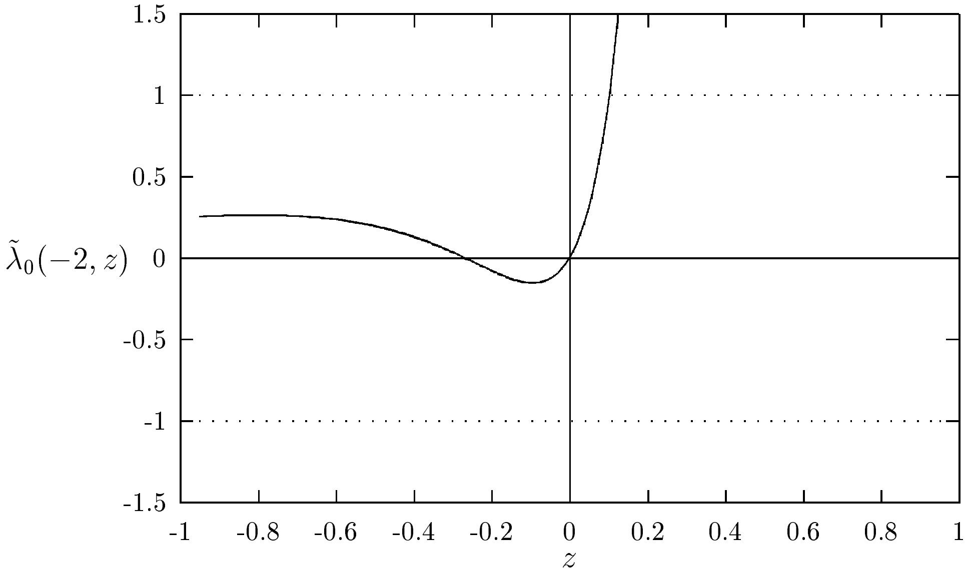


Figure 5.8: Eigenvalue of LFM vs z for $\beta = -1$

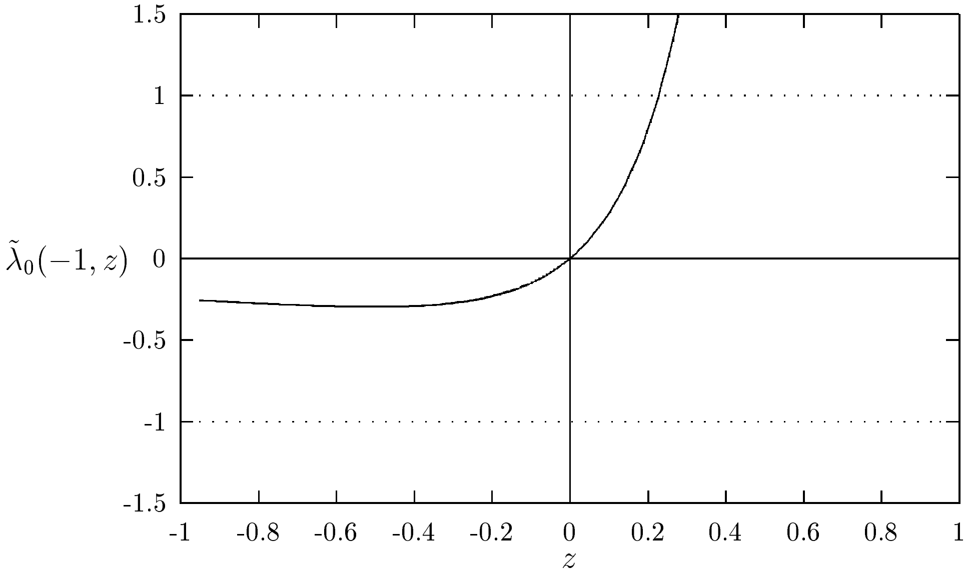


Figure 5.9: Eigenvalue of LFM vs z for $\beta = 0$

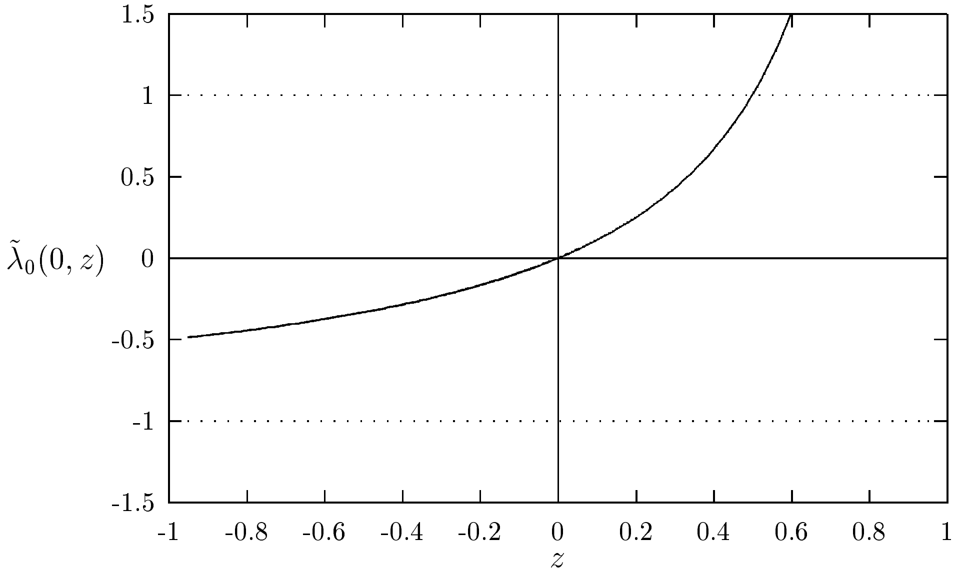


Figure 5.10: Eigenvalue of LFM vs z for $\beta = +0.5$

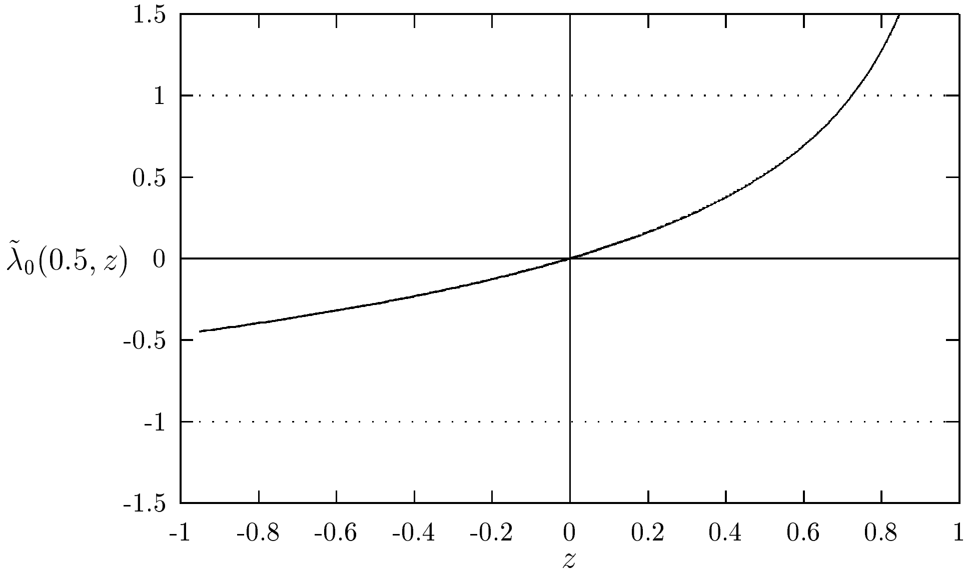
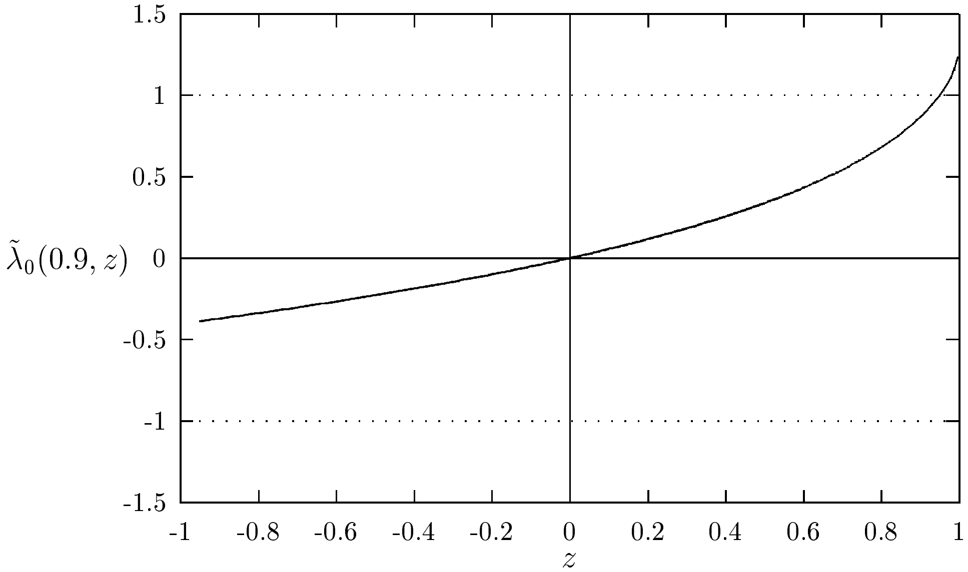


Figure 5.11: Eigenvalue of LFM vs z for $\beta = +0.9$



We next present the results of the search routine. The following plots show the solutions for the first two eigenvalues as well as all of the solutions for $\beta > -7$, see figures 5.12 5.14 and 5.13. The latter is simply a composition of the first two translated by $\beta = 0, -2, -4, \ldots$ which follows from previous comments about the recursive nature of the eigenvalues. Figure 5.15 compares the strength of the $\beta F(\beta)$ spectrum, which is given by the dominant eigenvalue, to that of the closest positive z solutions. Note that to the right of $\beta = 1$, the plot of $\beta F(\beta)$ stays constant at 0. In this region the indifferent fixed point totally dominates the multifractal characteristics, revealing itself by causing a phase transition. This is also true for the Farey map pressure function which will be presented in the following section.

With respect to the pressure function plot 5.15, note that $r(\Lambda^{(\beta)}) = 1$ so therefore the spectral gap is given by the vertical distance from the leading eigenvalue to the x-axis or the next eigenvalue if its magnitude is greater than one. We observe from figure 5.15 that there is no other eigenvalue outside of the spectral radius for $\beta > -1$. At $\beta = -1$, the positive solution for the third eigenvalue $\lambda_1(\beta, z)$, starts to appear. For increasing β, the positive z solutions move steadily to one. Analytically, it can be easily shown that $\lambda_0(1, 1) = 1$. This is exactly what we expect because of the Perron-Frobenius equation 3.7 where the eigenvalue must be one by the derivation. Here also, the eigenfunction is, of course, the density function $\rho(x)$, (see chapter 3).

Note that the first appearance of a solution due to a negative value of z occurs for β around -3.7 so its affect on the convergence of the operator is not important until then. However, when the negative z solutions do appear they are still dominated by the solution for the third eigenvalue. Thus, the latter solution provides us with a measure of the spectral gap. Since the solutions for the first and third eigenvalues of the MTO, which are the dominant and next dominant eigenvalues of the transfer operator respectively, appear to behave exponentially and are simply translations of each other, the spectral gap will also increase exponentially as β decreases.

The 'cusps' that occur in the structure are a result of an extremum of $\lambda_N(z, \beta)$ passing through 1 as β varies. This can be seen on comparing figures 5.3, 5.4 and 5.5. It is expected that these eigenvalues then move out into the complex plane but that is outside the present scope of our work. Also, the odd eigenvalues have less solutions and these all occur for more negative z and $\beta < -4$. Indeed, there are no solutions for $N = 5$ while for $N = 6$, some appear in the region considered.

Figure 5.12: $z(\beta)$ for first eigenvalue of LFM for $\beta > -7$

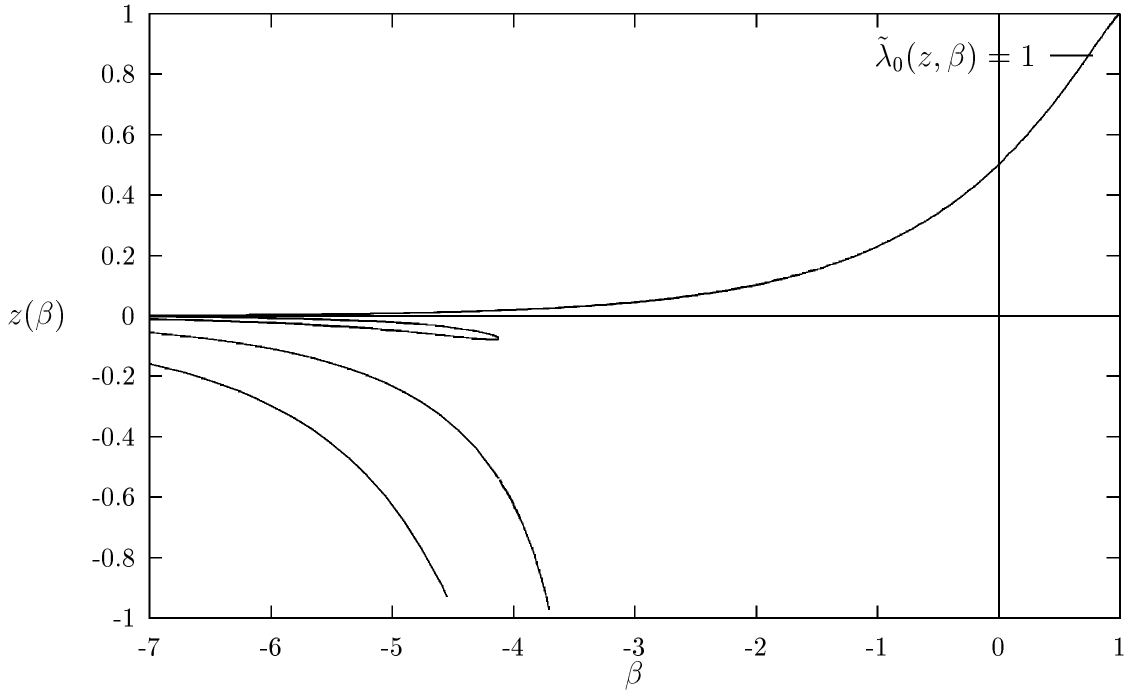


Figure 5.13: $z(\beta)$ for LFM for $\beta > -7$

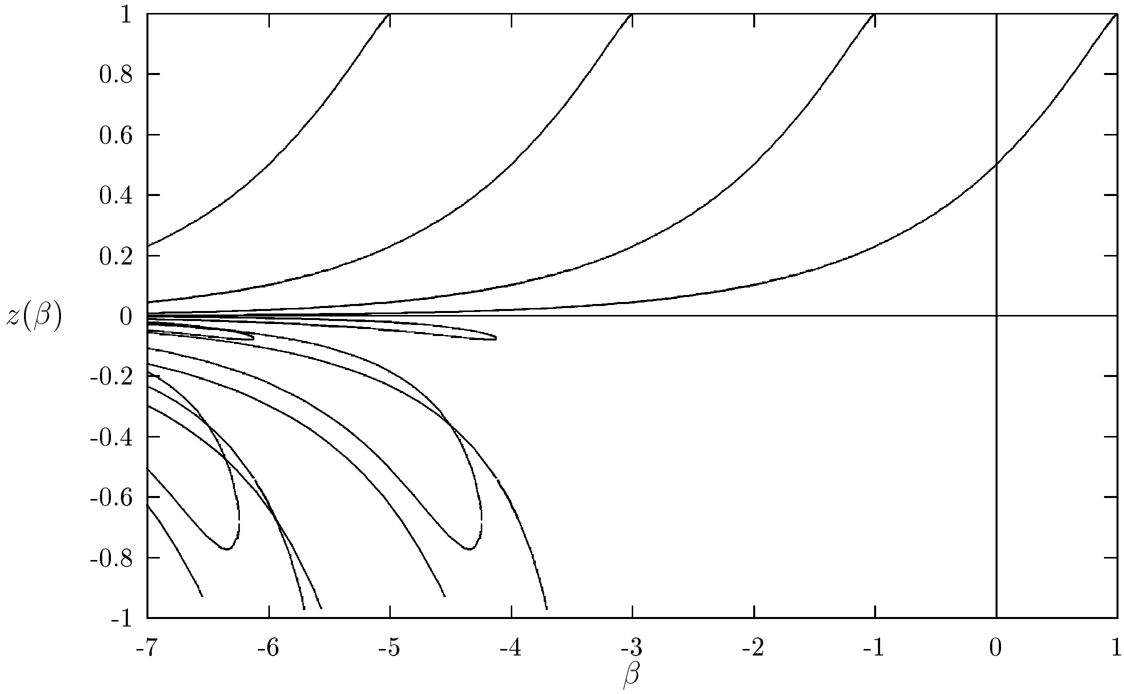


Figure 5.14: $z(\beta)$ for second eigenvalue of LFM for $\beta > -7$

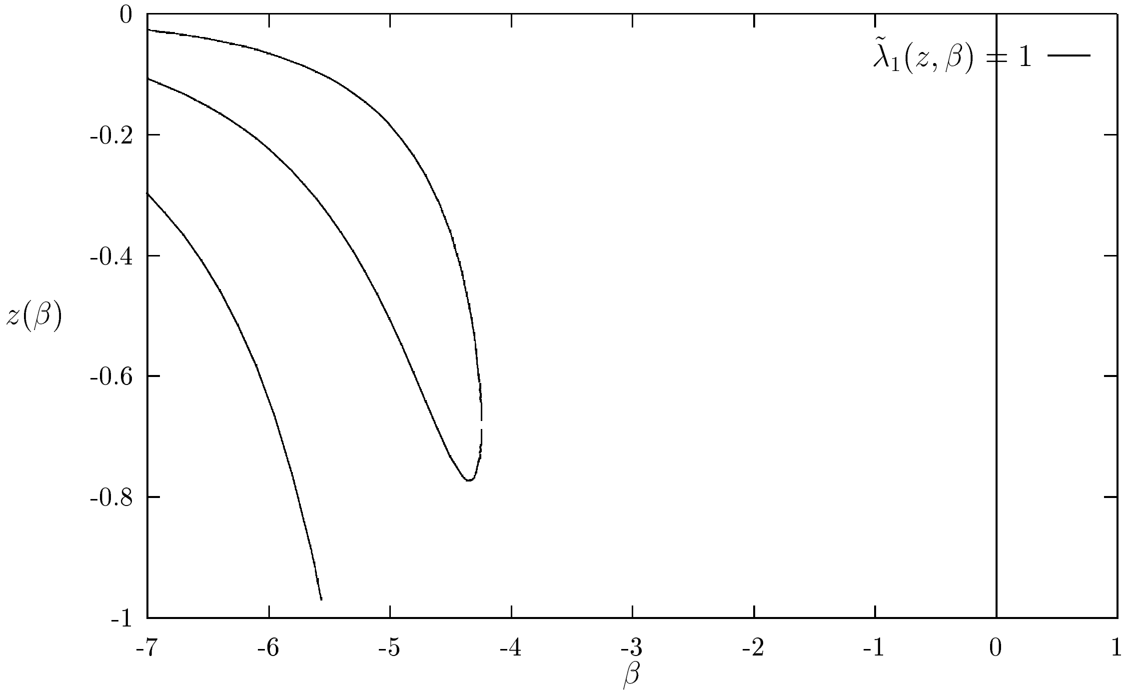
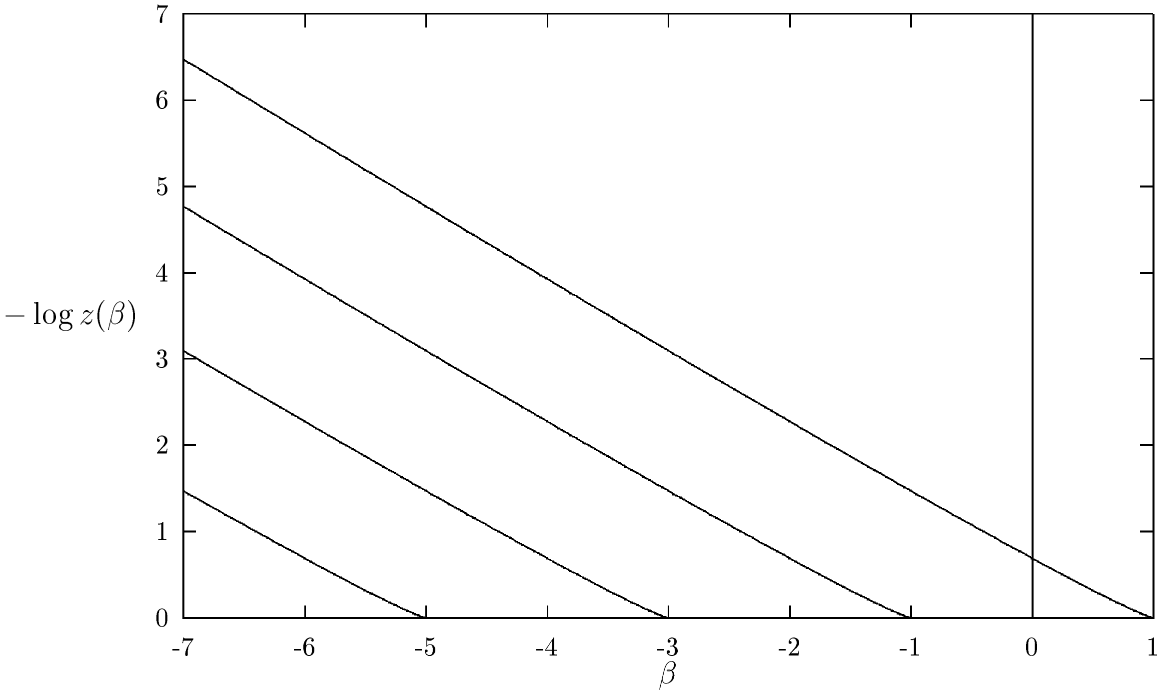


Figure 5.15: $-\beta F(\beta)$ for LFM for $\beta > -7$



5.3 The Modified Transfer Operator for the Farey Map

In the case of the Farey map, the eigenvalues of the modified transfer operator cannot, in general, be found in closed form and we resort to numerical methods for their estimation. However, for $\beta = -m/2$ where $m \in (0, 1, 2, \ldots)$ the situation is somewhat similar to the linear case in that some analytic solutions exist. Indeed, the eigenvalues may certainly be found explicitly for small m and used to check the numerics and more details are given later on. The algorithm can then be used confidently to examine the spectrum of the operator in greater depth. Firstly, we must construct a matrix representation for the modified transfer operator. The matrix is in general infinite dimensional so finite dimensional approximations are taken and LAPACK algorithms for Fortran 77 are used to extract the eigenvalues. The matrix entries are calculated using a special function package. Both of these tasks are controlled by the program *transfer*.

Firstly, the induced map for the Farey map with respect to the interval $J = [\frac{1}{2}, 1]$ is given by

$$g(y) = \begin{cases} \frac{1-y}{1+n(y-1)}, & \text{for } y \in (\frac{n}{n+1}, \frac{n+1}{n+2}] \quad n = 1, 2, 3, \ldots \\ 1 & \text{for} x = \frac{1}{2} \end{cases} \tag{5.9}$$

In order to construct a finite dimensional matrix representation of the MTO, we rephrase the operator equation in a similar fashion to the previous section. Using relationships 5.1 in equation 4.13, we have the eigenvalue problem

$$\begin{aligned} \mathcal{M}^{z\beta} \circ \phi(x) &= \sum_{n=1}^{\infty} \frac{z^n}{(1+nx)^{2\beta}} \phi\left(1 - \frac{x}{1+nx}\right) \\ &= \lambda(z,\beta)\phi(x) \end{aligned} \tag{5.10}$$

Note that the corresponding problem for the unmodified transfer operator can be solved analytically for $\beta = 1$ and $-m/2$ where $m = 1, 2, \ldots$, see [3]. For $\beta = 1$, Feigenbaum et al show that the eigenfunction has a simple pole, i.e. $\psi(x) \equiv 1/x$ and that $\lambda(1) = 1$. As mentioned before, this makes sense as the intermittent maps considered possesses singular behaviour only at the origin where iterations may move arbitrarily slowly. The eigenfunction for the modified transfer operator is thus expected to yield any singularities only when $x \sim 0$. We therefore write $\phi(x) = \sum_{m=0}^{\infty} (1-x)^m$ where we have safely expanded about $x = 1$. Substituting this into 5.11 above we are able to obtain a matrix representation for the modified transfer operator. This is done in the appendices.

Below is a collection of eigenvalue plots for various values of β, as for the linearised map. Again, they are include to give some idea of the movement of eigenvalues and solutions with varying β. It was found that the eigenvalues were generally alternately positive and negative when ordered with respect to magnitude. There were some problems when this ordering changed and a continuity check was implemented to sort the eigenvalues in a sensible way. We plot the 6 most significant eigenvalues in the following. Notice that only two eigenvalues, in the region considered, ever rise above 1. All other eigenvalues appear to be bounded by ± 1 and approach these limits more rapidly as β decreases. Note that these were never observed to actually reach 1, as this may not be clear from the plots which suffer from unavoidable resolution problems.

Figure 5.16: Leading 6 eigenvalues of MTO for Farey map vs z for $\beta = -4$

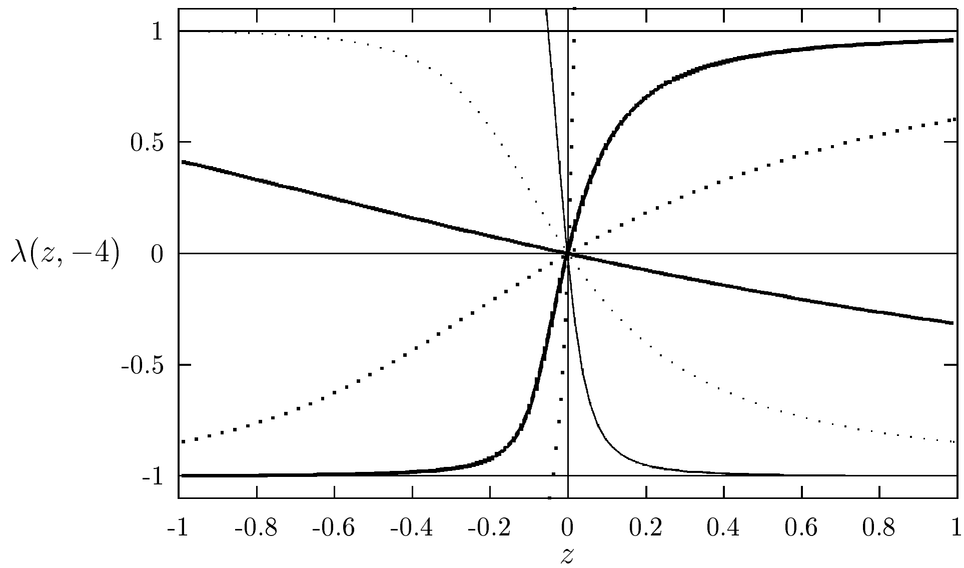


Figure 5.17: Leading 6 eigenvalues of MTO for Farey map vs z for $\beta = -3$

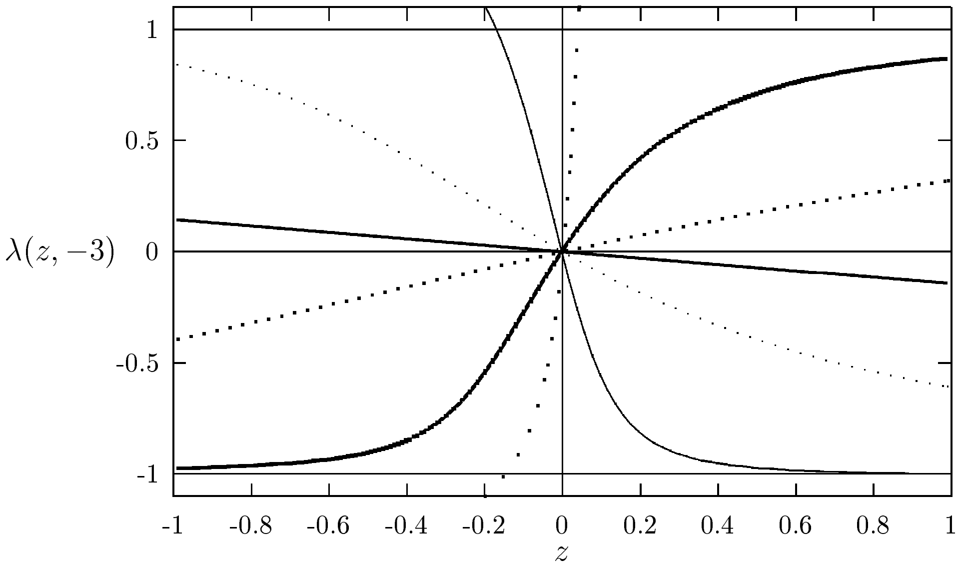


Figure 5.18: Leading 6 eigenvalues of MTO for Farey map vs z for $\beta = -2$

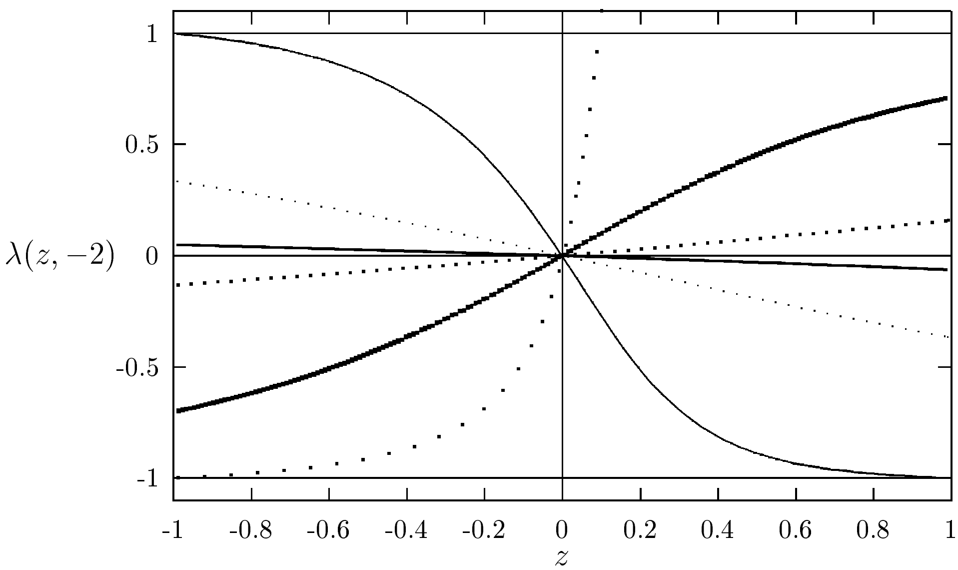


Figure 5.19: Leading 6 eigenvalues of MTO for Farey map vs z for $\beta = -1$

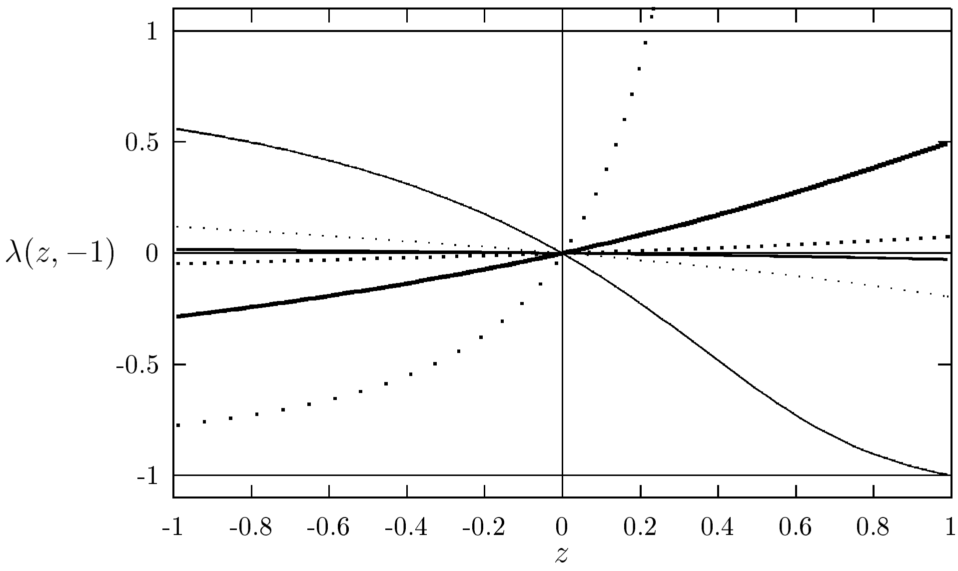


Figure 5.20: Leading 6 eigenvalues of MTO for Farey map vs z for $\beta = -0.5$

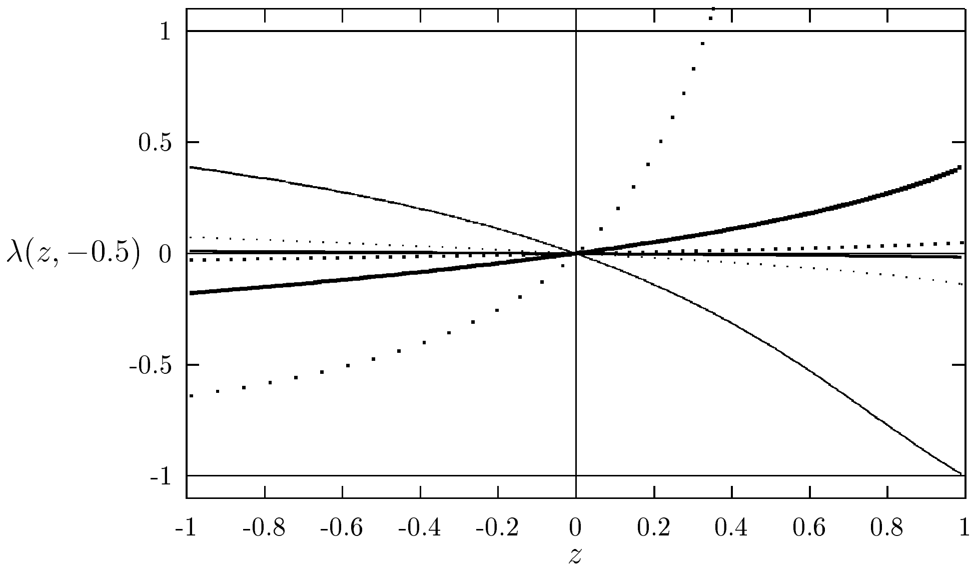


Figure 5.21: Leading 6 eigenvalues of MTO for Farey map vs z for $\beta = 0$

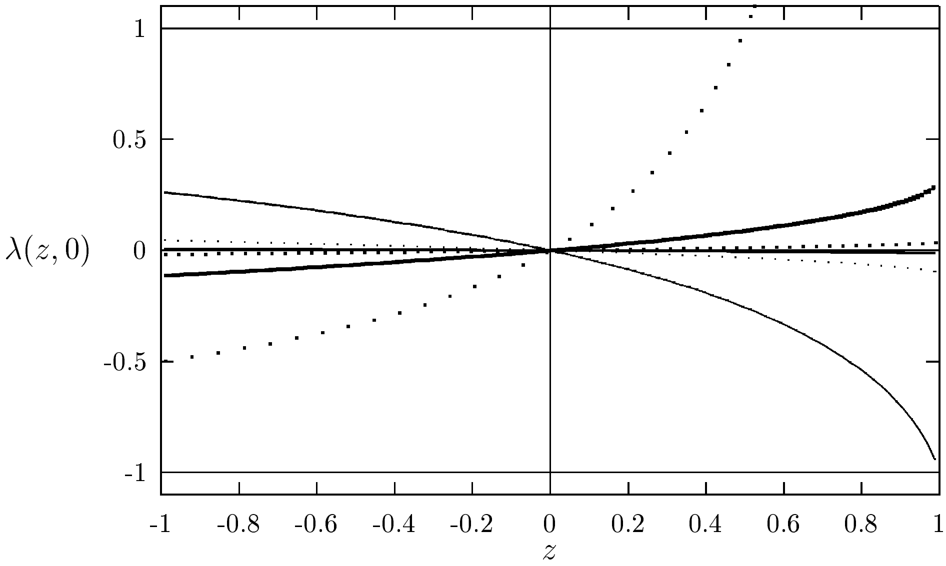
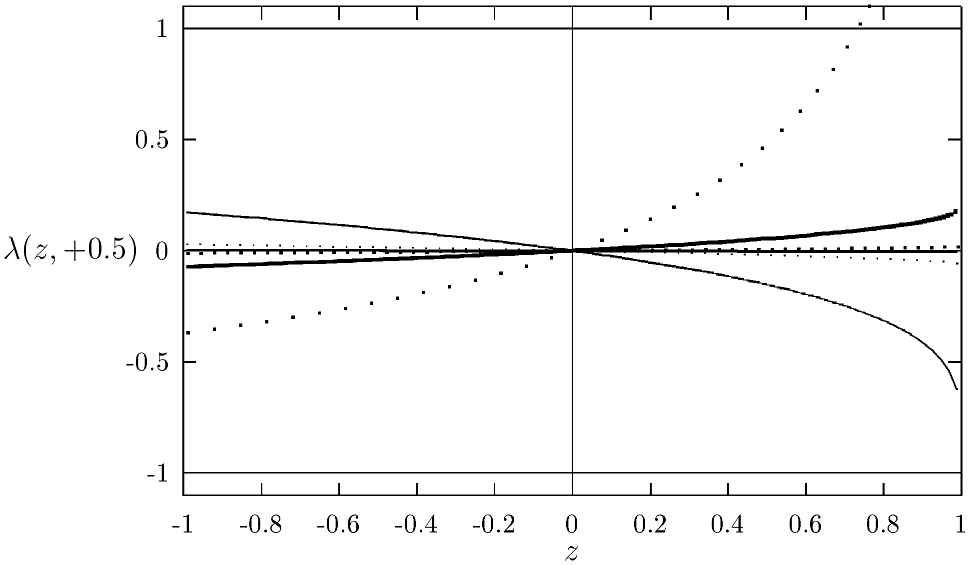


Figure 5.22: Leading 6 eigenvalues of MTO for Farey map vs z for $\beta = 0.5$



As mentioned previously, it is possible to solve the problem for the modified transfer operator analytically at some values of β. In particular, $\beta = -m/2$ where $m = 0, 1, 2, \ldots$, m+1 polynomial eigenfunctions of order m can be found. Of course, these are not the only eigenfunctions and there is no guarantee that they will have the largest eigenvalues. However, for $m = 0, 1$ and 2, exact solutions were obtained and the theoretical eigenvalues were found to match explicitly[1] with those determined numerically. For $m > 2$, exact solutions become much more infeasible. The actual eigenvalues for $\beta = 0, -1/2$ were found to be:

$$\lambda(z, 0) = \frac{z}{1-z} \tag{5.11}$$

and

$$\lambda_{\pm}(z, -1/2) = \frac{1}{2}\left(\frac{z \pm z(5 + 4z^2 - 8z)^{1/2}}{(z-1)^2}\right) \tag{5.12}$$

Note that the $\lambda_+(z, -1/2)$ diverges while $\lambda_-(z, -1/2)$ approaches -1 in the limit $z \to 1$. These may be easily compared by eye with figures 5.21 and 5.19 for various simple choices of z to show the equivalence. Satisfied by the quality of the numerics, search routines of a similar nature (see *transfer* program) to those used for the linearised case were used to find solutions to the problem $\lambda(z, \beta) = 1$ and hence the pressure function $\beta F(\beta)$, figures 5.23 and 5.24. There was noticeably less structure in the transfer operator's spectrum and within the region of β investigated only one positive solution was found. Recall that for the LFM, positive solutions appeared recursively, see figure 5.13. However, a comparison of the two pressure functions, figure 5.25, shows that the two maps express very similar behaviour at the level of their multifractal descriptions. The asymptotic behaviour of both maps as $\beta \to 1$ is certainly very similar.

It appears then, that the structure of the spectrum of the LFM's transfer operator is an artefact of the linearisation process. In the region considered, the transfer operator of the Farey map has only two eigenvalues outside the spectral radius. As for the LFM, the spectral gap appears to be exponentially increasing with decreasing β. The way the magnitudes of the other eigenvalues appear to be bounded by one is very curious

[1] that is, with a relative error at most 10^{-10}

and while no analytic explanations are offered here it is hoped that something may come of these present observations.

Figure 5.23: Solutions for $\lambda(z, \beta) = 1$ for MTO of Farey map

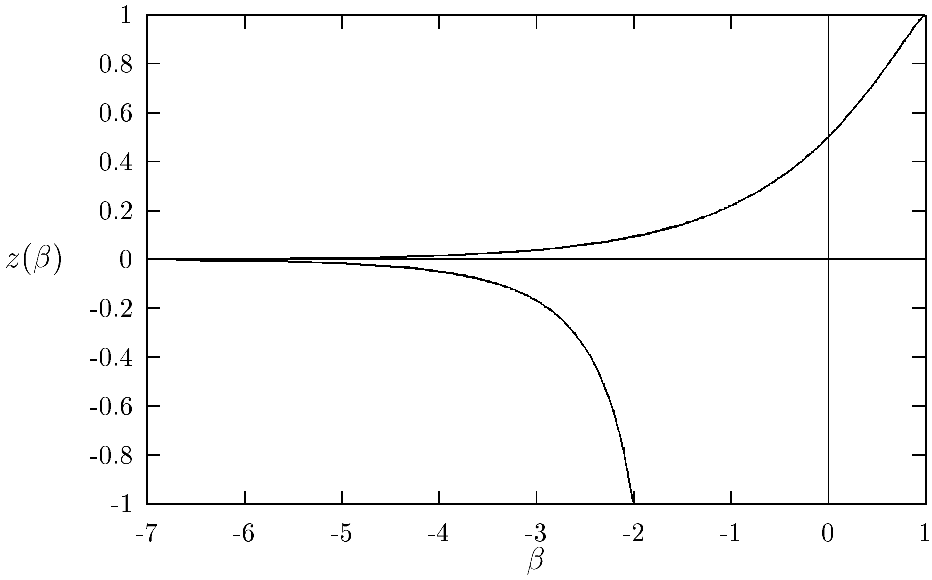


Figure 5.24: Pressure function for Farey map

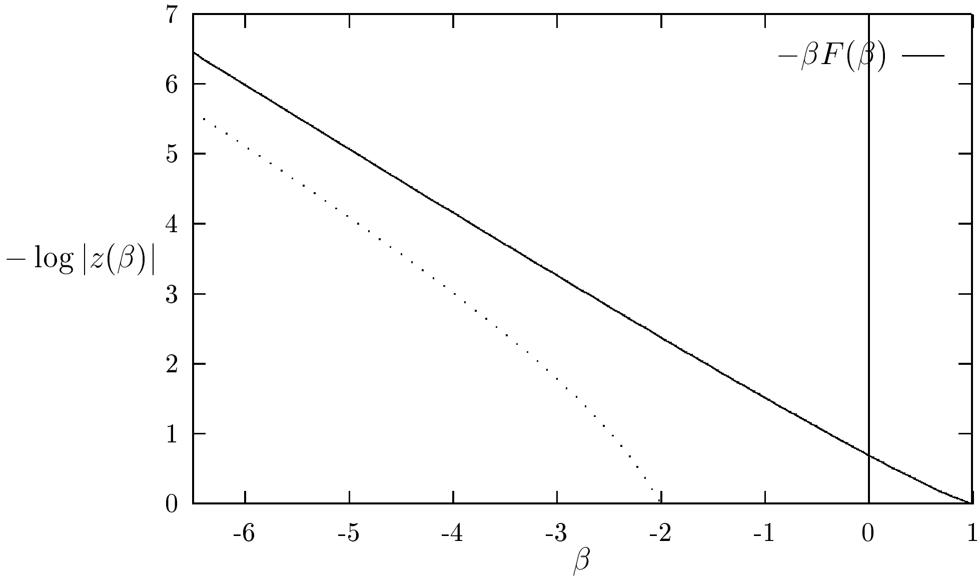
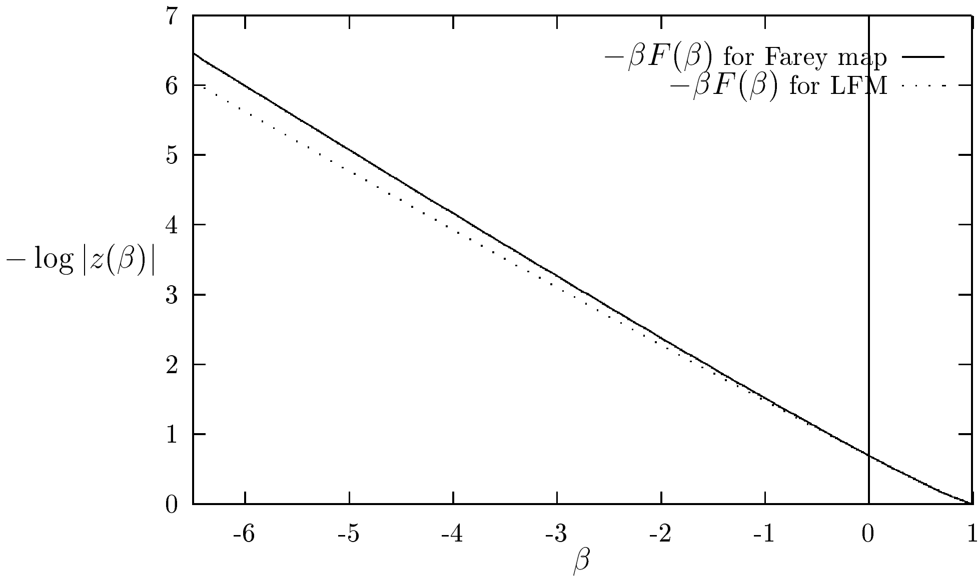
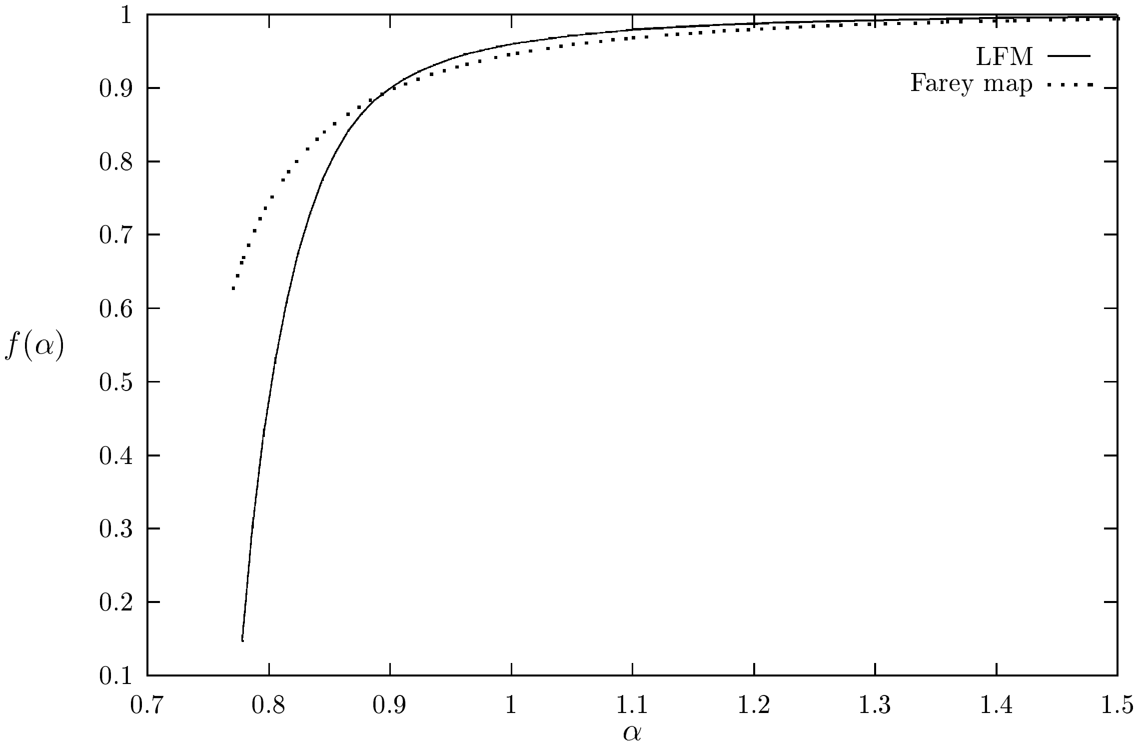


Figure 5.25: Comparison of Pressure functions for Farey map and LFM



Finally, we present the $f(\alpha)$ spectrum for the LFM and the Farey map. This was obtained via a very simple search algorithm written to process the output of the major program *transfer*. Once the pressure function $\beta F(\beta)$ is known, the entropy $S(E)$ may be determined by a Legendre transformation, see 2.15. The $f(\alpha)$spectrum was then calculated and plotted using relationship 4.1. Since the pressure functions for the LFM and the Farey map were found to be very similar, so too are their $f(\alpha)$spectra. It was with much interest that we observed the peculiar spectra of figure 2.9. Unlike the common single hump function that Tél suggests, we found the spectrum to asymptotically approach one. Also, while there exists a minimum α, it appears that all α greater than this feature in the measure we have constructed. While it may appear odd that an enormous amount of scaling values appear in the multifractals we have investigated and also with such density (because of their dimension being almost one), this can be analytically shown to be the case.

Figure 5.26: $f(\alpha)$ spectra for the LFM and the Farey map



We are interested in what happens to $f(\alpha)$ for large α which is equivalent to examining the $S(E)/E$ for small $E = \ln 2/\alpha$ (please refer to chapter 4). Also, considering the shape of $\beta F(\beta)$ we must have, for small E, the minimum of $\beta - \beta F(\beta)/E$ with respect to β, which is $f(\alpha)$ itself, occurring very near $\beta = 1$. Thus we may use Prellberg's asymptotic results for $\beta F(\beta)$ in the case of functions in the class $\mathcal{C}_1$ [6] which says

$$-\beta F(\beta) \sim c\frac{1-\beta}{-\log(1-\beta)} \tag{5.13}$$

where c is a constant. Using this and differentiating $(\beta - \beta F(\beta)/E)$ with respect to β we find the minimum occurs at

$$E = -\frac{c}{\log(1-\beta)}\left(\frac{1}{\log(1-\beta)} + 1\right) \tag{5.14}$$

We invert this expression for β by setting $a = \log(1-\beta)$ and solve the resultant quadratic in a to obtain:

$$a = \frac{-1 \pm (1-4E/c)^{\frac{1}{2}}}{2E/c} = \log(1-\beta) \tag{5.15}$$

Writing $\beta = 1 - \mathrm{e}^a$ we now have

$$S(E)/E = 1 - \mathrm{e}^a + c\frac{\mathrm{e}^a}{-aE} \tag{5.16}$$

Now, in the limit E goes to 0, we have $a \to -\infty$, so it is clear that

$$f(\alpha) = \left.\frac{S(E)}{E}\right|_{E=\ln 2/\alpha} \to 1 \quad \text{as} \quad \alpha \to \infty \tag{5.17}$$

So we have shown that the numerical results agree with the expected result due to the theory of Prellberg. The $f(\alpha)$ spectra are indeed pathological but this is reflective of the existence of the phase transition which, in turn, reflects the singular behaviour of the indifferent fixed point. As this was a late inclusion in the work, it is not yet clear what such an $f(\alpha)$ spectrum implies.

Chapter 6

Conclusion

The long term statistical behaviour of intermittent maps was shown to be a difficult characteristic to detect. The conventional idea of invariant density was shown to provide little information due to the domination of the indifferent fixed point at $x = 0$. A connection to the new area of multifractal formalism was made that extended the machinery of the invariant density to the more general setting of the Ruelle-Perron-Frobenius operator. The problems caused by the indifferent fixed point were then able to be removed by using the so-called induced map and the modified transfer operator. It was found with this stronger technique that it is possible to measure the long term behaviour of intermittent maps in a more subtle way.

The numerical results for both the Farey map and the LFM proved to be very interesting as well as very accurate. In particular, the spectrum of the Farey map's transfer operator showed that all bar two eigenvalues were were confined in the essential spectrum. All results showed the existence of a spectral gap which could be easily quantified. It is hoped that some analytic progress may be made in the future that may shed some light on this subdominant spectrum. Also, the resultant $f(\alpha)$ spectra showed the pathological nature of intermittent maps with an infinite spread of crowding indices corresponding to sets of dimension 1. This was also shown to agree with the theory of Prellberg [6].

This thesis touched on several seemingly disparate areas and it is acknowledged that there is much more mathematical and physical depth that may be explored. However, the work has provided some good insights into statistical mechanics, multifractals, intermittency and research in general, which will no doubt be of great benefit to future work.

Appendix A

Matrix Elements for the Modified Transfer Operators

The following two sections present the somewhat complicated decompositions of the eigenvalues of the piecewise linear Farey map's MTO and the normal matrix elements of the normal Farey map's MTO into sums of Lerch transcendent functions.

The Lerch transcendent function, $\Phi(z, s, d)$ is defined by the following series:

$$\Phi(z, s, d) = \sum_{n=0}^{\infty} \frac{z^n}{(n+s)^d} \tag{A.1}$$

A.1 Piecewise Linear Farey Map

The eigenvalues for the piecewise linear Farey map were found explicitly in chapter 5. Below is a decomposition of their general form into a series of Lerch transcendent functions.

$$\lambda_N(z,\beta) = (-1)^N \sum_{n=1}^{\infty} \frac{z^n}{(n^2(1+1/n))^{\beta+N}} \tag{A.2}$$

$$= \quad (-1)^N \left[\frac{z}{x^{\beta+N}} + \sum_{n=2}^{\infty} \frac{z^n}{(n^2(1+1/n))^{\beta+N}}\right]$$

$$= \quad (-1)^N \left[\frac{z}{x^{\beta+N}} + \sum_{n=2}^{\infty} \frac{z^n}{n^{2(\beta+N)}} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \frac{1}{n}^m \frac{(\beta+N+m-1)!}{(\beta+N-1)!} \right]$$

$$= \quad (-1)^N \left[\frac{z}{x^{\beta+N}} + z \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \frac{1}{n}^m \frac{(\beta+N+m-1)!}{(\beta+N-1)!} \left(\sum_{n=0}^{\infty} \frac{z^n}{(n+1)^{2\beta+2N+m}} - 1 \right) \right]$$

$$= \quad (-1)^N z \left[\frac{1}{x^{\beta+N}} + \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \frac{1}{n}^m \frac{(\beta+N+m-1)!}{(\beta+N-1)!} \left(\Phi(z, 2\beta+2N+m, 1) - 1 \right) \right]$$

where $\Phi(z, 2\beta + 2N + m, 1)$ is the Lerch transcendent function for $s = 1$ and $d = 2\beta + 2N + m$. The sum was found to converge very quickly. The following relationship was used in the calculation of Φ :

$$\begin{aligned} \Phi(z, 2\beta + 2N + m, 1) \quad &= \quad \frac{\Gamma(1 - 2\beta - 2N - m)}{z} (-\log z)^{2\beta+2N+m-1} \\ &+ z^{-1} \sum_{r=0}^{\infty} \zeta(2\beta + 2N + m - r) \frac{(\log z)^r}{r} \end{aligned} \tag{A.3}$$

where ζ is the Hurwitz-Zeta function and Γ is the well known Gamma function. See [1] for more details on these functions.

A.2 Farey Map

A similar approach was used to decompose the matrix elements into expansions involving Lerch transcendent functions (as compared with the actual eigenvalues of the LFM of the previous section)

$$\mathcal{M}^{z\beta} \circ \phi(1-x) = \sum_{n=1}^{\infty} \frac{z^n}{(1+n-nx)^{-2\beta}} \sum_{m=0}^{\infty} \phi_m \left(\frac{1-x}{1+n-nx} \right)^m \tag{A.4}$$

$$= \quad \sum_{m=0}^{\infty} \phi_m \sum_{n=1}^{\infty} \frac{z^n}{(1+n)^{-m-2\beta}} (1-x)^m (1 - \frac{n}{n+1} x)^{-m-2\beta}$$

$$= \quad \sum_{M=0}^{\infty} (-t)^M \sum_{m=0}^{\infty} \phi_m \sum_{\rho=0}^{M} \binom{-m-2\beta}{\rho} \binom{m}{M-\rho}$$

$$\times \sum_{n=1}^{\infty} z^n n^\rho (1+n)^{-m-\rho-2\beta}$$

We identify the matrix element of $\mathcal{M}^{z\beta}$:

$$\begin{aligned}
\mathcal{M}^{z\beta}_{Mm} &\equiv (-1)^M \sum_{\rho=0}^{M} \binom{-m-2\beta}{\rho} \binom{m}{M-\rho} \\
&\quad \times \sum_{n=1}^{\infty} z^n (1+n)^{-m-\rho-2\beta} ((1+n)-1)^{\rho} \\
&= (-1)^M \sum_{\rho=0}^{M} \binom{-m-2\beta}{\rho} \binom{m}{M-\rho} \\
&\quad \times \sum_{q=0}^{\rho} \binom{p}{q} (-1)^q (1+n)^{p-q} \sum_{n=1}^{\infty} z^n (1+n)^{-m-\rho-2\beta} \\
&= (-1)^M \sum_{\rho=0}^{M} (-1)^q \sum_{\rho=0}^{M-q} \binom{-m-2\beta}{M-\rho} \binom{m}{p} \binom{M-\rho}{q} \\
&\quad \times (\Phi(z, m+q+2\beta, 1) - 1)
\end{aligned} \tag{A.5}$$

where $\Phi(z, m+q+2\beta, 1)$ is the Lerch transcendent function defined in the previous section.

Appendix B

Program for Farey Map Analysis

Inclusion of algorithms used to obtain data.

```
c
c transfer.m4
c To regenerate Fortran versions
c m4 -DX file.m4 >file.f
c where X=S,D or Q for single, double, or quadruple.
c Link with LAPACK
c234567890123456789012345678901234567890123456789012345678901234567890
undefine(‘index’)
undefine(‘format’)
undefine(‘shift’)
undefine(‘include’)
undefine(‘len’)
ifdef(‘LAHEY’,‘
define(‘IMAG’,‘dimag’)
’)
ifdef(‘S’,‘
define(‘MACHEPS’,‘2.0d-7’)
define(‘CTYPE’,‘complex’)
define(‘RTYPE’,‘real’)
define(‘EXPLETTER’,‘e’)
define(‘ZERO’,‘0.0’)
define(‘HALF’,‘0.5’)
define(‘ONE’,‘1.0’)
define(‘TWO’,‘2.0’)
define(‘THREE’,‘3.0’)
define(‘FOUR’,‘4.0’)
define(‘PION2’,‘1.57079 63267’)
define(‘PI’,‘3.14159 26535’)
define(‘TWOPI’,‘6.28318 53072’)
define(‘PRECISION’,‘"single"’)
’)
ifdef(‘D’,‘
define(‘MACHEPS’,‘2.22d-16’)
define(‘CTYPE’,‘complex*16’)
define(‘RTYPE’,‘real*8’)
define(‘EXPLETTER’,‘d’)
```

```
define(`ZERO',`0.0d0')
define(`HALF',`0.5d0')
define(`ONE',`1.0d0')
define(`TWO',`2.0d0')
define(`THREE',`3.0d0')
define(`FOUR',`4.0d0')
define(`PION2',`1.57079 63267 94896 61923d0')
define(`PI',`3.14159 26535 89793 23846d0')
define(`TWOPI',`6.28318 53071 79586 47692d0')
define(`PRECISION',`"double"')
')
ifdef(`Q',`
define(`MACHEPS',`2.0q-32')
define(`CTYPE',`complex*32')
define(`RTYPE',`real*16')
define(`EXPLETTER',`q')
define(`ZERO',`0.0q0')
define(`HALF',`0.5q0')
define(`ONE',`1.0q0')
define(`TWO',`2.0q0')
define(`THREE',`3.0q0')
define(`FOUR',`4.0q0')
define(`PION2',`1.570796326794896619231321691639751442q0')
define(`PI',`3.141592653589793238462643383279502884q0')
define(`TWOPI',`6.283185307179586476925286766559005768q0')
define(`PRECISION',`"quadruple"')
')
c     transfer computes data for the Farey map and its linearized
c     version, LFM.
c

      program scan

      implicit RTYPE  (a-h,o-y)
      implicit CTYPE  (z)

1     continue
      write (6,*) 'Please enter a choice:'
      write (6,*) '1. exit'
      write (6,*) '2. F map: |eigenvalues| as fn of z,beta'
      write (6,*) '3. LFM: soln for evalue = 1 : z positive'
      write (6,*) '4. LFM: soln for evalue = 1 : z negative'
      write (6,*) '5. LFM: eigenvalues(z), fixed beta'
      write (6,*) '6. LFM: eigenvalues(beta), fixed z'
      write (6,*) '7. F map: soln for evalue = 1 : z positive'
      write (6,*) '8. F map: soln for evalue = 1 : z negative'
      read(6,'(i2)') j
      if (j.eq. 1) goto 100
      if (j.eq. 2) goto 200
      if (j.eq. 3) goto 300
      if (j.eq. 4) goto 400
      if (j.eq. 5) goto 500
      if (j.eq. 6) goto 600
      if (j.eq. 7) goto 700
      if (j.eq. 8) goto 800
      write (*,*) 'Try again....'
```

```
      goto 1

100   write (*,*) 'Bye....'
      stop
200   call zbetasurface
      goto 1
300   call lineareigposzed
      goto 1
400   call lineareignegzed
      goto 1
500   call linbetzed
      goto 1
600   call linzedbet
      goto 1
700   call fareyeigposzed
      goto 1
800   call fareyeignegzed
      goto 1
      end


c----------------------------------------------------------------------

      subroutine lineareigposzed

      implicit RTYPE (a-h,o-y)
      implicit CTYPE (z)
      parameter(nbmax=100)

c     search for zeig=1 and -1,  store z,beta     1/z=beta F(beta)
c     check abs(zeig-ONE) to do this

      open(10,file='linear/linear+.dat')

      write(10,'("# linear map data from 0th eigenvalue")')
      write(10,'("# data is beta,z,-log(abs(z))")')
      write(10,'("# output for gnuplot from transfer")')

      write(6,*) 'Search for 0th evalue = 1'
      write(6,'("Please enter no. beta cuts <",i3)') nbmax
      read*,rnb

      write(6,'("Please enter beta minimum and maximum:")')
      read*,betamin,betamax
      betadd=(betamax-betamin)/rnb

c search routine to find where zeig(z,beta)=1
c please make sure zplus=1/(integer)

      do beta=betamin,betamax+betadd/2,betadd
        print*,'beta = ',beta
        z=2EXPLETTER-1
        zplus=z
        zeig=ZERO

c search routine to find where abs(zeig(z,beta))> 1 and < 1
```

```
c know zeig=0 when z=0 and zeig increases with z
c so take steps of .1 out from origin until zeig >1
c backtrack and take steps of .05, .0025 and so on


        do while (abs(zeig).le.ONE.and.
     +             abs(z).lt.(ONE-abs(zplus)/2))
         call lineareigs(z,beta,0,zeig)
            z=z+zplus
        enddo

        zgood=z
        z=z-2*zplus
        zplus=zplus*HALF
        z=z+zplus
        if (abs(zeig).le.ONE) z=z+2*zplus
c        print*,zplus
c        print*,zgood,zeig

        do while(abs(zeig-ONE).ge.(1.0EXPLETTER-6).
     +            and.converged.le.HALF)
          call lineareigs(z,beta,0,zeig)
          zgood=z
c          print*,'z = ',z
c          print*,'zeig = ',zeig
          if (abs(zeig).le.ONE) then
            z=z+HALF*zplus
          else
            z=z-HALF*zplus
          endif
          zplus=zplus*HALF
c          print*,'zplus = ',zplus
c          print*,'z = ',zgood
c          print*,'zeig = ',zeig
        enddo

        rzgood = zgood
        write (6,'("z(beta) = ",g36.30)') rzgood
        write(6,'("zeig = ",3g12.5,/)') zeig
        alogofz = - log (abs(zgood))
        write(10,'(5(g25.15))') beta,rzgood,alogofz


      enddo

      close(10)

      return
      end

c-----------------------------------------------------------------

      subroutine lineareignegzed


      implicit RTYPE (a-h,o-y)
```

```
      implicit CTYPE (z)
      parameter(nbmax=100)

      write(6,*) 'Which eigenvalue?'
      read*,neig
      write(6,*) 'Which zero?'
      read*,nzero
      open(11,file='linear/linear-.dat')
      write(11,'("# Data from ",i2," eigenvalue")') neig
      write(11,'("# beta, z, log(abs(z))")')
      write(11,'("# Output for gnuplot from transfer")')
      write(11,'("# ",i2,"th zero ")') nzero

      write(6,'("Please enter no. beta cuts <",i3)') nbmax
      read*,rnb
      write(6,'("Please enter beta minimum and maximum:")')
      read*,betamin,betamax
      betadd=(betamax-betamin)/rnb
      if (betadd.eq.0) betadd=1.0

c search routine to find where zeig_neig(z,beta)=0

      do beta=betamin,betamax+betadd/2,betadd
        print*,'beta = ',beta
        z=-1EXPLETTER-3
        zminus=z
        check=z
        zeig=ZERO
        rzeig=ZERO
        change=ZERO
signold=ZERO
nzilch=0
c search routine to find where abs(zeig(z,beta))-1=0
c know zeig=0 when z=0
c so take steps of -.1(say) out from origin until
c zeig passes through 1
c backtrack and take steps of .05, .0025 and so on


        do while (change.le.HALF.and.
     +      abs(z).le.(ONE-HALF*abs(zminus)))
         call lineareigs(z,beta,neig,zeig)
            z=z+zminus
    rzeig=zeig
    sign=(rzeig-ONE)/abs(rzeig-ONE)
            if (abs(sign-signold).ge.(1.5)) nzilch=nzilch+1
    if ((nzilch+.5).gt.nzero) change=ONE
    signold=sign
        enddo
        if (change.le.HALF) goto 100
        zgood=z
        z=z-2*zminus
        zminus=zminus*HALF
        z=z+zminus

        do while(abs(zeig-ONE).ge.(1.0EXPLETTER-6))
```

```
          call lineareigs(z,beta,neig,zeig)
          zgood=z
  rzeig=zeig
          if (sign*(rzeig-ONE).le.ZERO) then
            z=z+HALF*zminus
          else
            z=z-HALF*zminus
    found=ONE
          endif
          zminus=zminus*HALF
c          print*,'zminus = ',zminus
c          print*,'z = ',zgood
c          print*,'zeig = ',zeig
        enddo
100     continue
        if (change.le.HALF) found=ZERO
        if (found.ge.HALF) then
          rzgood=zgood
          write(6,'("z(beta) = ",g36.30)') rzgood
          write(6,'("zeig = ",2g25.15,/)') zeig
          alogofz = - log (abs(zgood))
          write(11,'(5(g11.5))') beta,rzgood,alogofz
        else
  write(11,'("# no soln found for beta =",g11.5)') beta
          write(6,'("no soln found",/)')
        endif

      enddo


      close(11)


      return
      end

c-----------------------------------------------------------------------

      subroutine linbetzed

      implicit RTYPE (a-h,o-y)
      implicit CTYPE (z)

      open(12,file='linear/linear.dat')
      print*,'Please enter beta'
      read*,beta
      print*,'Please enter number of z cuts'
      read*,rnz
      print*,'Please enter no. of eigenvalue'
      read*,neig
      write(12,'("# eigenvalue vs zed plot")')
      write(12,'("# beta = ",g25.15)') beta
      write(12,'("# number ",i2," eigenvalue")') neig

      do rz=-.950,.999,(1.949/rnz)
z=rz
```

```
        call lineareigs(z,beta,neig,zeig)
        write(12,'(4(g25.15))') rz,zeig
        print*,'z = ',rz
        print*,'zeig = ',zeig
      enddo

      close(12)

      return
      end

c-----------------------------------------------------------------------

      subroutine linzedbet

      implicit RTYPE (a-h,o-y)
      implicit CTYPE (z)

      open(8,file='linear/linearb.dat')

      print*,'Please enter zed'
      read*,z
      print*,'Please enter number of beta cuts'
      read*,rnbeta
      print*,'Please enter no. of eigenvalue'
      read*,neig
      write(8,'("# eigenvalue vs beta plot")')
      write(8,'("# zed = ",g25.15)') z
      write(8,'("# number ",i2," eigenvalue")')

      do beta=-1,1,(2/rnbeta)
        call lineareigs(z,beta,neig,zeig)
        write(8,'(4(g25.15))') beta,zeig
        print*,'beta = ',beta
        print*,'zeig = ',zeig
      enddo

      close(8)

      return
      end



c-----------------------------------------------------------------------

      subroutine fareyeigposzed

      implicit RTYPE (a-h,o-y)
      implicit CTYPE (z)
      parameter(nbmax=100,nsigma=6)
      dimension zsigma(nsigma)
      neig=1
      neigenvalue=4
      num=neigenvalue
      controlerr=1.0EXPLETTER-7
```

```
c     search for zeig=1 and -1,  store z,beta     1/z=beta F(beta)
c     check abs(zeig-ONE) to do this

      open(10,file='farey/farey+.dat')

      write(10,'("# farey map data from 0th eigenvalue")')
      write(10,'("# data is beta,z,-log(abs(z))")')
      write(10,'("# output for gnuplot from transfer")')

      write(6,*) 'Search for where 0th evalue = 1'
      write(6,'("Please enter no. beta cuts <")')
      read*,rnb

      write(6,'("Please enter beta minimum and maximum:")')
      read*,betamin,betamax
      if (betamax.ge.ONE) then
        betamax=ONE-1.0EXPLETTER-2
        write(6,'("betamax cut to ",g11.5)') betamax
      endif
      betadd=(betamax-betamin)/rnb

c search routine to find where zeig(z,beta)=1
c please make sure zplus=1/(integer)

      do beta=betamin,betamax+betadd/2,betadd
        print*,'beta = ',beta
        z=2EXPLETTER-2
        zplus=z
        zsigma(num)=ZERO

c search routine to find where abs(zsigma(num))> 1 and < 1
c know zsigma(num)=0 when z=0 and zeig increases with z
c so take steps of .1 out from origin until zeig >1
c backtrack and take steps of .05, .0025 and so on

        do while (abs(zsigma(num)).le.ONE.and.
     +            abs(z).lt.(ONE-abs(zplus)/2))
          call prespectrum (z,beta,ztrace,nsigma,zsigma,errortrace,
     +  info,neig,neigenvalue,controlerr)
          print*,'z=',z
          print*,'eigenvalue is ',zsigma(num)
          z=z+zplus
        enddo

        zgood=z
        z=z-2*zplus
        zplus=zplus*HALF
        z=z+zplus
        if (abs(zsigma(num)).le.ONE) z=z+2*zplus

        do while(abs(zsigma(num)-ONE).ge.(controlerr*10))
          call prespectrum (z,beta,ztrace,nsigma,zsigma,errortrace,
     +  info,neig,neigenvalue,controlerr)
          print*,'z=',z
          print*,'eigenvalue is ',zsigma(num)
          zgood=z
```

```
          if (abs(zsigma(num)).le.ONE) then
            z=z+HALF*zplus
          else
            z=z-HALF*zplus
          endif
          zplus=zplus*HALF
          if (abs(zplus).le.(1EXPLETTER-25)) goto 50
        enddo

        rzgood = zgood
        write (6,'("z(beta) = ",g36.30)') rzgood
        write(6,'("zsigma(num) = ",3g12.5,/)') zsigma(num)
        alogofz = - log (abs(zgood))
        write(10,'(5(g25.15))') beta,rzgood,alogofz
      enddo

50    continue
      if (abs(zplus).le.1EXPLETTER-25)
     +  write(10,'("# no soln found for beta =",g11.5)') beta

      close(10)
      return
      end

c-----------------------------------------------------------------------

      subroutine fareyeignegzed

      implicit RTYPE (a-h,o-y)
      implicit CTYPE (z)
      parameter(nbmax=100,nsigma=6)
      dimension zsigma(nsigma)
      neig=2
      neigenvalue=4
      num=1
      controlerr=1.0EXPLETTER-7
c     search for zeig=1 and -1,  store z,beta     1/z=beta F(beta)
c     check abs(zeig-ONE) to do this

      open(10,file='farey/farey-.dat')

      write(10,'("# farey map data from 1st eigenvalue")')
      write(10,'("# data is beta,z,-log(abs(z))")')
      write(10,'("# output for gnuplot from transfer")')

      write(6,*) 'Search for where 0th evalue = 1'
      write(6,'("Please enter no. beta cuts <")')
      read*,rnb

      write(6,'("Please enter beta minimum and maximum:")')
      read*,betamin,betamax
      betadd=(betamax-betamin)/rnb

c search routine to find where zeig(z,beta)=1
c please make sure zplus=1/(integer)
```

```
      do beta=betamin,betamax+betadd/2,betadd
        print*,'beta = ',beta
        z=-2EXPLETTER-1
        zplus=z
        zsigma(num)=ZERO

c search routine to find where abs(zsigma(num))> 1 and < 1
c know zsigma(num)=0 when z=0 and zeig increases with z
c so take steps of .1 out from origin until zeig >1
c backtrack and take steps of .05, .0025 and so on

        do while (abs(zsigma(num)).le.ONE.and.
     +             abs(z).lt.(ONE-abs(zplus)/2))
          call prespectrum (z,beta,ztrace,nsigma,zsigma,errortrace,
     +  info,neig,neigenvalue,controlerr)
          print*,'z=',z
          print*,'eigenvalue is ',zsigma(num)
          z=z+zplus
        enddo

        zgood=z
        z=z-2*zplus
        zplus=zplus*HALF
        z=z+zplus
        if (abs(zsigma(num)).le.ONE) z=z+2*zplus

        do while(abs(zsigma(num)-ONE).ge.(controlerr*10))
          call prespectrum (z,beta,ztrace,nsigma,zsigma,errortrace,
     +  info,neig,neigenvalue,controlerr)
          print*,'z=',z
          print*,'eigenvalue is ',zsigma(num)
          zgood=z
          if (abs(zsigma(num)).le.ONE) then
            z=z+HALF*zplus
          else
            z=z-HALF*zplus
          endif
          zplus=zplus*HALF
          if (abs(zplus).le.1EXPLETTER-25) goto 50
        enddo

        rzgood = zgood
        write (6,'("z(beta) = ",g36.30)') rzgood
        write(6,'("zsigma(num) = ",3g12.5,/)') zsigma(num)
        alogofz = - log (abs(zgood))
        write(10,'(5(g25.15))') beta,rzgood,alogofz
      enddo

50    continue
      if (abs(zplus).le.1EXPLETTER-25)
     +  write(10,'("# no soln found for beta =",g11.5)') beta

      close(10)
      return
      end
```

```
c-----------------------------------------------------------------------
      subroutine zbetasurface

      implicit RTYPE  (a-h,o-y)
      implicit CTYPE  (z)
      parameter(nzmax=100,nbmax=100,nsigma=10)
      dimension zsigma(nsigma)
      controlerr=1.0EXPLETTER-10
      open (unit=10,file='surface/surface4.dat')
      write(10,'("# output for gnuplot from transfer")')
      write(6,'("Please enter no. zed cuts; 0=line cut, const z")')
      read*,rnzed
      write(6,'("Please enter no. beta cuts; 0=line cut, const beta")')
      read*,rnb
      nb=rnb
      nzed=rnzed
      if (nzed.eq.0) then
        print*,'Please enter z'
        read*,realzed
      else
        print*,'Please enter zmin and zmax of magnitude < 1 '
        read*,rminz,rmaxz
        addzed=(rmaxz-rminz)/rnzed
realzed=rminz
      endif
      if (nb.eq.0) then
print*,'Please enter beta'
read*,beta
        betamin=beta
      else
        print*,'Please enter beta min > -2, beta max < 2'
        read*,betamin,betamax
        addbeta=(betamax-betamin)/rnb
beta=betamin
      endif
      print*,'Please enter # eigenvalues'
      read*,neigenvalue
      print*,'Converge w.r.t. which evalue (0 for trace)?'
      read*,neig
      if (nzed.eq.0.and.nb.eq.0) then
write(10,'("# zed,beta = ",e11.5,1x,e11.5)') realzed,beta
      else
        if (nzed.eq.0) then
          write(10,'("# line cut for zed = ",e11.5)') realzed
        else
         write(10,'("# zed range is from ",e11.5," to ",
     +    e11.5)') rminz,rmaxz
        endif
        if (nb.eq.0) then
          write(10,'("# line cut for beta = ",e11.5)') beta
        else
          write(10,'("# beta range is from ",e11.5," to "
     +        ,e11.5)') betamin,betamax
        endif
      endif
```

```
      write(10,'("# first ",i3," eigenvalues")') neigenvalue

      do i=1,nzed+1,1
        write(6,'("zed is ",2e25.15)') realzed
        do j=1,nb+1,1
         write(6,'("beta is ",e25.15)') beta
         zed=realzed
         call prespectrum (zed,beta,ztrace,nsigma,zsigma,errortrace,
     +   info,neig,neigenvalue,controlerr)

          if (nb.eq.0) then
            write(10,'(20(1x,g12.5))') realzed,
     + (abs(zsigma(k)),k=1,neigenvalue,1),(zsigma(k),k=1,neigenvalue,1)
          elseif (nzed.eq.0) then
            write(10,'(20(1x,g12.5))') beta,
     + (abs(zsigma(k)),k=1,neigenvalue,1),(zsigma(k),k=1,neigenvalue,1)
          endif

          beta=beta+addbeta
        enddo
       if (nb.ne.0) write(10,'(1x)')
       beta=betamin
       realzed=realzed+addzed
      enddo

      write(6,*) 'Output stored in surface/surface4.dat'

100   format(' eig is ',e11.5,/)
      close(10)

      return
      end

c-----------------------------------------------------------------------

      subroutine lineareigs(z,beta,N,zeig)


      implicit RTYPE  (a-h,o-y)
      implicit CTYPE  (z)
      logical converged

      converged = .false.
      m=0
      coeff=ONE
      oldiff=ONE

      zeig=2**(-beta-N)

      do while (.not.converged)
        zeigold=zeig
        zmfactor=zlerch(z,TWO*beta+2*N+m,ONE)-ONE
        zeig=zeig+zmfactor*coeff
        diff=abs((zeig-zeigold)/zeig)
        if (diff.le.MACHEPS.or.(diff.ge.oldiff.and.m.gt.15))
     +        converged=.true.
```

```
        m=m+1
        coeff=coeff*(-1.0)*(beta+N+m-1)/m
        oldiff=diff
      enddo

      zeig=z*zeig*(-1)**N

      return
      end

c ------------------------------------------------------------------------

      subroutine prespectrum (z,beta,ztrace,nsigma,zsigma,errortrace,
     +  info,neig,neigenvalue,controlerr)

      implicit RTYPE  (a-h,o-y)
      implicit CTYPE  (z)
      parameter (nsigmamax=20)
      dimension zsigma(nsigma)
      dimension zsig(nsigmamax)
      dimension index(nsigmamax)

c Resort eigenvalues in order of real part (for z real calcs only).
c Sorting by magnitude is not enough as evalues do not keep the
c same order. However they appear to be distinct wrt real values.
c observe that they must swap as they pass through zero by
c continuity of derivative.  There is another sorting
c by magnitude first to make sure of things.  This sorting
c works very well for the Farey map.

         call spectrum (z,beta,ztrace,nsigma,zsigma,errortrace,
     +   info,neig,controlerr)
          if (info.ne.0) then
           print*,'error in spectrum, info = ',info
           stop
          endif
          realzed=z
          do k=1,nsigma-1,1
            do l=1,nsigma-1,1
              rz1=zsigma(l)
              rz2=zsigma(l+1)
              if (abs(zsigma(l)).lt.abs(zsigma(l+1))) then
                zed=zsigma(l)
                zsigma(l)=zsigma(l+1)
                zsigma(l+1)=zed
              endif
            enddo
          enddo
          do k=1,nsigma,1
            index(k)=k
            zsig(k)=ZERO
          enddo
          do k=1,nsigma-1,1
            do l=1,nsigma-1,1
              rz1=zsigma(index(l))
              rz2=zsigma(index(l+1))
```

```
            if (rz1.lt.rz2) then
              ind=index(l)
              index(l)=index(l+1)
              index(l+1)=ind
            endif
          enddo
        enddo

        num=1
        do k=1,nsigma,1
          if (index(k).le.neigenvalue) then
            zsig(num)=zsigma(index(k))
            num=num+1
          endif
        enddo

        if (realzed.ge.ZERO) then
          do k=1,neigenvalue,1
            zsigma(k)=zsig(neigenvalue-k+1)
c              print*,zsigma(k),k,index(k)
          enddo
        else
          do k=1,neigenvalue,1
            zsigma(k)=zsig(k)
c              print*,zsigma(k),k,index(k)
          enddo
        endif

      return
      end

c -----------------------------------------------------------------------

      subroutine spectrum (z,beta,ztrace,nsigma,zsigma,errortrace,
     +  info,neig,controlerr)


c     uses special function package specfunc for lerch's transcendent
c     jan 22, 1993, tp
c
c     info= 0: ok
c         = 1: trace not converged
c         =-1: error in computation of matrix entries

      implicit RTYPE  (a-h,o-y)
      implicit CTYPE  (z)
      parameter(maxn = 100) ! maximum matrix size
      parameter(nmax = 50) ! maximum matrix size
      parameter(tol=1.0EXPLETTER-6)  ! abs tolerance for trace convergence
      parameter(fac=.25EXPLETTER 0)  ! shrink factor for new z
      parameter(lwork = maxn*(maxn+6), ldvl=maxn, ldvr=maxn)
      dimension zsigma(nsigma)
      dimension za(maxn,maxn),zb(maxn,maxn),zw(maxn)
      dimension rwork(lwork),work(lwork)
      dimension vl(ldvl,maxn), vr(ldvr,maxn), scale(maxn)
      dimension rconde(maxn), rcondv(maxn), indx(maxn), absw(maxn)
```

```
      dimension z_l(maxn*2+1)
      character jobvl,jobvr,balanc,sense
      common bbeta,zz,z_l
      external zelement
      logical converged,matrixerror
      bbeta=beta
      zz=z

      eps = dlamch('P')

c        write(6,2) beta,z
2        format('# beta=',f25.15,', z=',2f25.15)
         maxw1 = 0
c If next are 'V', vl, vr etc. must be dimensioned.
         jobvl = 'V'
         jobvr = 'V'
         balanc = 'B'
         sense = 'E' ! 'E' means compute eigenvalue rec cond numbers
         zoldtrace = ZERO
 zwold = ZERO
         olderr=ONE


c loop over size n ( main loop begins here )
         do n = 1,nmax-1,1
    converged = .false.
c prepare new pseudo_lerch entries for use in zelement.
c each iteration a further row and column are added to the
c matrix and the eigenvalues extracted. Here z_1 is calculated and
c made a common variable for convenience.

            z_l(2*n-1)=zlerch(z,TWO*beta+2*n-2,ONE)-ONE
c      print*,2*n-1,z_l(2*n-1)
            z_l(2*n)=zlerch(z,TWO*beta+2*n-1,ONE)-ONE
c      print*,2*n,z_l(2*n)

c matrix is now augmented with the values of zelement which basically calls up
c Lerch transcendent fns, gamma fns, riemann fns, combinatorial values
c and other things ( see tp's decomp of M matrix )  nb. this must be done for
c each distinct pair (z,beta)

            do i = 1,n
               zb(i,n) = zelement(i,n)
            enddo
            do j = n-1,1,-1
               zb(n,j) = zelement(n,j)
            enddo
c           write(6,'("# Size=",i3,"; ",f8.2," seconds.")')
c    &            n, mclock()/60.0

c Copy matrix from saved version
    matrixerror = .false.
c     print*,'matrix...'
            do i = 1,n
               do j = 1,n
                  za(i,j) = zb(i,j)
```

```
c   print*,i,j,za(i,j)
          matrixerror = zb(i,j).ne.zb(i,j)
          if (matrixerror) goto 10
               enddo
            enddo
c   Compute eigenvalues of matrix
            call  zgeevx( balanc, jobvl, jobvr, sense, n, za, maxn, zw,
     $                       vl, ldvl, vr, ldvr, ilo, ihi, scale, abnrm,
     $                        rconde, rcondv, work, lwork, rwork, info )
            if(info.ne.0) then
              write(6,*)'# Error in eigenvalue computation'
              write(6,*)'# zgeevx returns info=',info
              stop '# Error in eigenvalue computation'
            endif
    maxw1 = max(int(work(1)),maxw1)
            do i = 1, n
       absw(i) = abs(zw(i))
    enddo

c output
            ztrace = ZERO
c sort evalues and sum them to obtain the trace which is used to
c determine convergence...

    call dindex(absw,n,indx)

            do j = 1, n
       i = indx(n+1-j) ! get evs in descending || order
               if (i.le.nsigma) then
c                 write(6,40) i,zw(i),rconde(i),eps*abnrm/rconde(i)
       endif
               ztrace = ztrace+zw(i)
            enddo
c      print*,'neig = ',neig
    if (neig.ne.0) then
              err=abs(zw(neig)-zwold)
      errmin=abs(eps*abnrm/rconde(neig))
    else
              err=abs((ztrace-zoldtrace)/ztrace)
              if (abs(ztrace).eq.ZERO) err=ZERO
      errmin=-ONE
            endif
    errortrace=abs((ztrace-zoldtrace)/ztrace)
c     print*,'err',err
c     print*,'errmin',errmin
c           print*,'olderr',olderr

c
c check to see if trace is close enough to previous one or if the error
c has actually become greater; in the latter case we assume that this is still
c OK and use the evalues so obtained - empirically it seems that the trace
c converges and then goes berserk so this is as near as we may get.
c Also, the convergence may be changed to be with respect to a
c particular eigenvalue. eigenvalues appear to converge uniformly.
c
c
```

```
c

    if (n.gt.15.and.(err.ge.olderr.or.err.lt.
     +   errmin.or.abs(ztrace).lt.eps.or.err.le.controlerr)) then
c             write(6,2) beta,z
c             write(6,'("# index",8x,"ev",27x,"rec cond",4x,"error")')
c              print*,'err',err
c        print*,'errmin',errmin
c              print*,'olderr',olderr
              do j = n, 1, -1
                i = indx(j) ! get evs in descending || order
                 if (i.le.nsigma) then
c                   write(6,40) i,zw(i),rconde(i),eps*abnrm/rconde(i)
         endif
              enddo
              if (neig.eq.0) write(6,55) ztrace,err
c              write(6,'("# max work(1)=",i6)') maxw1
c              write(6,'("# gnuplot output (modulus of evs)...")')
c output of absolute value only!
c          write(6,'(15(1x,g11.5))') z,(abs(zw(indx(j))),j=n,n-9,-1)
c              write(6,'(/)')
       converged = .true.
       goto 10
    else
              do j = n, 1, -1
                i = indx(j) ! get evs in descending || order
                 if (i.le.nsigma) then
                   zsigma(i)=zw(i)
c                   write(6,40)  i,zw(i),rconde(i),eps*abnrm/rconde(i)
                 endif
              enddo
c              print*,'trace =', ztrace
c              print*,'error =', err
       zoldtrace = ztrace
       olderr=err
              if (neig.ne.0) zwold=zw(neig)
c              write(6,'("# time ",f10.2," seconds.")') mclock()/60.0
    endif
         enddo

10       continue
         info=0
         if (matrixerror) then
   info=-1
c          write(6,'("# Error in matrix entries!")')
         elseif (.not.converged) then
   info=1
c          write(6,'("# Trace not converged!")')
         endif
 return

40    format('# ',i5,2(1x,e25.15),1x,f15.10,1x,1pe10.3)
55    format('# trace=',2g25.15,' error=',1pe10.3)
      end

c -----------------------------------------------------------------------
```

```
c evaluation of actual elements of matrix representing the modified
c transfer operator M.

      function zelement(i,j)
      implicit RTYPE  (a-h,o-y)
      implicit CTYPE  (z)
      parameter(maxn=100)
      dimension z_l(2*maxn+1)
      logical odd
      common beta,z,z_l
      mm = i-1
      m  = j-1
c      min=mm
c      if (m.lt.mm) min=m
      zel = ZERO
      a = -m-TWO*beta
      b = m
c      do k = 0,min
      do k = 0,mm
         term = ZERO
 do l = 0,mm-k
    c = mm-l
    term = term+choose(a,mm-l)*choose(b,l)*choose(c,k)
         enddo
         odd = mod(k,2).eq.1
         if (odd) term = -term
         zterm = term*z_l(m+k+1)
         zel = zel+zterm
      enddo
      odd = mod(mm,2).eq.1
      if (odd) zel = -zel
      zelement = zel
c     print*,'#&',m,mm,zelement
      return
      end

c -----------------------------------------------------------------------

      function choose(a,n)
      implicit RTYPE  (a-h,o-y)
      implicit CTYPE  (z)
      choose = ONE
      a1 = a+ONE
      do m = 1,n
        choose = choose*(a1-m)/m
      enddo
      return
      end
```

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