On the Thermodynamic Formalism for the Farey Map

Masters Thesis

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Abstract

The chaotic phenomenon of intermittency is modeled by a simple map of the unit interval, the Farey map. The long term dynamical behaviour of a point under iteration of the map is translated into a spin system via symbolic dynamics. Powerful methods from dynamical systems theory and statistical mechanics may then be used to analyse the map, respectively the zeta function and the transfer operator. Intermittency is seen to be problematic to analyze due to the presence of an 'indifferent fixed point'. Points under iteration of the map move away from this point extremely slowly creating pathological convergence times for calculations. This difficulty is removed by going to an appropriate induced subsystem. which also leads to an induced zeta function and an induced transfer operator. The main work is then divided into two sections. The first demonstrates a connection between the induced versions of the zeta function and the transfer operator providing useful results regarding the analyticity of the zeta function. The second provides a detailed analysis of the pressure function for the induced system and hence the original by considering bounds on the radius of convergence of the induced zeta function. In particular, the asymptotic behaviour of the pressure function in the limit β, the inverse temperature, tends to negative infinity is determined and the existence and nature of a phase transition at $\beta = 1$ is also discussed.

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Chapter 1

General Overview

1.1 Introduction

The long term behaviour of chaotic systems may be investigated through the use of the thermodynamic formalism for dynamical systems ([28], [5] and [3]). Sequences of iterates of the map may be likened to one-dimensional spin chains through the use of symbolic dynamics, [30]. This converts the problem of examining the dynamics of iterates of the map into the language of statistical mechanics. An important mechanism for the transition from order to chaos is *intermittency*, [29]. One such map that exhibits intermittent behaviour is the Farey map, [9]. The examination of the thermodynamics of the spin system corresponding to this map will be the main focus of this thesis.

Two well known methods for calculating the partition function of this thermodynamic 'Farey system' are presented: the Ruelle zeta function and the transfer operator method. Both are developed heuristically and are seen to give the same thermodynamics even though they appear to be quite disparate methods. It is observed that intermittency provides a problem for these techniques in the form of an indifferent fixed point. Points do not separate fast enough near this fixed point and the system is said to be non-hyperbolic. More will be said on this towards the end of this introductory chapter and it is sufficient to state here that in general the techniques require hyperbolic systems. A method of inducing the system is used to obtain a hyperbolic one that is still strongly linked to the original. Correspondingly, an induced zeta function and an induced transfer operator are developed for maps of an interval. Examination of these two well behaved objects then yields information on the original system.

The work is then broken into two main sections. The first of these sections concerns a proof of the meromorphic qualities of the induced zeta function. This is done by first establishing a connection between trace formulas involving the induced transfer operator and the induced zeta function itself. The work requires the introduction of nuclear operators for Banach spaces developed in the 50's by Alexander Grothendieck and standard techniques regarding analytic continuation.

The second part concerns the finding and analysis of an explicit form of the induced zeta function. The key is to extract the radius of convergence of the induced zeta function which yields the pressure function of the system. Several bounds are produced along with some exact values. The existence of a phase transition is observed and the scaling behaviour discussed; this has already been done in a more general setting in [22]. The behaviour of the pressure function for inverse temperature approaching negative infinity is also found analytically. Along the way, several results involving periodic continued fractions, Fibonacci numbers and the golden ratio are also obtained.

1.2 The Farey Map, Intermittency, Symbolic Dynamics and Statistical Mechanics

The process of intermittency is a well recognised route to chaos first observed by Pomeau and Manneville [18]. It describes a typical phenomenon observed at the boundary between chaos and order. Physically, it may characterise some measured signal that moves randomly between states of unpredictability and regularity. It is observed that with the tuning of some external parameter the period of occurrence and duration of these chaotic bursts continuously changes. At one extreme of the parameter range, the signal will become totally chaotic and at the other, fully periodic.

The process of intermittency may be modelled by simple 'toy' maps of the unit interval. Apart from being simple to investigate, the behaviour of such maps may be qualitatively generalised to more complex systems possessing intermittent behaviour — this is really just the paradigm of 'universality'. Toy models have certainly proved helpful in understanding more complicated systems as in the seminal work of Lorenz's model of the atmosphere[17]. Some examples of intermittent signals are the flow of highway traffic, the electric potential of a nerve membrane, the many currents and voltages observed in electronic components and circuits such as a junction diode and the relative velocities of the eddies of turbulence [29]. One particularly simple toy model is provided by the Farey map which is otherwise well known for its number theoretic properties.The Farey map, $f : I \to I$ where I is the unit interval $[0, 1]$ is defined as

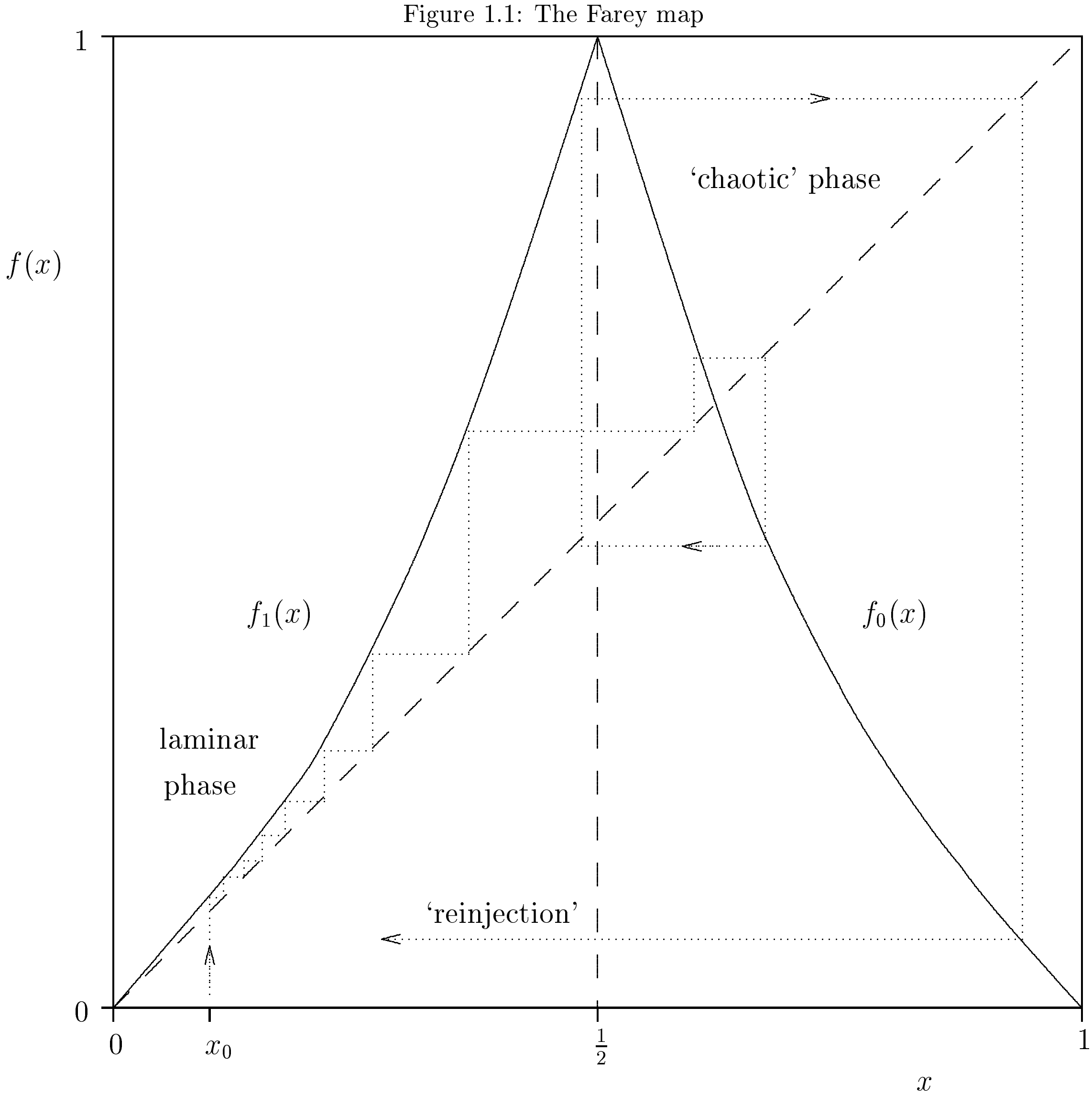
$$f(x) = \begin{cases} f_1(x) = \frac{x}{1-x} \; x \in [0, \frac{1}{2}] \\ \\ f_0(x) = \frac{1-x}{x} \; x \in [\frac{1}{2}, 1] \end{cases} \tag{1.1}$$

The inverses of the two branches are written as F_1 and F_0 and are given by

$$\begin{aligned} F_1(x) &= \tfrac{x}{1+x} \; x \in [0, 1] \\ F_0(x) &= \tfrac{1}{1+x} \; x \in [0, 1] \end{aligned} \tag{1.2}$$

Figure (1.1) displays the Farey map.

Figure 1.1: The Farey map



This figure also shows a typical intermittent orbit[1], i.e. the 'motion' of the point $x_0 \in I$ under iteration of the Farey map. Initially, the point moves slowly away from the point $x = 0$. This is equated with the 'laminar' or regular phase of an intermittent process. After numerous iterations, it moves to the righthand side of the unit interval where it is acted upon by the 'reinjection' branch of the map. Here it may bounce around irregularly (the chaotic phase) until it eventually returns to a point near 0 starting the laminar phase again.

1.2.1 Long term behaviour of intermittent processes

Given that the Farey map is simple model of intermittency, the question arises how best to analyse its dynamics and, indeed, what information about these dynamics is of interest? As with many chaotic processes the short term behaviour is inherently unpredictable and does not always provide tractable problems. In the long term however, average behaviour may be detectable and this will be the focus of this work.

Notice that the Farey map has two branches which have been indirectly referred to as the 'laminar' branch, f_1, and the 'chaotic/reinjection' branch f_0. An important question about a sequence of iterates is how many have been mapped through each branch? In other words, given a *seed* point x_0, how many of its iterates are, on average, to the left of $x = \frac{1}{2}$ (i.e. undergoing smooth motion), and how many are to the

[1]Note that all but countably many points in the interval $(0, 1)$ give rise to 'typical' orbits which are those points whose iterates densely fill the unit interval or more generally the underlying manifold of the system. This is because all rationals, and only these points (a countable set), eventually get mapped to the fixed point at 0 since the Farey map provides an enumeration of the rationals by back iteration of 0. Also, only countably many points end up stuck at the other fixed point, $\frac{\sqrt{5}-1}{2}$.

right (i.e. undergoing chaotic motion)? Translating these ideas into a mathematical setting, it is useful to label a point by a '1' if it is less than $\frac{1}{2}$ and by a '0' if it is greater. This idea of labelling of orbits is generally referred to as symbolic dynamics. The movement of a point under iteration can then be represented by a string of bits. In figure (1.1) for example, iteration of the point x_0 produces the sequence $\{1,1,1,1,1,1,1,1,1,0,0,1,0,1,\ldots\}$.

Naturally, there are many initial points whose iterates are encoded in a sequence beginning in the same way. However, as more and more iterates of x_0 are taken, fewer points remain that fit its particular sequence. In this sense, the apparent coarseness of this technique is removed by taking the length of the sequence to infinity.

It is now possible to make an analogy to a spin system[2]. In particular, the symbolic dynamics of the Farey map can be equated with a one-sided one-dimensional 2-spin system (that is, a one dimensional lattice of particles which may be spin up (1) or spin down (0)). However, it is not yet a thermodynamical system as the probabilities of these states and some notion of temperature have to be introduced. The former is rather simple: consider one iteration of the Farey map. The interval $[0, \frac{1}{2})$ is mapped through the branch f_1 and these points are thus represented by the singleton $\{1\}$. Similarly, all the points in $(\frac{1}{2}, 1]$ are represented by $\{0\}$. So the probability of a random number between 0 and 1 being in the 'up' state is $\frac{1}{2}$, which is the Lebesgue measure of $[0, \frac{1}{2})$. The same idea applies to the 'down' state. Consider then two iterations of the Farey map. There are now four possible spin states: $\{1,1\}$,$\{1,0\}$,$\{0,1\}$ and $\{0,0\}$; note that

[2] It has also been observed that the Farey system is reminiscent of a second-quantized Fermi gas [8].

the n^{th} iterate of the Farey map will provide 2^n states. The state $\{1,1\}$ corresponds to the interval $[0,\frac{1}{3})$ and the probability of its occurrence is $\frac{1}{3}$ and so on. Thus, the probability of a state is given by the Lebesgue measure of the interval corresponding to that state.

Furthering the analogy, the idea of considering the symbolic dynamics of a point in the limit of the number of iterates going to infinity is equivalent to taking the thermodynamic limit of the corresponding spin system. Also in this limit, a finer and finer partition of the unit interval will be constructed based on the symbolic dynamics. Let the 2^n lengths at the n^{th} stage of construction of the partition be written as $\ell_i, i = 1, \ldots, 2^n$. Note that the probability p_i of a state i in a thermodynamic system may be written as $\exp -\beta E_i$ where E_i is the energy of the state and β is the inverse of temperature T. Since the probabilities of the states being considered are also equal to ℓ_i the partition function for this system may be written as follows[3]

$$\sum_{n=1}^{2^n} \exp -\beta E_i \equiv \sum_{n=1}^{2^n} \ell_i^{\beta} \sim \exp -n\beta F(\beta) \tag{1.3}$$

where $F(\beta)$ is the free energy of the system and $\beta F(\beta)$ is the pressure function.

1.2.2 Methods for calculating the Pressure Function

This section presents some heuristic arguments for the construction of the zeta function and the transfer operator. The conclusions will then be compared with formal

[3] The lengths (probabilities) of the elements (spin states) of this partition will scale in a consistent way much like a fractal. However, the partition will exhibit a spectrum of scaling exponents and thus the partition may be thought of as a multifractal. Indeed, the theory of multifractals is intimately linked with the interpretation of dynamical systems as thermodynamic ones (see [30] and [8]).

definitions of these objects. Before going on to these two methods for the evaluation of the partition function, the partition itself will be presented in more detail. There are three important ways to view the partition's construction: forward iteration of the Farey map, back iteration of the Farey map and the enumeration of the rationals via Farey addition. Note that the first method applies to the zeta function, the second the transfer operator, and the third demonstrates the number theoretic properties of the map.

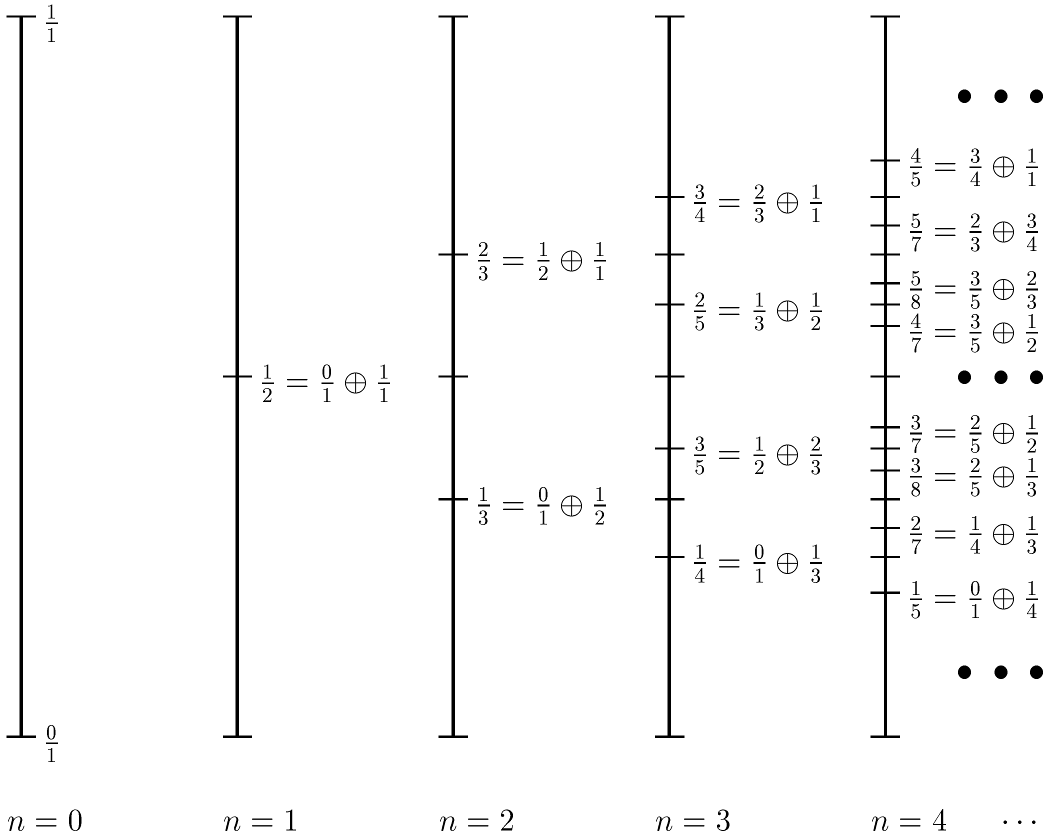
Consider the n^{th} iterate of the Farey map, f^n. This function will have 2^n branches. Since these branches are composed of f_1 and f_0, each branch will lie precisely above the corresponding element of the partition in its n^{th} stage of construction. Consider secondly the preimages of $x = 1$. The first preimage is just $x = \frac{1}{2}$ which divides the unit interval as per the partition in its first stage. Next, the preimage of the point $\frac{1}{2}$ is $\frac{1}{3}$ and $\frac{2}{3}$. These three points give the partition in its second stage and so on.

Finally, the endpoints of the intervals making up the partition may be formed by Farey addition of the rationals. The first few steps of this process are demonstrated in figure (1.2). The operation of the Farey addition of two rational numbers is represented by the symbol $\oplus$ and is defined by

$$\frac{a}{b} \oplus \frac{c}{d} = \frac{a+c}{b+d} \tag{1.4}$$

So, starting with $0 = \frac{0}{1}$ and $1 = \frac{1}{1}$, the added endpoint for the first stage is $\frac{0}{1} \oplus \frac{1}{1} = \frac{1}{2}$. Now, $\frac{0}{1} \oplus \frac{1}{2} = \frac{1}{3}$ and $\frac{1}{2} \oplus \frac{1}{1} = \frac{2}{3}$ which give the extra endpoints required for the second stage. In general, adding consecutive endpoints of the current stage via Farey addition gives the additional endpoints for the next stage.

Figure 1.2: Creation of the Partition of the Farey System



The Ruelle Zeta Function

One way of approximating the partition is as follows: at the n^{th} stage of construction take any point x_i in each interval of the partition and estimate the width of the interval using the slope of f^n at that point; i.e.

$$\ell_i \sim \Big((f^n)'(x_i)\Big)^{-1} \text{ for large } n \tag{1.5}$$

Intuitively, it can be seen that in the limit n approaches infinity, the slope of each branch becomes more and more uniform throughout the interval. Hence any point inside an interval will do and, for convenience, the fixed points of f^n may be used for the x_i [4]. Thus the partition sum in equation (1.3) may be estimated by

$$\sum_{i=1}^{2^n} \ell_i^\beta \sim \sum_{f^n x = x} \Big((f^n)'(x)\Big)^{-\beta} \text{ for large } n \tag{1.6}$$

It is expected that this approximation and the actual partition will match up in the thermodynamic limit.

It will be now be useful to introduce the zeta function that is used in fields of statistical mechanics and dynamical systems. Firstly, a brief historical note: certainly the most famous of the so-called zeta functions is that of Riemann. It still manages to keep the secrets of its inner workings from hungry mathematicians. Pure Number Theory is not the sole realm of the zeta function and it serves as a powerful medium of information in many other areas. In particular, it was the work of Artin and Mazur [2] in 1967 that opened the door for dynamical systems to have their own zeta functions. Ruelle [26] noticed that this idea could be generalised to systems with interactions by simply taking into account the change to having a non-trivial probability for a system to be in a given state. The zeta function is defined as follows

Definition 1.1

$$\zeta(z, w) = \exp \sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{f^n(x)=x} \prod_{k=0}^{n-1} w(f^k x) \tag{1.7}$$

where w is the *weight function* of the system. A more convenient definition of the zeta function for the work here is provided when w is replaced by $\exp \phi$; ϕ is the *interaction* of the system.

[4] This may cause some difficulties at $x = 0$

Definition 1.2

$$\zeta(z,\phi) = \exp \sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{f^n(x)=x} \exp \sum_{k=0}^{n-1} \phi(f^k x) \tag{1.8}$$

Throughout this work, the second definition will be used.

Now, the radius of convergence of the zeta function, see [27], is equal to $\exp -\beta F(\beta)$, i.e.,

$$\lim_{n\to\infty} \left| \frac{1}{n} \sum_{f^n(x)=x} \exp \sum_{k=0}^{n-1} \phi(f^k x) \right|^{\frac{1}{n}} = \exp -\beta F(\beta) \tag{1.9}$$

Note that $\left(\frac{1}{n}\right)^{\frac{1}{n}} \to 1$ so this term is unimportant. Equations (1.3) and (1.6) together show that

$$\left| \sum_{f^n(x)=x} \left((f^n)'(x)\right)^{-\beta} \right|^{\frac{1}{n}} \sim \exp -\beta F(\beta) \tag{1.10}$$

Equations (1.9) and (1.10) then imply that for the Farey system a natural choice of the interaction ϕ is

$$\phi(x) = -\beta \log |f'(x)| \tag{1.11}$$

since $(f^n)'(x) = \prod_{k=0}^{n-1} f'(f^k x)$ by the chain rule. Thus the interaction of a state[5], i, is the energy of the state, E_i, weighted by the inverse of the temperature, β.

Finally, the Farey system may be conveniently and compactly represented by the triplet

$$X = (I, f, \phi)$$

I.e., in the form of (state space, evolution operator, interaction).

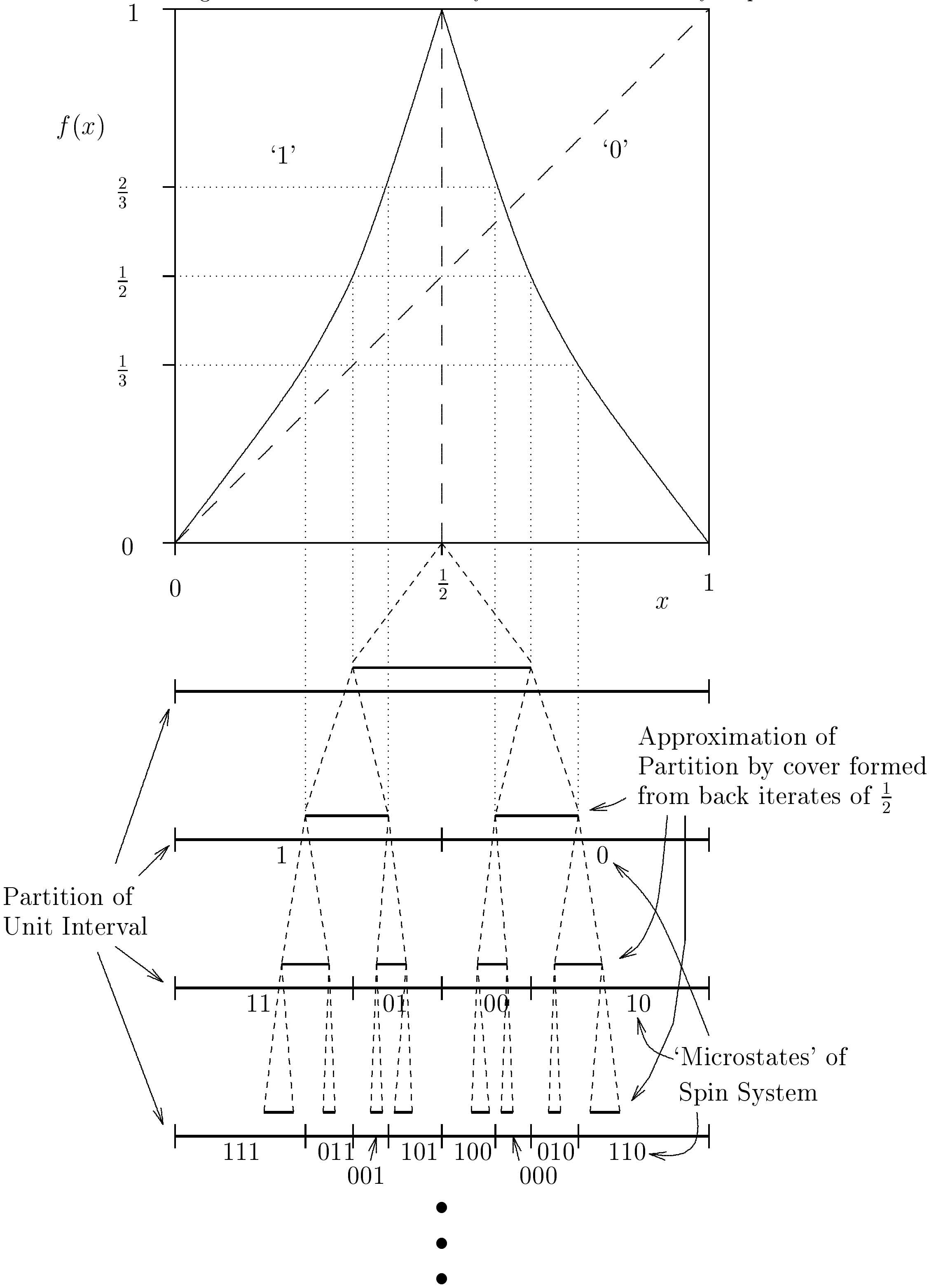
[5]note that [22] employs this same interaction

The Transfer Operator

The transfer operator method is based on idea of finding an operator that creates the partition function of the n^{th} stage by n iterations upon an appropriate initial function. Since the partition function grows like $\exp -n\beta F(\beta)$ it is expected that the largest eigenvalue of such an operator would be precisely $\exp -\beta F(\beta)$. Thus the choice of the initial function becomes unimportant as it is now the spectrum of this operator that contains information about the pressure function.

As with the zeta function method, an approximation for the actual partition is also employed here, the statistical mechanics of both expected to be the same [8]. The construction of this second approximation is demonstrated in figure (1.3) and is described as follows.

Figure 1.3: Partition created by back iteration of Farey map



Consider a *seed point*, x^* in the interval $(0, 1)$ (note that x^* in the example figure has been chosen to be $x = \frac{1}{2}$). Back iteration of this point by the Farey map gives the two points $F_1(x^*)$ and $F_0(x^*)$. The interval $[F_1(x^*), F_0(x^*)]$ then serves as an approximation for the unit interval (i.e. the 0^{th} stage of construction). The preimages of these two points are, in order along the unit interval, $F_1 \circ F_1(x^*)$, $F_1 \circ F_0(x^*)$, $F_0 \circ F_0(x^*)$ and $F_0 \circ F_1(x^*)$. The two intervals defined by $[F_1 \circ F_1(x^*), F_1 \circ F_0(x^*)]$ and $[F_0 \circ F_0(x^*), F_0 \circ F_1(x^*)]$ then give the approximation at the first stage of construction. Note that the closer x^* is to 0, the more precise the approximation. In general, the n^{th} stage of construction will be made up of 2^{n+1} points of the form

$$x_i^{(n)} = F_{e_n} \circ F_{e_{n-1}} \circ \cdots \circ F_{e_0}(x^*) \tag{1.12}$$

where $i = 1, \ldots, 2^{n+1}$ and $e_j = 0, 1$. As per the symbolic dynamics, each point may be represented by a sequence of 0's and 1's: $\{e_n, e_{n-1}, \ldots, e_0\}$. The $x_i^{(n)}$ are paired off as endpoints of the 2^n intervals needed to give an estimation of the true partition with two points being endpoints of the same interval if their sequence representations match up except for the entry e_0 (which must be different). So the approximation for the true length of the partition, ℓ_i, is given by the difference between these two points and will be denoted by l_i:

$$\ell_i \sim l_i = F_{e_n} \circ F_{e_{n-1}} \circ \cdots \circ F_{e_1} \circ F_0(x^*) - F_{e_n} \circ F_{e_{n-1}} \circ \cdots \circ F_{e_1} \circ F_1(x^*) \tag{1.13}$$

The l_i may now be themselves approximated in the following way. Note that for $x \in (0, 1)$, F_1 and F_0 are both contractive mappings, i.e. $|F_1'(x)|, |F_0'(x)| < 1$. The two endpoints of an interval must approach each other with increasing n. Thus a taylor approximation may be used for large n:

$$\begin{aligned}
& F_{e_n} \circ F_{e_{n-1}} \circ \cdots \circ F_{e_1} \circ F_0(x^*) \\
&= F_{e_n} \circ F_{e_{n-1}} \circ \cdots \circ F_{e_1} \circ \left(F_1(x^*) + (F_0(x^*) - F_1(x^*))\right) \\
&\approx F_{e_n} \circ F_{e_{n-1}} \circ \cdots \circ F_{e_1} \circ F_1(x^*) \\
&\quad + (F_0(x^*) - F_1(x^*)) \left. \frac{d}{dy} F_{e_n} \circ F_{e_{n-1}} \circ \cdots \circ F_{e_1}(y) \right|_{y=F_1(x^*)} \\
&= F_{e_n} \circ F_{e_{n-1}} \circ \cdots \circ F_{e_1} \circ F_1(x^*) \\
&\quad + (F_0(x^*) - F_1(x^*)) \prod_{k=1}^{n} F'_{e_k} \left(F_{e_{k-1}} \circ \cdots \circ F_{e_1} \circ F_1(x^*) \right)
\end{aligned} \tag{1.14}$$

Substituting this into equation (1.13), the partition function, equation (1.3), becomes

$$\sum_{i=1}^{2^n} \ell_i^\beta \sim \sum_{i=1}^{2^n} l_i^\beta \sim \sum_{e_n,\ldots,e_1} \prod_{k=1}^{n} \left| F'_{e_k} \left(F_{e_{k-1}} \circ \cdots \circ F_{e_1} \circ F_1(x^*) \right) \right|^\beta \tag{1.15}$$

The transfer operator will now appear as a way of calculating the expression on the righthand side of equation (1.15). Consider the following function:

$$\psi_j^{(\beta)}(x) \equiv \psi^{(\beta)}(x; e_j, e_{j-1}, \ldots, e_1) \equiv \sum_{e_n,\ldots,e_j} \prod_{k=j}^{n} \left| F'_{e_k} \left(F_{e_{k-1}} \circ \cdots \circ F_{e_1} \circ F_1(x) \right) \right|^\beta \tag{1.16}$$

Note that $\psi_1^{(\beta)}(x^*)$ is actually the partition function. Now set $y = F_{e_{j-1}} \circ \cdots \circ F_{e_1} \circ F_1(x)$. The definition of the $\psi_j^{(\beta)}$ becomes:

$$\psi_j^{(\beta)}(y) \equiv \psi^{(\beta)}(y; e_j, e_{j-1}, \ldots, e_1) \equiv \sum_{e_n,\ldots,e_j} \prod_{k=j}^{n} \left| F'_{e_k} \left(F_{e_{k-1}} \circ \cdots \circ F_j(y) \right) \right|^\beta \tag{1.17}$$

Next, consider the following manipulation of the function $\psi_j^{(\beta)}$:

$$\begin{aligned}
\psi_j^{(\beta)}(y) &= \psi^{(\beta)}(y; e_j, e_{j-1}, \ldots, e_1) \\
&= \sum_{e_n,\ldots,e_j} \prod_{k=j}^{n} \left| F'_{e_k} \left(F_{e_{k-1}} \circ \cdots \circ F_j(y) \right) \right|^\beta \\
&= \sum_{e_j} \left| F'_{e_j}(y) \right|^\beta \sum_{e_n,\ldots,e_{j+1}} \prod_{k=j+1}^{n} \left| F'_{e_k} \left(F_{e_{k-1}} \circ \cdots \circ F_j(y) \right) \right|^\beta
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{e_j} \left|F'_{e_j}(y)\right|^{\beta} \psi^{(\beta)}(F_{e_j}(y); e_{j+1}, e_j, \ldots, e_1) \\
&= \sum_{e_j} \left|F'_{e_j}(y)\right|^{\beta} \psi^{(\beta)}_{j+1}(F_{e_j}(y)) \qquad (1.18)
\end{aligned}
$$

Thus, a method for producing the partition function by iteration of an operator on some initial function has been obtained. For large n, the partition function is of course expected to behave like $\exp -n\beta F(\beta)$. Thus it is expected from inspection of equation (1.18) that in the limit of large n and large $n-j$, that $\psi^{(\beta)}_j \sim \exp(-\beta F(\beta))\psi^{(\beta)}_{j+1}$. The substitution of this into the final line of equation (1.18), writing $\lambda(\beta) = \exp(-\beta F(\beta))$, e_j simply as e, x for y and $\psi^{(\beta)}_j$ as ψ, then yields the following eigenvalue equation

$$
\lambda(\beta)\psi(x) = \sum_{e=0,1} |F'_e(x)|^{\beta} \psi(F_e(x)) \qquad (1.19)
$$

and also the definition of a transfer operator $\mathcal{L}^{(\beta)}$,

$$
\mathcal{L}^{(\beta)}\psi(x) = \sum_{e=0,1} |F'_e(x)|^{\beta} \psi(F_e(x)) \qquad (1.20)
$$

By derivation, the largest eigenvalue of this operator is the exponential of the pressure function. More generally, the exponential of the pressure function is seen to correspond with the *spectral radius* of the transfer operator. The spectral radius of an operator $\mathcal{O}$, $r(\mathcal{O})$, is defined as the supremum over the magnitudes of all the elements of $\sigma(\mathcal{O})$, the spectrum of $\mathcal{O}$ ([28]). Noting that $F'_e(x) = \frac{1}{f'(F_e(x))}$ and setting $y = F_e(x)$, this definition can be recast in another useful form:

Definition 1.3

$$
\mathcal{L}^{(\beta)} \circ \psi(x) = \sum_{f(y)=x} \frac{\psi(y)}{|f'(y)|^{\beta}}
$$

A formal definition of the transfer operator is given as follows [22]:

$$\mathcal{L}^{(\beta)} \circ \psi(x) = \sum_{f(y)=x} \psi(y) \exp \phi(y) \tag{1.21}$$

Using the interaction found in the discussion of the zeta function, equation (1.11), the transfer operator for the Farey map given in definition (1.3) is seen to agree with this more formal statement.

Note that for $\beta = 1$, the transfer operator is just the Perron-Frobenius operator. The corresponding eigenfunction is referred to as the *invariant density*, ψ_I. This density is actually a probability measure for the long term behaviour of the iterates of the map concerned. In the case of the Farey map, $\psi_I = \frac{1}{x}$, (see [8]). However, an attempt to normalize this function is problematic as it is of infinite measure on $[0, 1]$. Thus, the normalization of ψ_I is actually the Dirac delta function $\delta(x)$. So, in the long term iterates are expected to be on average at the fixed point 0, a typical feature of intermittent maps. Definition (1.3) becomes the Perron-Frobenius equation $\mathcal{L} \circ \psi(x) = \sum_{f(y)=x} \frac{\psi(y)}{|f'(y)|}$. The transfer operator is thus also known as the Ruelle-Perron-Frobenius operator since it was the work of Ruelle that extended the original idea ([28]).

It is important to impress that the thermodynamics calculated from the zeta function and the transfer operator methods are the same. At first it may appear that they are two different problems. However, as has been shown in this section, they are based on the same thermodynamic system and are expected to deliver the same results and in particular the same pressure function. A much deeper connection between the two methods will be presented in the first half of this work.

1.2.3 A problem

The above methods of calculating the pressure function are very efficient for hyperbolic systems. However, for the Farey map they become somewhat ineffective. In the case of a simple map of an interval like the Farey map, the term hyperbolic means the map must be *uniformly expanding.* If a map is expanding then the iterates of any two nearby points will separate in exponential fashion; i.e., it is required that on all points of the interval $|f'| \geq 1+\epsilon$ where $\epsilon > 0$. This is not true for the Farey map and, in particular, breaks down at the fixed point at 0 which is referred to as an *indifferent fixed point.* Clearly it is an unstable fixed point as points eventually move away, something that can be intuited by the shape of the function around 0, see figure (1.1). Iterates do not separate exponentially from 0 however and can even be so chosen as to take arbitrarily long times to do so, [22]. As was mentioned previously, the invariant density of the Farey map is the Dirac delta function $\delta(x)$, further demonstrating the singular nature of intermittency. Indeed, the generic feature of such intermittent maps is that their derivative approaches and becomes 1 at $x = 0$, creating an indifferent fixed point there. In particular, for the Farey map, $f'(x) = (1-x)^{-2} \sim 1 + 2x$ for x near 0. The presence of such an indifferent fixed point numerically leads to slow convergence to the thermodynamic limit, if at all, and also may bring about singularities in the pressure function. This problem is dealt with by the so-called method of inducing which is discussed in the following chapter.

Chapter 2

The Induced System

In order to remove the pathology of the fixed point at 0, a method of inducing is employed. The system is said to be 'induced' onto a subset J of I. Inducing was first used in this setting in a paper by Prellberg and Slawny, [22]. The main point of this technique is that an expanding map is produced which can be analysed via the devices of the zeta function and transfer operator with none of the problems associated with the indifferent fixed point. Also, if the interaction of the system is chosen appropriately, very strong connections between the induced system and the original system can be demonstrated and utilized forthwith. These connections will be presented at the end of the chapter. Note that this work applies to maps of intervals in general and is in no way particular to the Farey map. The induced map or the first return map, g, is defined in the following way: first consider the function $n(x) : J \to N$ where

$$n(x) = n \quad \text{if} \quad f^k(x) \notin J \quad \forall \quad k = 1, \ldots, n-1 \quad \text{and} \quad f^n(x) \in J \tag{2.1}$$

The induced map[1] is then simply given by $g : J \to J$ where

$$g(x) = f^{n(x)}(x) \tag{2.2}$$

Also, defining J^c to be the complement of J in I, i.e. $J^c = I \setminus J$, consider the following set of points:

$$K = \{x \in J^c \mid \ \forall\, n : \ f^n(x) \in J^c\} \tag{2.3}$$

These are exactly the points that do not mix with the induced system. Recall that the Farey system was written as the triplet $X = (I, f, \phi)$. The method of inducing can now be seen to split this system into two new non-interacting dynamical systems: the induced system $Y = (J, g, \phi_Y)$ and the complementary system $Y^c = (K, f\mid_K, \phi_{Y^c})$. The interactions of these two systems, ϕ_Y and ϕ_{Y^c}, must be chosen appropriately so that information regarding the original system is not lost; the work that follows will determine these functions.

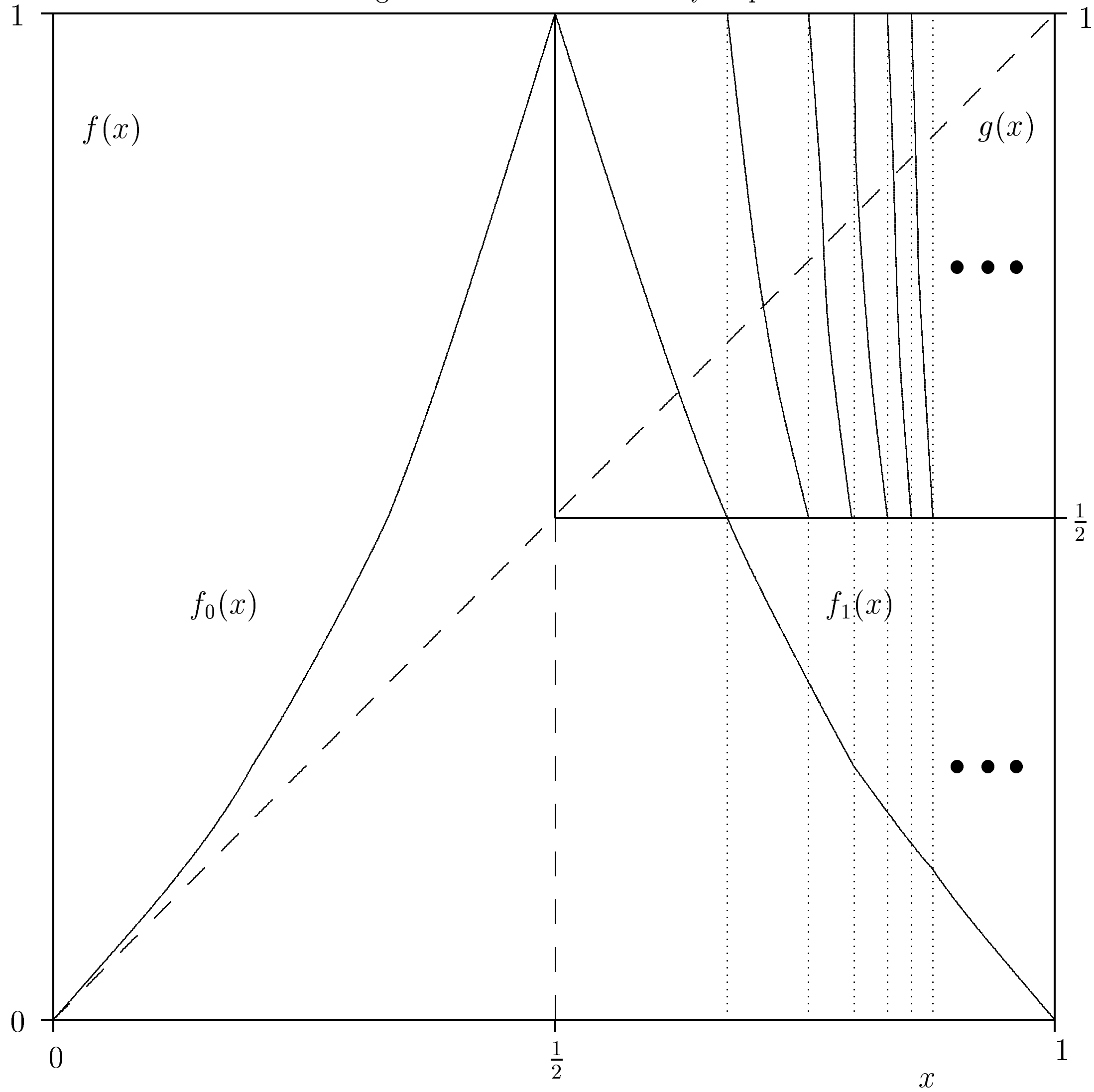
For the Farey map, the induced version with respect to the interval $J = [\frac{1}{2}, 1]$ is given by

$$g(y) = \begin{cases} \frac{1-y}{1+n(y-1)}, & \text{for } y \in (\frac{n}{n+1}, \frac{n+1}{n+2}] \quad n = 1, 2, 3, \ldots \\ 1 & \text{for} x = \frac{1}{2} \end{cases} \tag{2.4}$$

This new function is displayed along with the Farey map in figure (2.1).

[1] Note that the number of points in J where no finite n exists such that the orbit returns to J is at most countable and g may be defined arbitrarily at these points with no added problems, see [22].

Figure 2.1: The induced Farey map



2.1 The Induced Zeta Function and the Induced Transfer Operator

The concept of the induced system naturally gives rise to an induced zeta function and an induced transfer operator. The following is a derivation of the form of the induced zeta function.

The logarithm of the zeta function of the Farey map can be decomposed into two parts: another zeta function relating to the induced system and a simple term relating to the indifferent fixed point at 0. More generally, the zeta function for any system X can be thought of as the product of the zeta functions $\zeta_1(z)$ and $\zeta_2(z)$ relating to the systems Y and the induced system Y^c respectively. In fact, they will demonstrate natural choices for the interactions of these systems, ϕ_Y and ϕ_{Y^c} based on the fact that $\phi(x) = -\beta \log |f'(x)|$ from equation (1.11). Explicitly, we have from the definition of the zeta function (1.2):

$$\begin{aligned}
\log \zeta_X(z,\phi) &= \sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{f^n(x)=x} \exp \sum_{k=0}^{n-1} \phi(f^k x) \\
&= \sum_{n=1}^{\infty} \frac{z^n}{n} \left(\sum_{f^n(x)=x, \exists k | f^k x \in J} + \sum_{f^n(x)=x, \nexists k | f^k x \in J} \right) \exp \sum_{k=0}^{n-1} \phi(f^k x) \\
&= \sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{f^n(x)=x, \exists k | f^k x \in J} \exp \sum_{k=0}^{n-1} \phi(f^k x) + \sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{f^n(x)=x, \nexists k | f^k x \in J} \exp \sum_{k=0}^{n-1} \phi(f^k x) \\
&= \log \zeta_1(z) + \log \zeta_2(z)
\end{aligned}$$

Now consider the complementary system $Y^c = (K, f\ |_K, \phi_{Y^c})$ which was defined at the start of this chapter with the interacion $\phi_{Y^c} = \phi$. The fixed points of f^n in K are by definition precisely those that are part of cycles that have no point in J. The second term is therefore the logarithm of the zeta function $\zeta_{Y^c}(z)$. In the case of the Farey map, $\log \zeta_{Y^c}(z)$ is very simple. K is just the indifferent fixed point at 0. Since $\phi(f^k 0) = \phi(0) = -\log f'(0) = -\log 1 = 0$ the zeta function reduces to $\log \zeta_{Y^c}(z) = \sum_{n=1}^{\infty} \frac{z^n}{n}(1)$. This series is simply the expansion of $-\log(1-z)$ about $z = 0$. Thus, the series for $\zeta_{Y^c}(z)$ can be analytically extended to a meromorphic function in the whole of the z plane independent of β.

$$\log \zeta_{Y^c}(z) = \log \frac{1}{1-z} \Rightarrow \zeta_{Y^c}(z) = \frac{1}{1-z} \tag{2.5}$$

Returning to the first term, it can be rewritten as follows:

$$
\begin{aligned}
\log\zeta_1(z) &= \sum_{n=1}^{\infty}\frac{z^n}{n}\sum_{f^n(x)=x,\exists k|f^kx\in J}\exp\sum_{k=0}^{n-1}\phi(f^kx) \\
&= \sum_{n=1}^{\infty}\frac{z^n}{n}\sum_{f^n(x)=x,\exists k|f^kx\in J}\exp\sum_{k=0}^{n-1}-\beta\log|f'(f^kx)| \\
&= \sum_{n=1}^{\infty}\frac{1}{n}\sum_{f^n(x)=x,\exists k|f^kx\in J}\exp\sum_{k=0}^{n-1}\Big(\log z-\beta\log|f'(f^kx)|\Big) \\
&= \sum_{n=1}^{\infty}\sum_{m=1}^{n}\frac{1}{m}\sum_{\substack{f^n(x)=x,\, x\in J \text{ s.t.}\\ n\text{-cycle has } m \text{ different points in } J}}\exp\sum_{k=0}^{n-1}\Big(\log z-\beta\log|f'(f^kx)|\Big) \\
&= \sum_{m=1}^{\infty}\frac{1}{m}\sum_{n=m}^{\infty}\sum_{\substack{g^m(x)=x\\ \sum_{k=0}^{m-1}n(g^k(x))=n}}\exp\sum_{k=0}^{n-1}\Big(\log z-\beta\log|f'(f^kx)|\Big)
\end{aligned}
$$

Since $g^m(x)=x$, the sum in the exponential of the previous line can be rewritten as follows:

$$
\begin{aligned}
&\sum_{k=0}^{n-1}\Big(\log z-\beta\log|f'(f^kx)|\Big) \\
&\quad= n(x)\log z-\beta\log\prod_{k=0}^{n(x)-1}|f'(f^k\circ g^0(x))| \\
&\quad+ n(g^1x)\log z-\beta\log\prod_{k=0}^{n(g^1(x))-1}|f'(f^k\circ g^1(x))| \\
&\quad\ \vdots\quad\vdots \\
&\quad+ n(g^{m-1}x)\log z-\beta\log\prod_{k=0}^{n(g^{m-1}(x))-1}|f'(f^k\circ g^{m-1}(x))| \\
&\quad= \sum_{k=0}^{m-1}\Big(n(g^kx)\log z-\beta\log|g'(g^kx)|\Big)
\end{aligned}
$$

where the chain rule has been used: $g'(y)=\frac{d}{dy}f^{n(y)}(y)=\prod_{k=0}^{n(y)-1}f'(f^ky)$.

So:

$$\begin{aligned}
\log\zeta_1(z) &= \sum_{m=1}^{\infty}\frac{1}{m}\sum_{n=m}^{\infty}\sum_{\substack{g^m(x)=x\\ \sum_{k=0}^{m-1}n(g^k(x))=n}}\exp\sum_{k=0}^{m-1}\Big(n(g^kx)\log z-\beta\log|g'(g^kx)|\Big)\\
&= \sum_{m=1}^{\infty}\frac{1}{m}\sum_{g^m(x)=x}\exp\sum_{k=0}^{m-1}\Big(n(g^kx)\log z-\beta\log|g'(g^kx)|\Big)
\end{aligned}$$

Now, define a new interaction, which depends explicitly on z,

$$\phi_z(y)=n(y)\log z-\beta\log|g'(y)| \tag{2.6}$$

To demonstrate the link to the induced system, it is useful to to introduce a parameter σ in the following way:

$$\log\zeta_1(z)=\zeta_1(z,\sigma)\big|_{\sigma=1}=\sum_{m=1}^{\infty}\frac{\sigma^m}{m}\sum_{g^m(x)=x}\exp\sum_{k=0}^{m-1}\phi_z(g^kx)\Bigg|_{\sigma=1} \tag{2.7}$$

It is clear then that $\zeta_1(z,\sigma)=\zeta_Y(\sigma)$ if the induced system is given the interaction $\phi_Y=\phi_z$ defined above. The notation $\zeta_{\text{ind}}(z,\beta)$ will also be used to denote this function. Thus, the definition of the induced system is given by the triplet

$$Y_z=(J,g,\phi_z)$$

where the z subscript has been introduced to emphasize the fact that this system has an explicit z dependence. Finally, considering the identifications of $\zeta_1(z)$ and $\zeta_2(z)$ the following relationship between the zeta functions of the induced and the original system may now be stated as the following result

Lemma 2.1 *The zeta functions of the original, the induced and the complementary system are related in the following way:*

$$\zeta_X(z)\equiv\zeta_{Y_z}(1)\zeta_{Y^c}(z)$$

Using the interaction found in the discussion of the zeta function and the definition of the transfer operator (1.21), the transfer operator for the induced Farey map, which will be denoted by $\mathcal{M}^{(z,\beta)}$, will have the following definition:

Definition 2.2

$$\mathcal{M}^{(z,\beta)} \circ \phi(x) = \sum_{f(y)=x} \frac{z^{n(y)}\phi(y)}{|g'(y)|^{\beta}}$$

Note, that a naive definition of the transfer operator would not include the $z^{n(y)}$. This factor provides the link back to the original system as it carries the information about how many iterates of the Farey map it takes for a point to return to J. Thus, knowledge of time has been retained but has been removed from the actual dynamics of g making it a nicer map to investigate.

Consider now a decomposition of $\mathcal{L}^{(\beta)}$ into $\mathcal{L}_1^{(\beta)} + \mathcal{L}_0^{(\beta)}$ where $\mathcal{L}_1^{(\beta)}\psi = \mathcal{L}^{(\beta)}(\chi_{J^c}\psi)$ and $\mathcal{L}_0^{(\beta)}\psi = \mathcal{L}^{(\beta)}(\chi_J\psi)$. Here, χ is the characteristic map; $\chi_A x = 1$ if $x \in J$ and $\chi_A x = 0$ if $x \notin A$ where A is some set.

The main connection to the thermodynamics of the original map as far as the the transfer operator is concerned is the following result (which will be stated without proof).

Theorem 2.3 ([22]) *Suppose $0 < |z| < 1/r(\mathcal{L}_1^{(\beta)})$. Then $\frac{1}{z}$ is an eigenvalue of $\mathcal{L}^{(\beta)}$ if and only if 1 is an eigenvalue of $\mathcal{M}^{(z,\beta)}$. In addition, these corresponding eigenvalues are of the same geometric multiplicity.*

where $r(\mathcal{O})$ is the spectral radius of the operator $\mathcal{O}$ and is given by

$$r(\mathcal{O}) = \sup_{\lambda \in \sigma(\mathcal{O})} |\lambda|$$

Note that $\sigma(\mathcal{O})$ denotes the spectrum of the operator $\mathcal{O}$ which are all the values λ such that $(\lambda I - \mathcal{O})$ does not have a continuous inverse, ([23]). The value of $r(\mathcal{L}_1^{(\beta)})$ for the induced Farey map is 1 and this is shown in the appendices in section (B.1) More details on these observations and further connections regarding the eigenfunctions of $\mathcal{L}^{(\beta)}$ and $\mathcal{M}^{(z,\beta)}$ can be found in[22].

Chapter 3

The connection between the Induced Transfer Operator $\mathcal{M}^{(z,\beta)}$ and the Induced ζ-function

This section provides a result linking the Fredholm determinant of the induced transfer operator to the zeta function. Using this connection, results on the meromorphic properties of the zeta function will then be obtained. The method applied has been used in a general setting by Ruelle for expanding maps and Anosov flows in [27] and for the Gauss map by Mayer in [19]. A generalisation of the work done here would hopefully demonstrate the same relationship for induced transfer operators and the corresponding induced zeta functions in a more general setting.

3.1 Nuclear Operators according to Grothendieck

This section presents some important results from the theory of Nuclear Operators which was developed by Grothendieck in the 1950's [11]. Much of what is presented here comes from a useful summary of Grothendieck's work provided by Mayer in [20] and also in [4]. The basic notion of the work is to provide a way of finding operators with well defined traces in the setting of general Banach spaces.

The first step is to associate a linear operator, acting from any Banach space to another, with a Fredholm kernel. A brief description of the latter is thus provided here.

3.1.1 Fredholm kernels

Consider any two Banach spaces $(E, \| \ \|_E)$ and $(F, \| \ \|_F)$ and their tensor product $E \otimes F$. Next take the so-called π-norm, $\| \ \|_\pi$, which is defined as follows:

$$\|X\|_\pi \equiv \inf \sum_{\{i\}} \|e_i\|_E \|f_i\|_F \tag{3.1}$$

where the infimum is taken over all finite sets of $\{e_i\} \in E$ and $\{f_i \in F\}$ such that $X = \sum_{\{i\}} e_i \otimes f_i$. The completion of $E \otimes F$ with respect to the π-norm is the 'projective topological tensor product' of E and F. This new Banach space will be denoted by $E \widehat{\otimes}_\pi F$; its elements are referred to as the Fredholm kernels. Note that by choosing $a_i \neq 0$ and $b_i \neq 0$ so that $\|a_i^{-1} e_i\|_E = 1$ and $\|b_i^{-1} f_i\|_F = 1$ and then setting $\lambda_i = a_i b_i$, each Fredholm kernel has a representation

$$X = \sum_{\{i\}} \lambda_i e_i' \otimes f_i' \tag{3.2}$$

such that $\|e_i'\|_E = 1$ and $\|f_i'\|_F = 1$. This also implies that $\sum_{\{i\}} \|\lambda_i\| < \infty$.

3.1.2 Nuclear Operators

Again, consider any two Banach spaces $(E, \| \|_E)$ and $(F, \| \|_F)$ and now also the space $E^* \widehat{\otimes}_\pi F$. Here, E^* is the dual of the Banach space E which is defined as the set of all bounded linear functionals on E. The norm of any $f \in E^*$ is given by

$$\|f\|_{E^*} \equiv \sup_{e \in E^*; \|e\| \leq 1} |f(e)| \tag{3.3}$$

The contraction of any element of $E^* \widehat{\otimes}_\pi F$ with an element of E yields an element of F. In this sense, every element of $E^* \widehat{\otimes}_\pi F$ is equivalent to a bounded linear operator $\mathcal{L}_X : E \to F$, i.e. $\mathcal{L}_X \in B(E, F)$[1]. Considering the representation of X in equation (3.2), the operation of $\mathcal{L}_X$ may always be written as

$$\mathcal{L}e \equiv \sum_{\{i\}} \lambda_i e_i^*(e) f_i \quad \text{forall } e \in E \tag{3.4}$$

where $\|e_i\|_E = 1$ and $\|f_i\|_F = 1$. So, the above indicates there is a natural mapping $\phi : E^* \widehat{\otimes}_\pi F \to B(E, F)$. It is important at this point to simply note that ϕ is not always an injective mapping.

Now, for any Banach space B, the definition of a nuclear operator $\mathcal{L}$ is given as follows:

Definition 3.1 *Take any $\mathcal{L} : E \to F$ such that $\mathcal{L}$ is a linear bounded operator and B is any Banach space. $\mathcal{L}$ is said to be nuclear if there exists a Fredholm kernel $X \in E \widehat{\otimes}_\pi F$ with $\mathcal{L} = \mathcal{L}_X$.*

[1] $B(E, F)$ is the space of all linear bounded maps of E to F

One of the most important features of nuclear operators is the possible existence of a trace. Firstly, the trace of a Fredholm kernel $X \in E^*\widehat{\otimes}_\pi E$ with the representation $X = \sum_{\{i\}} \lambda_i e_i^* \otimes e_i$ is given by

$$\text{trace}X \equiv \sum_{\{i\}} \lambda_i e_i^*(e_i) \tag{3.5}$$

where $e_i^* \in E^*$, $e_i \in E$, $\|e_i^*\|_E^* = 1$, $\|e_i\|_E = 1$ and $\{\lambda_i\} \in \ell_1$. While this is known to be a well defined function, problems arise when considering an operator $\mathcal{L} \in B(E, F)$. This is due to the fact that ϕ is not necessarily injective — if it is not injective than there will be ambiguity in trying to define a trace of $\mathcal{L}$ based on the trace of its Fredholm kernel, as it has more than one such kernel.[2]

The search for trace-class operators acting on Banach spaces led Grothendieck to the following classification of Fredholm kernels and thence nuclear operators.

Definition 3.2 *Take any Fredholm kernel $X \in E\widehat{\otimes}_\pi F$. X is said to be 'p-summable' if X has a representation $X = \sum_{\{i\}} \lambda_i e_i^* \otimes f_i$ such that $\sum_{i=1}^{\infty} |\lambda_i|^p < \infty$, i.e. $\{\lambda_i\} \in \ell_p$. The 'order' of the Fredholm kernel is the number q which is the infimum of all p such that $0 < p \leq 1$ and X is p-summable. Note that, in general, $0 \leq q \leq 1$ while $0 < p \leq 1$.*

Definition 3.3 *Take $\mathcal{L} : E \to F$, a nuclear operator. $\mathcal{L}$ is p-summable if there is a p-summable $X \in E\widehat{\otimes}_\pi F$ such that $\mathcal{L} = \mathcal{L}_X$. The order of $\mathcal{L}$ is the same as the order of X.*

[2]Note that a Fredholm kernel has many representations and hence the fact that two representations are different does not mean their corresponding kernels are different. However, two different kernels may easily provide different traces giving rise to the problem mentioned in the text.

Theorem 3.4 *Consider a nuclear operator $\mathcal{L} : E \to E$ of order p such that $0 \le p \le \frac{2}{3}$. Then $\mathcal{L}$ is of trace class:*

$$\mathrm{trace}\mathcal{L} = \sum_{\{i\}} \nu_i$$

where ν_i are the eigenvalues of $\mathcal{L}$ counted according to their algebraic multiplicity. The Fredholm determinant $\det(1 - \xi\mathcal{L})$ *is an entire function of ξ given by the formula*

$$\det(1 - \xi\mathcal{L}) = \prod_i (1 - \nu_i \xi) = \exp \mathrm{trace} \log(1 - \xi\mathcal{L})$$

Also, if $\mathcal{L} = \mathcal{L}(\sigma)$ and $\sigma \to \mathcal{L}(\sigma)$ is a holomorphic function of σ in some domain D then the function $\det(1 - \mathcal{L}(\sigma))$ *is holomorphic in D.*

Another important result concerns the composition of a nuclear operator with bounded operators:

Lemma 3.5 *Consider $\mathcal{L} : E \to F$, a nuclear operator of order q, $\mathcal{O}_1 \in L(F, G)$ and $\mathcal{O}_2 \in L(G, E)$. Then the composition mapping $\mathcal{O}_1 \cdot \mathcal{L} \cdot \mathcal{O}_2 : G \to G$ is also a nuclear operator of order q.*

3.1.3 Nuclear Spaces

A nuclear space is defined by the fact that any bounded linear operator mapping it to any Banach space is a nuclear operator. Nuclear spaces are actually a class of Fréchet spaces. A Fréchet space is a convex topological metric space that admits a metric such that its topology is reproduced by the metric and it is complete with respect to this same measure of distance. Note that not every Fréchet space is normable so this is not a direct generalization of Banach spaces. In fact, it is known that any infinite dimensional Banach space is not nuclear. One particularly useful nuclear space is $\mathcal{H}(D)$ where D is some open set in C^n. Indeed, this space will aid in the proof that $\mathcal{M}^{(z,\beta)}$ is a nuclear operator. $\mathcal{H}(D)$ is the space of holomorphic functions on D which are continuous on $\overline{D}$ together with the seminorms $\| \|_K$, where K is compact in D and

$$\|f\|_K = \sup_{\xi \in K} |f(\xi)| \tag{3.6}$$

Also, for certain special Banach spaces, every nuclear operator $\mathcal{L}$ is of order 0 and considering, theorem (3.4), must be therefore of trace class. One such example is any Banach space of holomorphic functions over a domain D in C^n.

3.1.4 Compact Operators

Compact operators are defined as per [23] which will be the main reference for this section:

Definition 3.6 (([23]), VIII, §1, p. 143) *Let $\mathcal{L}$ be a linear operator such that $\mathcal{L} : E \to F$ where E and F are convex vector spaces. $\mathcal{L}$ is said to be* compact *if for any neighbourhood of the origin $U \in E$, there exists a compact set $K \in F$ such that $\mathcal{L}(U) \subseteq K$.*

Compact operators have many nice properties and, with regards to spectrum, behave in many ways like finite dimensional matrix operators. The following results and observations are for a compact operator that maps a convex Hausdorff space E into itself. The operator's spectrum, apart from possibly 0[3], comprises entirely of eigenvalues and

[3] If E is infinite dimensional, 0 always belongs to the spectrum

the eigenspace associated with each eigenvalue is finite dimensional. These eigenvalues are also either a finite set of values or a sequence which is convergent to zero. In particular, one result which will be important for later on is recorded here as a lemma:

Lemma 3.7 (([23]), VIII, §1, Corollary 2, p. 147) *If $\mathcal{L}$ is a compact linear operator mapping a convex Hausdorff space E into itself and $W = \lambda I - \mathcal{L}$, where $0 \neq \lambda \in C$, then λ is not an eigenvalue of $\mathcal{L}$ iff W is bijective.*

Finally, note that any operator that has a countable set of eigenvalues converging to 0 is a compact operator. Thus all nuclear operators are compact but the reverse is not true since a sequence whose terms converge to zero is by no means an absolutely convergent series.

3.1.5 Nuclearity of $\mathcal{M}^{(z,\beta)}$

Using the definition of the induced system, the action of the induced transfer operator is defined as follows:

Definition 3.8

$$\mathcal{M}^{(z,\beta)} \circ \phi(\xi) = \sum_{n=1}^{\infty} \frac{z^n}{(1+n\xi)^{2\beta}} \phi\left(1 - \frac{\xi}{1+n\xi}\right)$$

where the composition operators will be denoted by G_n, i.e. $G_n(\xi) = 1 - \frac{\xi}{1+n\xi}$. For the induced Farey map, $\xi \in [\frac{1}{2}, 1]$. In order to analyse this operator more fully, however, it is natural to extend the domain into the complex plain where the full weight of complex analysis can be brought to bear on the problem. Note that each G_n has a simple pole at $\xi = -\frac{1}{n}$. It is thus advisable to stay away from these points and, in particular, away from $\xi = 0$. A suitable domain for the work here, which is by no means the best or only choice, is given by

$$D = \{\xi : |\xi - 1| < \frac{3}{4}\} \tag{3.7}$$

Next consider the domain $D' = \{\xi : |\xi - 1| < \frac{2}{3}\} \subset D$.

Lemma 3.9

$$\overline{G_n(D)} \subseteq \overline{D'}$$

Proof. Expressing G_n in the form

$$G_n(\xi) = \frac{1 + (n-1)\xi}{1 + n\xi} \tag{3.8}$$

it is seen to be a Möbius or a linear fractional transformation, see [25]. It is well known that these transformations map discs to either a disc or a halfspace in the complex plane. Since G_n is bounded on D' it cannot map it to a half space and so $G_n(D')$ must be an open disc. Note that

$$\overline{G_n(\xi)} = \frac{\overline{1 + (n-1)\xi}}{\overline{1 + n\xi}} = \frac{1 + (n-1)\overline{\xi}}{1 + n\overline{\xi}} = G_n(\overline{\xi}) \tag{3.9}$$

where $\overline{\xi}$ denotes the complex conjugate of ξ. This means that since D' is symmetric about the real axis, the disc $G_n(D')$ must also have its centre lying on the real axis. Further, note that the boundary of D', $\partial D'$ is mapped to the boundary of $G_n(D')$, $\partial G_n(D')$. Therefore, the two intersections of $\partial D'$ with the real axis are mapped to the two intersections of $\partial G_n(D')$ with the real axis. Thus, it is enough to find $G_n(\frac{1}{3})$ and $G_n(\frac{5}{3})$ to totally specify the region $G_n(D)$. Their absolute difference will give the diameter of $G_n(D)$ and their mean the position of its centre. The following is then observed:

$$G_n(\frac{1}{4}) = 1 - \frac{1}{n+4} \text{ and } G_n(\frac{7}{4}) = 1 - \frac{7}{7n+4} \tag{3.10}$$

It is clear then that each circle lies inside $\overline{D'}$ as $G_n(\frac{7}{4})$ and $G_n(\frac{1}{4})$ always lie in $[\frac{1}{3}, \frac{2}{3}]$. Since all points inside D are mapped inside each circle, the proof is complete. □

Let $H_\infty(D)$ be the Banach space comprising of all functions holomorphic on the domain D and continuous on the closure of D. It is clear from lemma (3.9) that $\mathcal{M}^{(z,\beta)}$ maps elements of $H_\infty(D)$ to functions on D. Since the G_n are holomorphic functions on $\overline{D}$, the composition of a holomorphic function $\phi \in H_\infty(D)$ and G_n is another holomorphic function. Providing the weighted sum over all such compositions is itself bounded, the function $\mathcal{M}^{(z,\beta)}(\phi)$ will be an element of $H_\infty(D)$. So it is clear then that $\mathcal{M}^{(z,\beta)}(H_\infty(D)) \subset H_\infty(D)$.

Lemma 3.10 $\mathcal{M}^{(z,\beta)} : \mathcal{H}(D) \to H_\infty(D)$ *is a nuclear operator for* $|z| < 1$ *and* $\beta \in C$.

Proof. $\mathcal{M}^{(z,\beta)}$ is a bounded operator for $|z| < 1$ and $\beta \in C$ since for any $\phi \in H_\infty(D)$ ($\|\phi\|_{H_\infty(D)} \neq 0$)

$$\begin{aligned} \|\mathcal{M}^{(z,\beta)} \circ \phi(\xi)\|_{H_\infty(D)} &= \sup_{\xi \in D} |\mathcal{M}^{(z,\beta)} \circ \phi(\xi)| \\ &\leq \sup_{\xi \in D} \sum_{n=1}^{\infty} |z|^n |(1+n\xi)^{-2\beta}| \left| \phi(1 - \frac{\xi}{1+n\xi}) \right| \\ &\leq \|\phi(\xi)\|_{H_\infty(D)} \sup_{\xi \in D} \sum_{n=1}^{\infty} |z|^n |(1+n\xi)^{-2\beta}| \end{aligned} \tag{3.11}$$

Via the ratio test, the sum on the right always converges for $|z| < 1$ irrespective of the value of β. Since $\mathcal{H}(D)$ is a nuclear space it follows from the discussion in section (3.1.3) that $\mathcal{M}^{(z,\beta)}$ is a nuclear operator.

$\square$

Corollary 3.11 *$\mathcal{M}^{(z,\beta)} : H_\infty(D) \to H_\infty(D)$ is a nuclear operator of order 0 for $|z| < 1$ and $\beta \in C$.*

Proof. Composing $\mathcal{M}^{(z,\beta)}$ with the injective function $i : H_\infty(D) \to \mathcal{H}(D)$ and noting lemma (3.5), it follows that $\mathcal{M}^{(z,\beta)} : H_\infty(D) \to H_\infty(D)$ is a nuclear operator. The last remark of section (3.1.3) then gives that $\mathcal{M}^{(z,\beta)}$ must be a nuclear operator of order 0.

$\square$

Nuclearity of $\mathcal{M}^{(z,\beta)}$ for $|z| = 1$ will be proven later with appropriate restrictions on β. Note that the above also shows that $\mathcal{M}^{(z,\beta)} : H_\infty(D) \to H_\infty(D)$ is a compact operator for $|z| < 1$ and $\beta \in C$.

3.2 The trace of $\mathcal{M}^{(z,\beta)}$

Proposition 3.12 *The trace of the induced transfer operator can be expressed as*

$$\mathrm{trace}\mathcal{M}^{(z,\beta)} = \sum_{n=1}^{\infty} \mathrm{trace}\mathcal{M}_n^{(z,\beta)} = \sum_{n=1}^{\infty} \frac{z^n([\overline{n}])^{-2\beta}}{1 + ([\overline{n}])^2}$$

where $[\overline{n}]$ is the continued fraction

$$\cfrac{1}{n + \cfrac{1}{n + \cfrac{1}{n + \cdots}}}$$

Proof. The following analysis is for z inside the unit disc and arbitrary β. Consider the operator $\mathcal{M}^{(z,\beta)}$ in the form of a sum of operators: $\mathcal{M}^{(z,\beta)} = \sum_{n=1}^{\infty} \mathcal{M}_n^{(z,\beta)}$ where $\mathcal{M}_n^{(z,\beta)} \circ \phi(\xi) = \frac{z^n}{(1+n\xi)^{2\beta}}\phi(1 - \frac{\xi}{1+n\xi}) = z^n(-G_n'(\xi))^\beta \phi(G_n(\xi))$. The point of this is that $\text{trace}\mathcal{M}^{(z,\beta)} = \sum_{n=1}^{\infty} \text{trace}\mathcal{M}_n^{(z,\beta)}$ since trace is a linear function on operators and the problem can thus be broken down into one of determining the trace of the simpler operators $\mathcal{M}_n^{(z,\beta)}$. The method used to determine the spectrum of $\mathcal{M}_n^{(z,\beta)}$ and hence $\text{trace}\mathcal{M}_n^{(z,\beta)}$ is based on the work of Kamowitz found in [15] and [14]. The same idea is also used for the Gauss map by Mayer [19].

Recall that G_n has exactly one fixed point ξ_n^* in D, see page 76, which was found to be

$$\xi_n^* = [1, \overline{n}] = \frac{n - 2 + \sqrt{n^2+4}}{2n} \tag{3.12}$$

The eigenvalue equation for each $\mathcal{M}_n^{(z,\beta)}$ is

$$\mathcal{M}_n^{(z,\beta)} \circ \phi(\xi) = \lambda_n \phi(\xi) = z^n(-G_n'(\xi))^\beta \phi(G_n(\xi)) \tag{3.13}$$

At the fixed point ξ_n^*, $G_n(\xi_n^*) = x_n^*$ and so equation (3.13) becomes

$$\lambda_n \phi(\xi_n^*) = z^n(-G_n'(\xi_n^*))^\beta \phi(xi_n^*) \tag{3.14}$$

Providing $\phi(\xi_n^*) \neq 0$, equation (3.14) shows that

$$\lambda_n = \lambda_{n,0} = z^n(-G_n'(\xi_n^*))^\beta \tag{3.15}$$

In the case that $\phi(\xi_n^*) = 0$, the differentiation of equation (3.13) with respect to ξ yields another possible value of λ_n, $\lambda_{n,1}$:

$$\lambda_{n,1}\phi'(\xi_n^*) = z^n \left(2\beta(-G_n'(\xi))^{\beta-1}(-G_n''(\xi))\phi(G_n(\xi)) + (-G_n'(\xi))^\beta \phi'(G_n(\xi))G_n'(\xi)\right) \tag{3.16}$$

Once again, at the fixed point ξ_n^*, this equation simplifies. The first term on the righthand side of equation (3.16) disappears as $\phi(G_n(\xi_n^*)) = \phi(\xi_n^*) = 0$ by assumption. This leads to the equation

$$\lambda_{n,1}\phi'(\xi_n^*) = z^n(-1)^1(-G_n'(\xi_n^*))^{\beta+1}\phi'(\xi_n^*) \tag{3.17}$$

Again, providing $\phi'(\xi_n^*) \neq 0$, equation (3.17) shows that

$$\lambda_{n,1} = z^n(-1)^1(-G_n'(\xi_n^*))^{\beta+1} \tag{3.18}$$

Clearly, for the k^{th} differentiation, only the term that has $\frac{d}{d^k\xi}\phi(\xi)|_{\xi=\xi_n^*}$ may be non-zero, as the assumption has been mad that $\frac{d}{d^l\xi}\phi(\xi)|_{\xi=\xi_n^*} = 0$ for all $l \in \{0, 1, \ldots, k-1\}$. So this process immediately generalises to show that each

$$\lambda_{n,j} = z^n(-1)^k(-G_n'(\xi_n^*))^{\beta+k} = \frac{z^n(-1)^k}{(1+n\xi_n^*))^{\beta+k}} \tag{3.19}$$

is in the spectrum of $\mathcal{M}_n^{(z,\beta)}$, where $k \in \{0, 1, 2, \ldots\}$. Finally, the only other possiblity for the eigenvalue equation (3.13) is that $\lambda_n = 0$. From the remark in a footnote to section (3.1.4), 0 is actually part of the spectrum and hence an eigenvalue since $H_\infty(D)$ is infinite dimensional. So the spectrum of $\mathcal{M}_n^{(z,\beta)}$, denoted by $\sigma(\mathcal{M}_n^{(z,\beta)})$ has been found to be at most this set of values. I.e.

$$\sigma(\mathcal{M}_n^{(z,\beta)}) \subset \{0\} \bigcup \left\{ z^n(-1)^k(-G_n'(\xi_n^*))^{\beta+k}; k \in \{0, 1, 2, \ldots\} \right\} \tag{3.20}$$

The next step is to show that these numbers are indeed eigenvalues of $\mathcal{M}_n^{(z,\beta)}$. One way of doing this is to show that $(\xi - \xi_n^*)^k$ is not in the range of the operator $(\lambda_{n,k} - \mathcal{M}_n^{(z,\beta)})$. Lemma (3.7) then implies that $\lambda_{n,k}$ must be an eigenvalue since otherwise $(\lambda_{n,k} - \mathcal{M}_n^{(z,\beta)})$ would be bijective. A result from Kamowitz, lemma 2 in [14], is followed here as a means of proving this fact.

Lemma 3.13

$$\sigma(\mathcal{M}_n^{(z,\beta)}) = \{z^n(-1)^k(-G_n'(\xi_n^*))^{\beta+k}; k \in \{0,1,2,\ldots\}\} \tag{3.21}$$

In addition, these eigenvalues have algebraic multiplicity of one.

Proof The observation that $(\xi - \xi_n^*)^k \notin R(\lambda_{n,k}\mathcal{M}^{(z,\beta)})$ may be shown by way of a contradiction. Assume that there exists a function $\phi \in H_\infty(D)$ such that $(\lambda_{n,k} - \mathcal{M}^{(z,\beta)})\phi(\xi) = (\xi - \xi_n^*)^k$. Inserting the expressions for $\lambda_{n,k}$ and $\mathcal{M}^{(z,\beta)}$ this assumption becomes

$$z^n(-1)^k(-G_n'(\xi_n^*))^{\beta+k}\phi(\xi) - z^n(-G_n'(\xi))^\beta\phi(G_n(\xi)) = (\xi - \xi_n^*)^k \tag{3.22}$$

If $k = 0$ and $\xi = \xi_n^*$, equation (3.22) reduces to the following:

$$z^n(-G_n'(\xi_n^*))^\beta\phi(\xi_n^*) - z^n(-G_n'(\xi_n^*))^\beta\phi(\xi_n^*) = 1 \tag{3.23}$$

where the fact that $G_n(\xi_n^*) = \xi_n^*$ has been used. The left hand side of this is obviously 0 which means the assumption is incorrect for $k = 0$. In the case $k > 0$, a similar contradiction occurs. When ξ is set to ξ_n^*, equation (3.22) simplifies with the righthand side equalling 0.

$$z^n(-1)^k(-G_n'(\xi_n^*))^{\beta+k}\phi(\xi_n^*) - z^n(-G_n'(\xi_n^*))^\beta\phi(\xi_n^*) = 0 \tag{3.24}$$

It follows immediately that $\phi(\xi_n^*)$ must be 0. Differentiating (3.22):

$$\begin{aligned} &z^n(-1)^k(-G_n'(\xi_n^*))^{\beta+k}\phi'(\xi) - z^n(-1)^1(-G_n'(\xi))^{\beta+1}\phi'(G_n(\xi)) \\ &\quad -z^n(-G_n''(\xi))\beta(-G_n'(\xi))^{\beta-1}\phi(G_n(\xi)) = k(\xi - \xi_n^*)^{k-1} \end{aligned} \tag{3.25}$$

If $k = 1$, the righthand side of equation (3.25) equals one. Setting $\xi = \xi_n^*$ and recalling that $\phi(\xi_n^*) = 0$, the lefthand side becomes 0, again a contradiction. If $k > 1$ however and $\xi = \xi_n^*$, equation (3.25) becomes

$$z^n(-1)^k(-G_n'(\xi_n^*))^{\beta+k}\phi'(\xi_n^*) - z^n(-1)^1(-G_n'(\xi_n^*))^{\beta+1}\phi'(\xi_n^*) = 0 \tag{3.26}$$

which indicates that $\phi'(\xi_n^*) = 0$.

In general, for all $j < k$ the j^{th} differentiation of equation (3.22) is

$$\begin{aligned} &z^n(-1)^k(-G_n'(\xi_n^*))^{\beta+k}\frac{d^j}{d\xi^j}\phi(\xi) - z^n(-1)^j(-G_n'(\xi))^{\beta+j}\frac{d^j}{d\xi^j}\phi(G_n(\xi)) \\ &+ \quad (\text{terms involving } \tfrac{d^l}{d\xi^l}\phi(\xi) \text{ where } l < j) = k(k-1)\cdots(k-j+1)(\xi-\xi_n^*)^{(k-j)} \end{aligned} \tag{3.27}$$

Again the righthand side vanishes when $\xi = \xi_n^*$. If it is assumed that for all $l < j$, $\frac{d^l}{d\xi^l}\phi(\xi_n^*) = 0$, then the term in brackets disappears. Equation (3.28) reduces to

$$z^n(-1)^k(-G_n'(\xi_n^*))^{\beta+k}\frac{d^j}{d\xi^j}\phi(\xi_n^*) - z^n(-1)^j(-G_n'(\xi_n^*))^{\beta+j}\frac{d^j}{d\xi^j}\phi(\xi_n^*) = 0 \tag{3.28}$$

Since $k \neq j$, $\frac{d^j}{d\xi^j}\phi(\xi_n^*)$ must also be equal to 0, i.e. $\frac{d^l}{d\xi^l}\phi(\xi_n^*) = 0$ for all $l < j+1$. Since this was shown to be true for $j = 0$, then, by induction, $\frac{d^j}{d\xi^j}\phi(\xi_n^*) = 0$ for all $j = 0, 1, \ldots, k-1$. Looking now at the k^{th} differentiation of equation (3.22)

$$\begin{aligned} k! \;=\; & z^n(-1)^k(-G_n'(\xi_n^*))^{\beta+k}\frac{d^k}{d\xi^k}\phi(\xi) - z^n(-1)^k(-G_n'(\xi))^{\beta+k}\frac{d^k}{d\xi^k}\phi(G_n(\xi)) \\ & + (\text{terms involving } \tfrac{d^j}{d\xi^j}\phi(\xi) \text{ where } j < k) \end{aligned} \tag{3.29}$$

Now, setting ξ to ξ_n^*, the terms in the brackets vanish as has been discussed and the first two terms of the equation cancel each other. So the righthand side of equation (3.29) equals 0 but as the lefthand side is $k!$, the assumption that $(\xi - \xi_n^*)^k$ is in the range of $(\lambda_{n,k} - \mathcal{M}_n^{(z,\beta)})$ is clearly false.

Finally, it remains to be shown that each eigenvalue pertains to a one dimensional eigenspace. This will be done by demonstrating that the eigenfunctions can be determined iteratively. It was shown in the above that for any eigenfunction, $\phi_{n,k}$, corresponding to $\lambda_{n,k}$, the first $k-1$ derivatives evaluated at ξ_n^* vanished. The k^{th} derivative, equation (3.29), reveals no information about the value of $\frac{d^k}{d\xi^k}\phi_{n,k}(\xi_n^*)$ except that it must not be equal to zero. Thus $\frac{d^k}{d\xi^k}\phi_{n,k}(\xi_n^*)$ is free to be any complex constant. Now, consider the $k+1^{\text{th}}$ derivative of the eigenvalue equation evaluated at $\xi = \xi_n^*$:

$$\begin{aligned} 0 &= z^n(-1)^k(-G_n'(\xi_n^*))^{\beta+k}\frac{d^{k+1}}{d\xi^{k+1}}\phi_{n,k}(\xi_n^*) \\ &- z^n(-1)^{k+1}(-G_n'(\xi_n^*))^{\beta+k+1}\frac{d^{k+1}}{d\xi^{k+1}}\phi_{n,k}(G_n(\xi_n^*)) \\ &- z^n(-1)^k(-G_n''(\xi_n^*))(\beta+k)(-G_n'(\xi_n^*))^{\beta+k-1}\frac{d^k}{d\xi^k}\phi_{n,k}(G_n(\xi_n^*)) \end{aligned} \tag{3.30}$$

Clearly, $\frac{d^{k+1}}{d\xi^{k+1}}\phi_{n,k}(\xi_n^*)$ is proportional to and is uniquely determined by the value of $\frac{d^k}{d\xi^k}\phi_{n,k}(\xi_n^*)$. In general, the value of $\frac{d^{k+l}}{d\xi^{k+l}}\phi_{n,k}(\xi_n^*)$ will be partly proportional to the values of $\frac{d^{k+j}}{d\xi^{k+j}}\phi_{n,k}(\xi_n^*)$ where $0 \leq j \leq l-1$ and therefore ultimately exactly proportional to the value of $\frac{d^k}{d\xi^k}\phi_{n,k}(\xi_n^*)$. Thus the eigenfunction is determined uniquely by the choice of value for $\frac{d^k}{d\xi^k}\phi_{n,k}(\xi_n^*)$ and is indeed proportional to this value. This means that up to a multiplicative constant, there is one eigenfunction for each $\lambda_{n,k}$, i.e. each eigenvalue is of algebraic multiplicity 1.

□

The trace of $\mathcal{M}_n^{(z,\beta)}$ is then simply the sum over all of the $\lambda_{n,k}$. Since $|G_n'| < 1$, (see page 76), this is a convergent geometric series:

$$\begin{aligned} \text{trace}\mathcal{M}_n^{(z,\beta)} &= \sum_{k=0}^{\infty}\lambda_{n,k} \\ &= \sum_{k=0}^{\infty}\frac{z^n(-1)^k}{(1+n\xi_n^*)^{2\beta+2k}} \\ &= \frac{z^n}{(1+n\xi_n^*)^{2\beta}}\sum_{k=0}^{\infty}\left(\frac{-1}{(1+n\xi_n^*)^2}\right)^k \end{aligned}$$

$$= \frac{z^n(1+n\xi_n^*)^{2-2\beta}}{1+(1+n\xi_n^*)^2} \tag{3.31}$$

The quantity $(1+n\xi_n^*)$ simplifies as follows. From the definition of ξ_n^* in (3.12), it may be written as

$$\xi_n^* = \frac{1}{1+[\overline{n}]} \tag{3.32}$$

where $[\overline{n}]$ is the continued fraction $[n, n, n, \ldots] = [n; n, n, \ldots]^{-1} = [\overline{n;}]^{-1}$. So, using the above representation of ξ_n^*, $1+n\xi_n^*)$ becomes:

$$\begin{aligned} 1+n\xi_n^* &= 1+n\frac{1}{1+[\overline{n}]} = \frac{1+\frac{1}{[\overline{n}]}}{1+[\overline{n}]} = \frac{1}{[\overline{n}]} \\ &= \frac{(1+[\overline{n}])}{(1+[\overline{n}])} = \frac{1}{[\overline{n}]} = [\overline{n;}] \end{aligned} \tag{3.33}$$

The formula for the trace of $\mathcal{M}_n^{(z,\beta)}$, equation (3.31), may then be written as

$$\text{trace}\mathcal{M}_n^{(z,\beta)} = \frac{z^n([\overline{n}])^{-2\beta}}{1+([\overline{n}])^2} = \frac{z^n([n;\overline{n}])^{2-2\beta}}{1+([n;\overline{n}])^2} \tag{3.34}$$

Returning to the object of this section, the trace of $\mathcal{M}^{(z,\beta)}$ itself is then found to be:

$$\text{trace}\mathcal{M}^{(z,\beta)} = \sum_{n=1}^{\infty} \text{trace}\mathcal{M}_n^{(z,\beta)} = \sum_{n=1}^{\infty} \frac{z^n([\overline{n}])^{-2\beta}}{1+([\overline{n}])^2} \tag{3.35}$$

□

3.3 A Generalized Induced Transfer Operator, $\mathcal{M}_{(k)}^{(z,\beta)}$

At this point, it is useful to introduce a generalization of the induced transfer operator, $\mathcal{M}_{(k)}^{(z,\beta)}$. The idea here follows from the work of Ruelle in [27] and Mayer in [19]. Ruelle shows that the ζ-function for an expanding map is given in terms of the Fredholm determinants of the generalized transfer operator.

$$\zeta(z,\beta) = \prod_{k=0}^{N} [\det(1 - z\mathcal{L}_k^{\beta})]^{(-1)^{k+1}} \tag{3.36}$$

where $\mathcal{L}_k^\beta$ is the generalized transfer operator of the system and N is the dimension of the compact manifold on which the system is defined. Note that for the Farey map and its induced version the action takes place on closed intervals, so this dimension N is 1. Therefore, the induced zeta function may be expected to be the quotient of the Fredholm determinants of $\mathcal{M}_1^{(z,\beta)}$ and $\mathcal{M}_0^{(z,\beta)} = \mathcal{M}^{(z,\beta)}$. The next few sections aim to show that this result is indeed true for the induced system.

This generalized induced transfer operator is defined for $k = 0, 1, 2, \ldots$ and is given by

Definition 3.14

$$\mathcal{M}_{(k)}^{(z,\beta)} \circ \phi(\xi) = (-1)^k \mathcal{M}^{(z,\beta+k)} \circ \phi(\xi) = (-1)^k \sum_{n=1}^{\infty} \frac{z^n}{(1+n\xi)^{2\beta+2k}} \phi(1 - \frac{\xi}{1+n\xi}) \quad (3.37)$$

Notice that to obtain $\mathcal{M}_{(k)}^{(z,\beta)}$ from $\mathcal{M}^{(z,\beta)}$ all that is needed is a multiplicative factor of $(-1)^k$ and the linear shift $\beta \to \beta + k$. The trace of $\mathcal{M}_{(k)}^{(z,\beta)}$ is therefore easily found from the trace of $\mathcal{M}^{(z,\beta)}$ given in (3.35)

$$\text{trace}\mathcal{M}_{(k)}^{(z,\beta)} = \text{trace}(-1)^k \mathcal{M}^{(z,\beta+k)} = (-1)^k \sum_{n=1}^{\infty} \frac{z^n ([\overline{n}])^{-2\beta-2k}}{1 + ([\overline{n}])^2} \quad (3.38)$$

These generalised operators are also nuclear operators for all $|z| < 1$ and $\beta \in C$. This follows immediately from lemma (3.10) due to the direct correspondence between $\mathcal{M}_{(k)}^{(z,\beta)}$ and $\mathcal{M}^{(z,\beta)}$.

3.4 The trace of $(\mathcal{M}^{(z,\beta)})^N$

Here the trace of $(\mathcal{M}^{(z,\beta)})^N$ is determined. Also, the trace of $(\mathcal{M}_1^{(z,\beta)})^N$ is calculated in the search for a connection between these generalized induced transfer operators and the induced ζ-function of $\mathcal{M}^{(z,\beta)}$. The method used is similar to the previous sections so some details will be spared.

Proposition 3.15 *The trace of the induced transfer operator composed with itself N times is given by*

$$trace\mathcal{M}^{(z,\beta)^N} = \sum_{i_1=1}^{\infty}\sum_{i_2=1}^{\infty}\cdots\sum_{i_N=1}^{\infty}\frac{\prod_{l=1}^N z^{i_l}[\overline{i_l, i_{l-1}, \ldots, i_1, i_N \ldots, i_{l+1}}]^{2\beta}}{1-(-1)^N\prod_{l=1}^N[\overline{i_l, i_{l-1}, \ldots, i_1, i_N \ldots, i_{l+1}}]^2}$$

Proof. It is useful to once more express $\mathcal{M}^{(z,\beta)}$ in the form $\mathcal{M}^{(z,\beta)} = \sum_{n=1}^{\infty}\mathcal{M}_n^{(z,\beta)}$. $(\mathcal{M}^{(z,\beta)})^N$ may itself then be written as a sum of simpler operators:

$$(\mathcal{M}^{(z,\beta)})^N = \sum_{i_1=1}^{\infty}\sum_{i_2=1}^{\infty}\cdots\sum_{i_N=1}^{\infty}\mathcal{M}_{i_1}^{(z,\beta)}\mathcal{M}_{i_2}^{(z,\beta)}\cdots\mathcal{M}_{i_N}^{(z,\beta)} \tag{3.39}$$

and its trace is then given by

$$\text{trace}(\mathcal{M}^{(z,\beta)})^N = \sum_{i_1=1}^{\infty}\sum_{i_2=1}^{\infty}\cdots\sum_{i_N=1}^{\infty}\text{trace}\left(\mathcal{M}_{i_1}^{(z,\beta)}\mathcal{M}_{i_2}^{(z,\beta)}\cdots\mathcal{M}_{i_N}^{(z,\beta)}\right) \tag{3.40}$$

Attention may now be focussed on the composition of operators on the righthand side of equation (3.40) which will be denoted by $\mathcal{M}_{I_N}^{(z,\beta)}$ where $I_N = \{i_1, i_2, \ldots, i_N\}$. To find an explicit form for $\mathcal{M}_{I_N}^{(z,\beta)}$, consider $\mathcal{M}_{i_{N-1}}^{(z,\beta)}\circ\mathcal{M}_{i_N}^{(z,\beta)}\circ\phi(\xi)$ and the representation $\mathcal{M}_n^{(z,\beta)}\phi(\xi) = z^n(-G_n'(\xi))^\beta\phi(G_n(\xi))$.

$$\begin{aligned}\mathcal{M}_{i_{N-1}}^{(z,\beta)}\circ\mathcal{M}_{i_N}^{(z,\beta)}\circ\phi(\xi) &= \mathcal{M}_{i_{N-1}}^{(z,\beta)}\circ z^{i_N}(-G_{i_N}'(\xi))^\beta\phi(G_{i_N}(\xi)) \\ &= z^{i_{N-1}+i_N}(-G_{i_{N-1}}'(\xi))^\beta(-G_{i_N}'(G_{i_{N-1}}(\xi)))^\beta\phi(G_{i_N}(G_{i_{N-1}}(\xi)))\end{aligned} \tag{3.41}$$

Continuing on in the same way by applying in order the operators $\mathcal{M}_{i_{N-2}}^{(z,\beta)}$, $\mathcal{M}_{i_{N-3}}^{(z,\beta)}$ and so on, the following expression for $\mathcal{M}_{I_N}^{(z,\beta)}$ is obtained:

$$\mathcal{M}_{I_N}^{(z,\beta)} \circ \phi(\xi) = \prod_{l=1}^{N} z^{i_l} \left[-G'_{i_l}(G_{i_{l-1}} G_{i_{l-2}} \cdots G_{i_1}(\xi))\right]^{\beta} \phi(G_{i_N} G_{i_{N-1}} \cdots G_{i_1}(\xi)) \tag{3.42}$$

Notice that the function ϕ in equation (3.42) has as its argument a composition of the inverse branches of the induced map g. This is exactly the object whose fixed point was needed to find the exact expression for the induced zeta function, see section (4.1). There it was shown that $G_{i_N} G_{i_{N-1}} \cdots G_{i_1}$ had 2 fixed points and only one inside D, see page 76. On examination of equation (4.12), the fixed point of $G_{i_N} G_{i_{N-1}} \cdots G_{i_1}$, denoted by $\xi^*_{I_N}$ is seen to be:

$$\xi^*_{I_N} = [1, \overline{i_N, i_{N-1}, \ldots, i_1}] \tag{3.43}$$

Now, the eigenvalue equation for $\mathcal{M}_{I_N}^{(z,\beta)}$ is the following

$$\begin{aligned} \lambda\phi(\xi) &= (-1)^N \prod_{l=1}^{N} z^{i_l} \left[G'_{i_l}(G_{i_{l-1}} G_{i_{l-2}} \cdots G_{i_1}(\xi))\right]^{\beta} \phi(G_{i_N} G_{i_{N-1}} \cdots G_{i_1}(\xi)) \\ &= \prod_{l=1}^{N} z^{i_l} \left[(-1)^N \frac{d}{d\xi}(G_{i_N} G_{i_{N-1}} \cdots G_{i_1}(\xi))\right]^{\beta} \phi(G_{i_N} G_{i_{N-1}} \cdots G_{i_1}(\xi)) \end{aligned} \tag{3.44}$$

Consider the product term in equation (3.44), $\prod_{l=1}^{N} z^{i_l} \left[-G'_{i_l}(G_{i_{l-1}} G_{i_{l-2}} \cdots G_{i_l}(\xi))\right]^{\beta}$. Differentiating both sides of the equation $g_n(G_n(\xi)) = \xi$ gives the result

$$G'_n(\xi) = \frac{1}{g'_n(G_n(\xi)} \tag{3.45}$$

Upon substitution of this result, the product term becomes

$$\prod_{l=1}^{N} z^{i_l} \left[-G'_{i_l}(G_{i_{l-1}} G_{i_{l-2}} \cdots G_{i_1}(\xi))\right]^{\beta} = \prod_{l=1}^{N} z^{i_l} \left[-g'_{i_l}(G_{i_l} G_{i_{l-1}} \cdots G_{i_1}(\xi))\right]^{-\beta} \tag{3.46}$$

Also, with the observation that

$$G_{i_l} G_{i_{l-1}} \cdots G_{i_1}(\xi) = [1, \overline{i_l, i_{l-1}, \ldots, i_1, i_N \ldots, i_{l+1}}] \tag{3.47}$$

equation (3.44) further simplifies when ξ is set to $\xi^*_{I_N}$, the unique fixed point.

$$\lambda\phi(\xi^*_{I_N}) = \prod_{l=1}^{N} z^{i_l}\left[-g'_{i_l}([1,\overline{i_l, i_{l-1},\ldots,i_1,i_N\ldots,i_{l+1}}])\right]^{-\beta}\phi(\xi^*_{I_N}) \qquad (3.48)$$

Using lemma (4.3) equation (3.48) becomes

$$\lambda\phi(\xi^*_{I_N}) = \prod_{l=1}^{N} z^{i_l}[\overline{i_l, i_{l-1},\ldots,i_1,i_N\ldots,i_{l+1}}]^{2\beta}\phi(\xi^*_{I_N}) \qquad (3.49)$$

So, providing $\phi(\xi^*_{I_N}) \neq 0$, equation (3.49) yields that

$$\lambda = \lambda_{I_N,0} = \prod_{l=1}^{N} z^{i_l}[\overline{i_l, i_{l-1},\ldots,i_1,i_N\ldots,i_{l+1}}]^{2\beta} \qquad (3.50)$$

is a possible eigenvalue.

Now, exactly the same procedure used in the previous section may be applied here. At the unique fixed point of $G_{i_N}G_{i_{N-1}}\cdots G_{i_1}$ equation (3.49) and the l^{th} differentiation of equation (3.44) show that if $\frac{d^j}{d\xi^j}\phi(\xi)|_{\xi=\xi^*_{I_N}} = 0 \ \forall \ j = 0,1,\ldots,k-1$ and $\frac{d^k}{d\xi^k}\phi(\xi)|_{\xi=\xi^*_{I_N}} \neq 0$ then

$$\lambda = \lambda_{I_N,k} = (-1)^{Nk}\prod_{l=1}^{N} z^{i_l}[\overline{i_l, i_{l-1},\ldots,i_1,i_N\ldots,i_{l+1}}]^{2\beta+2k} \qquad (3.51)$$

where $k = 0,1,2,\ldots$.

The fact that these values are actually eigenvalues needs to be then shown; the approach will be the same as that used in the previous section for M_n and not as much detail will be shown. In particular, the following work will demonstrate that $(\lambda_{I_N,k} - \mathcal{M}^{(z,\beta)}_{I_N})$ is not a bijective mapping.

Lemma 3.16

$$\sigma(\mathcal{M}^{(z,\beta)}_{I_N}) = (-1)^{Nk}\prod_{l=1}^{N} z^{i_l}[\overline{i_l, i_{l-1},\ldots,i_1,i_N\ldots,i_{l+1}}]^{2\beta+2k}; k \in \{0,1,2,\ldots\}\} \qquad (3.52)$$

These eigenvalues have algebraic multiplicity of one.

Proof Assume that there exists a $\phi \in H_\infty(D)$ such that

$$(\lambda_{I_N,k} - \mathcal{M}_{I_N}^{(z,\beta)})\phi(\xi) = (\xi - \xi_{I_N}^*)^k \tag{3.53}$$

Written out in its full glory, the above becomes

$$\begin{aligned} &(-1)^{Nk}\prod_{l=1}^{N} z^{i_l}[\overline{i_l, i_{l-1}, \ldots, i_1, i_N \ldots, i_{l+1}}]^{2\beta+2k}\phi(\xi) \qquad (3.54)\\ &- \prod_{l=1}^{N} z^{i_l}\left[(-1)^N \frac{d}{d\xi}(G_{i_N}G_{i_{N-1}}\cdots G_{i_1}(\xi))\right]^\beta \phi(G_{i_N}G_{i_{N-1}}\cdots G_{i_1}(\xi)) = (\xi - \xi_{I_N}^*)^k \end{aligned}$$

As before, assume that for any $0 < j < k$ that for all $l < j$, $\frac{d^l}{d\xi^l}\phi(\xi_{I_N}^*) = 0$. Differentiating equation (3.55) j times:

$$\begin{aligned} &(-1)^{Nk}\prod_{l=1}^{N} z^{i_l}[\overline{i_l, i_{l-1}, \ldots, i_1, i_N \ldots, i_{l+1}}]^{2\beta+2k}\frac{d^j}{d\xi^j}\phi(\xi)\\ -\ &\prod_{l=1}^{N} z^{i_l}\left[(-1)^N \frac{d}{d\xi}(G_{i_N}G_{i_{N-1}}\cdots G_{i_1}(\xi))\right]^{\beta+j} \frac{d^j}{d\xi^j}\phi(G_{i_N}G_{i_{N-1}}\cdots G_{i_1}(\xi))\\ +\ &(\text{terms involving } \tfrac{d^l}{d\xi^l}\phi(\xi) \text{ where } l < j)\\ =\ &k(k-1)\cdots(k-j+1)(\xi - \xi_{I_N}^*)^{k-j} \end{aligned} \tag{3.55}$$

and then setting ξ to $\xi_{I_N}^*$ yields the following equation:

$$\begin{aligned} &(-1)^{Nk}\prod_{l=1}^{N} z^{i_l}[\overline{i_l, i_{l-1}, \ldots, i_1, i_N \ldots, i_{l+1}}]^{2\beta+2k}\frac{d^j}{d\xi^j}\phi(\xi_{I_N}^*)\\ -\ &(-1)^{Nj}\prod_{l=1}^{N} z^{i_l}[\overline{i_l, i_{l-1}, \ldots, i_1, i_N \ldots, i_{l+1}}]^{2\beta+2j}\frac{d^j}{d\xi^j}\phi(\xi_{I_N}^*) = 0 \end{aligned} \tag{3.56}$$

Thus, since $j < k$, the two factors multiplying $\frac{d^j}{d\xi^j}\phi(\xi_{I_N}^*)$ are different and $\frac{d^j}{d\xi^j}\phi(\xi_{I_N}^*)$ must be itself 0. Therefore $\frac{d^l}{d\xi^l}\phi(\xi_{I_N}^*) = 0$ for all $l < j+1$. Since this is true for $j = 1$ it must be true by induction for all $0 < j < k$. Consider finally the k^{th} differentiation of equation (3.55):

$$(-1)^{Nk}\prod_{l=1}^{N} z^{i_l}[\overline{i_l, i_{l-1}, \ldots, i_1, i_N \ldots, i_{l+1}}]^{2\beta+2k}\frac{d^k}{d\xi^k}\phi(\xi)$$

$$
\begin{aligned}
&- \prod_{l=1}^{N} z^{i_l} \left[(-1)^N \frac{d}{d\xi}(G_{i_N} G_{i_{N-1}} \cdots G_{i_1}(\xi)) \right]^{\beta+k} \frac{d^k}{d\xi^k} \phi(G_{i_N} G_{i_{N-1}} \cdots G_{i_1}(\xi)) \\
&+ (\text{terms involving } \tfrac{d^l}{d\xi^l}\phi(\xi) \text{ where } l < k) \\
&= k! \qquad (3.57)
\end{aligned}
$$

At $\xi = \xi^*_{I_N}$ the first two terms on the left cancel each other, while the term in brackets has been shown to be 0. Since the righthand side is not equal to 0, the assumption has led to a contradiction. Therefore, $(\xi - \xi^*_{I_N})^k$ is not in the range of $(\lambda_{I_N,k} - \mathcal{M}^{(z,\beta)}_{I_N})$. Thus $(\lambda_{I_N,k} - \mathcal{M}^{(z,\beta)}_{I_N})$ is not bijective and lemma (3.7) implies that $\lambda_{I_N,k}$ must be an eigenvalue of M_{I_N}.

In showing that each eigenvalue is of algebraic multiplicity one, exactly the same argument used in lemma (3.13) may be employed here. It is seen that the eigenfunction is uniquely determined by the choice of the value of $\frac{d^k}{d\xi^k}\phi(\xi^*_n)$ and is in fact proportional to it. Thus, up to a multiplicative constant, each eigenvalue possesses one eigenfunction. This completes the proof of the lemma.

$\square$

The trace for each $\mathcal{M}^{(z,\beta)}_{I_N}$ is thus given by

$$
\begin{aligned}
\text{trace}\mathcal{M}^{(z,\beta)}_{I_N} &= \sum_{k=0}^{\infty} \lambda_{I_N,k} \\
&= \sum_{k=0}^{\infty} (-1)^{Nk} \prod_{l=1}^{N} z^{i_l} [\overline{i_l, i_{l-1}, \ldots, i_1, i_N \ldots, i_{l+1}}]^{2\beta+2k} \\
&= \prod_{l=1}^{N} z^{i_l} [\overline{i_l, i_{l-1}, \ldots, i_1, i_N \ldots, i_{l+1}}]^{2\beta} \sum_{k=0}^{\infty} \left((-1)^N \prod_{l=1}^{N} [\overline{i_l, i_{l-1}, \ldots, i_1, i_N \ldots, i_{l+1}}]^2 \right)^k \\
&= \frac{\prod_{l=1}^{N} z^{i_l} [\overline{i_l, i_{l-1}, \ldots, i_1, i_N \ldots, i_{l+1}}]^{2\beta}}{1 - (-1)^N \prod_{l=1}^{N} [\overline{i_l, i_{l-1}, \ldots, i_1, i_N \ldots, i_{l+1}}]^2} \qquad (3.58)
\end{aligned}
$$

Substituting the result of (3.58) into equation (3.40) the following expression for the trace of $\mathcal{M}^{(z,\beta)^N}$ is obtained:

$$\mathrm{trace}\mathcal{M}^{(z,\beta)^N} = \sum_{i_1=1}^{\infty}\sum_{i_2=1}^{\infty}\cdots\sum_{i_N=1}^{\infty}\frac{\prod_{l=1}^{N} z^{i_l}[\overline{i_l, i_{l-1}, \ldots, i_1, i_N \ldots, i_{l+1}}]^{2\beta}}{1-(-1)^N\prod_{l=1}^{N}[\overline{i_l, i_{l-1}, \ldots, i_1, i_N \ldots, i_{l+1}}]^2} \tag{3.59}$$

$\square$

The trace of N compositions of the generalized induced transfer operator with itself, $\mathcal{M}_{(k)}^{(z,\beta)^N}$, is then given by setting $\beta \to \beta + k$ in the trace formula (3.59) and multiplying by $(-1)^k N$.

$$\begin{aligned}\mathrm{trace}\mathcal{M}_{(k)}^{(z,\beta)^N} &= \mathrm{trace}(-1)^k\mathcal{M}^{(z,\beta+k)^N} = (-1)^{kN}\mathrm{trace}\mathcal{M}^{(z,\beta+k)^N}\\ &= (-1)^{kN}\sum_{i_1=1}^{\infty}\sum_{i_2=1}^{\infty}\cdots\sum_{i_N=1}^{\infty}\frac{\prod_{l=1}^{N} z^{i_l}[\overline{i_l, i_{l-1}, \ldots, i_1, i_N \ldots, i_{l+1}}]^{2\beta+2k}}{1-(-1)^N\prod_{l=1}^{N}[\overline{i_l, i_{l-1}, \ldots, i_1, i_N \ldots, i_{l+1}}]^2}\end{aligned} \tag{3.60}$$

and, in particular, for the first generalized operator, $k = 1$:

$$\mathrm{trace}\mathcal{M}_{(1)}^{(z,\beta)^N} = (-1)^N\sum_{i_1=1}^{\infty}\sum_{i_2=1}^{\infty}\cdots\sum_{i_N=1}^{\infty}\frac{\prod_{l=1}^{N} z^{i_l}[\overline{i_l, i_{l-1}, \ldots, i_1, i_N \ldots, i_{l+1}}]^{2\beta+2}}{1-(-1)^N\prod_{l=1}^{N}[\overline{i_l, i_{l-1}, \ldots, i_1, i_N \ldots, i_{l+1}}]^2} \tag{3.61}$$

In examining the various representation of the induced zeta function given in equations (4.14), 4.15, 4.22, 4.33 and (4.34), its form is seen to closely resemble the trace formulas given in (3.59) and (3.61). In analogy with the work of Ruelle in [27], the logarithm of the induced zeta function is found to be equal to be a sum over N of the difference between $\mathrm{trace}\mathcal{M}^{(z,\beta)^N}$ and $\mathrm{trace}\mathcal{M}_{(1)}^{(z,\beta)^N}$ divided by N. This difference is given by:

$$\begin{aligned}&\mathrm{trace}\mathcal{M}^{(z,\beta)^N} - \mathrm{trace}\mathcal{M}_{(1)}^{(z,\beta)^N}\\ &= \mathrm{trace}\mathcal{M}_{(0)}^{(z,\beta)^N} - \mathrm{trace}\mathcal{M}_{(1)}^{(z,\beta)^N}\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i_1=1}^{\infty}\sum_{i_2=1}^{\infty}\cdots\sum_{i_N=1}^{\infty}\frac{\prod_{l=1}^{N} z^{i_l}[\overline{i_l,i_{l-1},\ldots,i_1,i_N\ldots,i_{l+1}}]^{2\beta}}{1-(-1)^N\prod_{l=1}^{N}[\overline{i_l,i_{l-1},\ldots,i_1,i_N\ldots,i_{l+1}}]^2} \\
&\quad -(-1)^N\sum_{i_1=1}^{\infty}\sum_{i_2=1}^{\infty}\cdots\sum_{i_N=1}^{\infty}\frac{\prod_{l=1}^{N} z^{i_l}[\overline{i_l,i_{l-1},\ldots,i_1,i_N\ldots,i_{l+1}}]^{2\beta+2}}{1-(-1)^N\prod_{l=1}^{N}[\overline{i_l,i_{l-1},\ldots,i_1,i_N\ldots,i_{l+1}}]^2} \\
&= \sum_{i_1=1}^{\infty}\sum_{i_2=1}^{\infty}\cdots\sum_{i_N=1}^{\infty}\prod_{l=1}^{N} z^{i_l}[\overline{i_l,i_{l-1},\ldots,i_1,i_N\ldots,i_{l+1}}]^{2\beta} \\
&\quad \times\frac{1-(-1)^N\prod_{l=1}^{N}[\overline{i_l,i_{l-1},\ldots,i_1,i_N\ldots,i_{l+1}}]^2}{1-(-1)^N\prod_{l=1}^{N}[\overline{i_l,i_{l-1},\ldots,i_1,i_N\ldots,i_{l+1}}]^2} \\
&= \sum_{i_1=1}^{\infty}\sum_{i_2=1}^{\infty}\cdots\sum_{i_N=1}^{\infty}\prod_{l=1}^{N} z^{i_l}[\overline{i_l,i_{l-1},\ldots,i_1,i_N\ldots,i_{l+1}}]^{2\beta}
\end{aligned} \tag{3.62}
$$

For the sake of generality, a further parameter σ can be introduced as per equation (2.7) at the end of the derivation of the induced zeta function. The sum over N of these differences with a weighting $\frac{\sigma^N}{N}$ is then

$$
\begin{aligned}
&\sum_{N=1}^{\infty}\frac{\sigma^N}{N}\left(\mathrm{trace}\mathcal{M}^{(z,\beta)^N}-\mathrm{trace}\mathcal{M}_{(1)}^{(z,\beta)^N}\right) \\
&\quad = \sum_{N=1}^{\infty}\frac{\sigma^N}{N}\sum_{i_1=1}^{\infty}\sum_{i_2=1}^{\infty}\cdots\sum_{i_N=1}^{\infty}\prod_{l=1}^{N} z^{i_l}[\overline{i_l,i_{l-1},\ldots,i_1,i_N\ldots,i_{l+1}}]^{2\beta} \\
&\quad = \sum_{N=1}^{\infty}\frac{\sigma^N}{N}\sum_{n=N}^{\infty}\sum_{\{i_k\}_1^N;\sum_{k=1}^{N} i_k=n}\prod_{l=1}^{N} z^{i_l}[\overline{i_l,i_{l-1},\ldots,i_1,i_N\ldots,i_{l+1}}]^{2\beta} \\
&\quad = \sum_{n=1}^{\infty}z^n\sum_{N=1}^{n}\sum_{\{i_k\}_1^N;\sum_{k=1}^{N} i_k=n}\frac{\sigma^N}{N}\prod_{l=1}^{N}[\overline{i_l,i_{l-1},\ldots,i_1,i_N\ldots,i_{l+1}}]^{2\beta} \\
&\quad = \sum_{n=1}^{\infty}z^n\sum_{m=1}^{n}\sum_{\{i_k\}_1^m;\sum_{k=1}^{m} i_k=n}\frac{\sigma^m}{m}\prod_{l=1}^{m}[\overline{i_l,i_{l-1},\ldots,i_1,i_m\ldots,i_{l+1}}]^{2\beta}
\end{aligned} \tag{3.63}
$$

where the same method of enumeration used in arriving at expression (4.15) for the induced ζ-function has been employed and the dummy indices have been relabelled to match up with those of the previous section. Note, that the order of the entries in the continued fraction in the last line of equation (3.63) may be reversed. The reasoning for this is that for every sequence $\{i_k\}_1^m$ that appears in the summation, so too does the reverse one $\{i_k\}_m^1$, trivially so if they are the same. Therefore, in reversing the order of the entries, the product term of every sequence is still summed over.

$$\sum_{N=1}^{\infty} \frac{\sigma^N}{N} \left(\text{trace} \mathcal{M}^{(z,\beta)^N} - \text{trace} \mathcal{M}_{(1)}^{(z,\beta)^N} \right)$$
$$= \sum_{n=1}^{\infty} z^n \sum_{m=1}^{n} \sum_{\{i_k\}_1^m ; \sum_{k=1}^m i_k = n} \frac{\sigma^m}{m} \prod_{l=1}^{m} [\overline{i_l, i_{l+1}, \ldots, i_m, i_1 \ldots, i_{l-1}}]^{2\beta} \tag{3.64}$$

So, for $\sigma = 1$, this final expression is exactly the same as that of the logarithm of the induced zeta function found in (4.34). Therefore, the following relationship has been shown to be true:

$$\zeta_{ind}(z, \beta) = \exp \sum_{n=1}^{\infty} \frac{1}{n} \left(\text{trace} \mathcal{M}^{(z,\beta)^n} - \text{trace} \mathcal{M}_{(1)}^{(z,\beta)^n} \right) \tag{3.65}$$

A very important observation is that these sums of traces are Fredholm determinants since, by definition:

$$\det(1 - \sigma \mathcal{M}_{(k)}^{(z,\beta)}) = \exp \left(- \sum_{n=1}^{\infty} \frac{\sigma^n}{n} \text{trace} \mathcal{M}_{(k)}^{(z,\beta)^n} \right) \tag{3.66}$$

Thus, there is the following theorem:

Theorem 3.17 *The induced zeta function and the induced transfer operators are related in the following way:*

$$\zeta_{\text{ind}}(z, \beta) = \frac{\det(1 - \mathcal{M}_{(1)}^{(z,\beta)})}{\det(1 - \mathcal{M}^{(z,\beta)})} = \frac{\det(1 - \mathcal{M}_{(1)}^{(z,\beta)})}{\det(1 - \mathcal{M}_{(0)}^{(z,\beta)})}$$

Proof. The result follows immediately from equations (3.65) and (3.66). □

As $\mathcal{M}^{(z,\beta)}$ and $\mathcal{M}_{(1)}^{(z,\beta)}$ are nuclear operators, it is known that these determinants are entire functions of σ and are of order 0 for $|z| < 1$ and $\beta \in C$. A stronger and more important result regarding the analyticity of the Fredholm determinants of the $\mathcal{M}_{(k)}^{(z,\beta)}$

with respect to each of β and z within certain regions of (z, β) space, will be shown in a later section. The analyticity properties of the zeta function will then easily follow from relationship (3.17).

3.5 Meromorphy of the map $(z, \beta) \to \mathcal{M}^{(z,\beta)}$

Recall that D was defined as the set $\{\xi \in C : |\xi - 1| < \frac{3}{4}\}$.

Theorem 3.18 *The map $(z, \beta) \to \mathcal{M}^{(z,\beta)}$ is a holomorphic function in z for all $|z| < 1, \beta$ fixed $\in C$ and a holomorphic function in β for all $\beta \in C, |z|$ fixed $\leq 1, |z| \neq 1$. $\mathcal{M}^{(1\beta)}$ is a meromorphic function of β in the whole of the β plane with simple poles located at $\beta = \frac{1-k}{2}, k = 0, 1, 2, \ldots$ possessing as residues the operator $\mathcal{T}_k$ where $\mathcal{T}_k \circ \phi(\xi) = (-1)^k \phi^{(k)}(1)/2k!$. $\mathcal{M}^{(z,\beta)}$ is a nuclear operator of order 0 for all $|z| \leq 1, \beta \in C$.*

Proof. By lemma (3.10), $\mathcal{M}^{(z,\beta)}$ is a bounded operator for $|z| < 1$ and $\beta \in C$. Corollary (3.11) shows that it is therefore a nuclear operator of order 0 for these values of z and β. Hence the first part of the thereom has already been shown. Recall that specifically lemma (3.10) showed that

$$\|\mathcal{M}^{(z,\beta)} \circ \phi(\xi)\|_{H_\infty(D)} \leq \|\phi(\xi)\|_{H_\infty(D)} \sup_{\xi \in D} \sum_{n=1}^{\infty} |z|^n |(1 + n\xi)^{-2\beta}| \tag{3.67}$$

Thus, for $|z| = 1$, the terms in the above sum behave in the limit $n \to \infty$ like $|(n\xi)^{-2\beta}| = |\xi^{-2\beta} n^{-2\beta}| = \text{constant} \times |n^{-2\Re(\beta)} n^{-2i\Im(\beta)}| = \text{constant} \times |n^{-2\Re(\beta)} \exp -(2i\Im(\beta) \log n)| = \text{constant} \times |n^{-2\Re(\beta)}|$. So this is a bounded sum providing $\Re(\beta) > \frac{1}{2}$. The work of Mayer, see references [21] and [4], on the thermodynamic formalism for the Gauss map provides a method for examining the rest of the β-plane when $z=1$ and a similar approach is followed here.

To investigate the analyticity of $\mathcal{M}^{(z,\beta)}$ with respect to β for $|z|=1$ firstly consider any $\phi \in H_\infty(D)$. Trivially, ϕ is holomorphic in the disc $D' = \{\xi : |\xi - 1| \leq \frac{2}{3}\}$ since $D' \subset D$. Hence, ϕ has a well defined taylor series around $\xi = 1$ that is uniformly convergent in D'. ϕ may be broken up into two parts as follows. Let $\phi = (\phi - \phi_N) + \phi_N$ where

$$\phi_N(\xi) = \phi(\xi) - \sum_{k=0}^{N} \frac{\phi^{(k)}(1)}{k!}(\xi - 1)^k \tag{3.68}$$

Note that $\phi_N \in H_\infty(D)$ since it differs from ϕ by a polynomial of finite order N. Each ϕ is be uniformly convergent in D' and so ϕ_N satisfies (see p.? in [25])

$$|\phi_N(\xi)| \leq C|\xi - 1|^{N+1} \tag{3.69}$$

for $\xi \in D'$.

ϕ_N may be thought of as a projection of ϕ onto $H_\infty^N(D)$: this is the subspace of $H_\infty(D)$ that contains all the functions that behave like $(\xi - 1)^M$ where M is some integer greater than N, around $\xi = 1$. The projection map will be denoted by $\mathcal{P}_N : H_\infty(D) \to H_\infty^N(D)$ with the definition:

$$\mathcal{P}_N \circ \phi = \phi_N \tag{3.70}$$

$\mathcal{M}^{(z,\beta)} \circ \phi$ may now be rewritten for $|z| \leq 1$ and $\beta > \frac{1}{2}$

$$\mathcal{M}^{(z,\beta)} \circ \phi = \mathcal{M}^{(z,\beta)} \circ (\phi - \phi_N) + \mathcal{M}^{(z,\beta)} \circ \phi_N \tag{3.71}$$

The first term is easily evaluated:

$$\mathcal{M}^{(z,\beta)} \left(\sum_{k=0}^{N} \frac{\phi^{(k)}(1)}{k!}(\xi - 1)^k \right) = \sum_{k=0}^{N} (-1)^k \frac{\phi^{(k)}(1)}{k!} \sum_{n=1}^{\infty} \frac{z^n}{(1+n\xi)^{2\beta}} \left(\frac{\xi}{1+n\xi} \right)^k \tag{3.72}$$

$$= \sum_{k=0}^{N} (-1)^k \frac{\phi^{(k)}(1)}{k!} \frac{z}{\xi^{2\beta}} \Phi(z, 2\beta + k, \frac{1}{\xi} + 1)$$

where $\Phi(z, s, \nu)$ is the so-called Lerch transcendent [7] defined by

$$\Phi(z, s, \nu) = \sum_{n=0}^{\infty} \frac{z^n}{(\nu + n)^s} \tag{3.73}$$

for $|z| \leq 1$ and $\nu > 1$.

This can be thought of as the action of the operators $\mathcal{R}_k^{(z,\beta)}, k = 0, 1, \ldots, N$, acting on ϕ where

$$\mathcal{R}_k^{(z,\beta)} \circ \phi(\xi) = \frac{(-1)^k \phi^{(k)}(1)}{k!} \frac{z}{\xi^{2\beta}} \Phi(z, 2\beta + k, \frac{1}{\xi} + 1) \tag{3.74}$$

$\mathcal{M}^{(z,\beta)}$ may now be decomposed in the following way:

$$\mathcal{M}^{(z,\beta)} = \sum_{k=0}^{N} \mathcal{R}_k^{(z,\beta)} + \mathcal{M}^{(z,\beta)} \circ \mathcal{P}_N \tag{3.75}$$

These pieces will be examined individually. The map $(z, \beta) \to \mathcal{R}_k^{(z,\beta)}$: the $\mathcal{R}_k^{(z,\beta)}$'s are relatively simple operators in that they map $H_\infty(D)$ to a one dimensional subspace of itself. I.e., the $\mathcal{R}_k^{(z,\beta)}$'s are rank 1 operators, map any $\phi(\xi)$ to a multiple of the function $\frac{z}{\xi^{2\beta}} \Phi(z, 2\beta + k, \frac{1}{\xi} + 1)$. The singularity structure of the map $(z, \beta) \to \mathcal{R}_k^{(z,\beta)}$ will be determined by that of the analytic continuation of Φ found in [7]. For $|z| = 1$ there are two possibilites:

- Case 1: $z = e^{i\theta}, 0 < \theta < 2\pi$

$$\Phi(e^{i\theta}, s, \nu) = \frac{1}{2\nu^s} + \int_0^\infty \frac{e^{it\theta}}{(\nu + t)^s} dt - 2 \int_0^\infty \sin[it\theta - s \tan^{-1}(t/\nu)] \frac{dt}{(\nu^2 + t^2)^{\frac{s}{2}} (e^{2\pi t} - 1)} \tag{3.76}$$

for $\Re(\nu) > 0$. Note that ν corresponds to $\frac{1}{\xi} + 1$. This is a Möebius transformation of the domain D and maps it to the half plane $\{\xi : \Re(\xi) > 3/2\}$. Hence $\Re(\frac{1}{\xi} + 1) > 0$.

The first term on the right hand side of equation (3.76) is clearly holomorphic in β in the whole β-plane. The third is also well defined for all s: the integrand dies away exponentially and behaves like $\frac{i\theta - s/\nu}{2\pi\nu^s} + O(t)$ around $t = 0$. The second term is a possible source of 'trouble'. Substituting s and ν for $2\beta + k$ and $\frac{1}{\xi} + 1$ respectively, the integral becomes:

$$\int_0^\infty \frac{e^{it\theta} dt}{(\frac{1}{\xi} + 1 + t)^{2\beta+k}} \tag{3.77}$$

Bearing a close resemblance to the Gamma function as it does, it is no surprise that the above may be expressed in terms of the incomplete Gamma function, $\Gamma(p, q)$, which is defined as

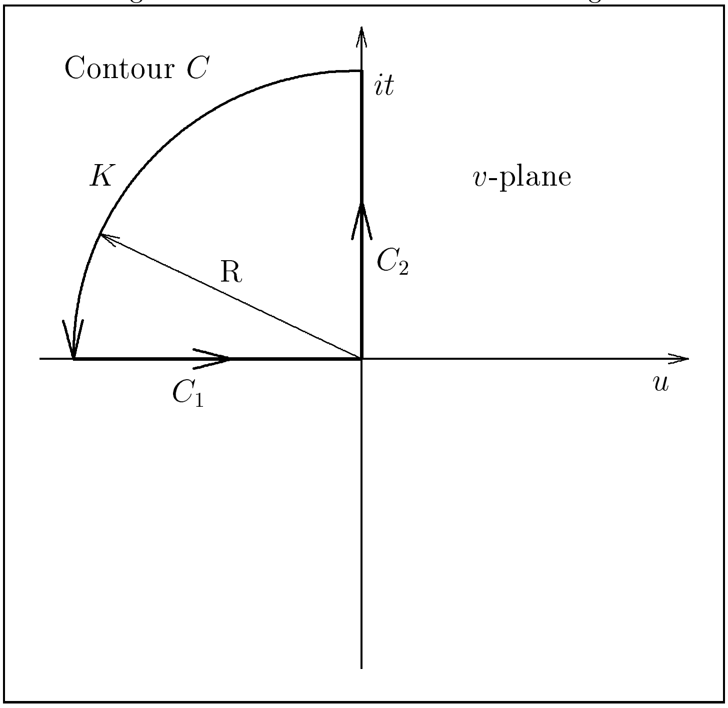
$$\Gamma(p, q) \equiv \int_q^\infty e^{-t} t^{p-1} dt \tag{3.78}$$

and can be analytically continued to complex values of p and q. Indeed, for fixed $q \neq 0$, $\Gamma(p, q)$ is an entire function of p,(see [10]), which will be the important factor in determining the analyticity of this second term. Firstly the integral must be manipulated to a more tractable form. Consider the complex variable $v = u + it$. The integral can now be thought of as an integration of the function $f(v; \theta, 2\beta + k, \xi) = \frac{(i)^{2\beta+k-1} e^{v\theta}}{(v + i(\frac{1}{\xi}+1))^{2\beta+k}}$ along the positive imaginary axis beginning at $v = 0$. Next consider the contour C shown in figure (3.1).

For fixed ξ, the pole of $f(v; \theta, 2\beta + k, \xi)$ occurs at $v = -i(\frac{1}{\xi} + 1)$. These points are outside of the contour C as the set $\{\xi' \in C : \xi' = -i(\frac{1}{\xi} + 1) : \xi \in D\} \equiv \{\xi' \in C : \Im(\xi') < -3/2\}$. Thus by Cauchy's theorem, the integral around the contour is 0. Also, the integral around K vanishes in the limit $R \to 0$. Putting this together,

$$\int_0^\infty \frac{e^{it\theta} dt}{(\frac{1}{\xi} + 1 + t)^{2\beta+k}}$$

Figure 3.1: Contour for evaluation of integral



$$
\begin{aligned}
&= \int_{C_2} f(v;\theta,2\beta+k,\xi)dv \\
&= -\int_{C_1} f(v;\theta,2\beta+k,\xi)dv \\
&= -(-i)^{2\beta+k-1}\int_{-\infty}^{0}\frac{e^{u\theta}du}{(u+i(\frac{1}{\xi}+1))^{2\beta+k}} \\
&= (-i)^{2\beta+k-1}\int_{0}^{\infty}\frac{e^{-u\theta}du}{(u-i(\frac{1}{\xi}+1))^{2\beta+k}} \\
&= (-i\theta)^{2\beta+k-1}e^{-i(\frac{1}{\xi}+1)\theta}\Gamma(2\beta+k+1,-i(\frac{1}{\xi}+1)\theta) \qquad (3.79)
\end{aligned}
$$

where the final evaluation of the integral is taken from [10] subject to the conditions $|\arg(-i(\frac{1}{\xi}+1))| < \pi$ and $\Re(\theta) > 0$. The second criterion is trivially met while for the first, $\Im(-i(\frac{1}{\xi}+1)) < -3/2$ always, so this is also satisfied. As mentioned previously, $\Gamma(2\beta+k+1,-i\nu\theta)$ is entire in $2\beta+k+1$, hence in β, for all $-i\nu\theta \neq 0$. As both θ and ν are never 0, it has now been shown that the map $(z,\beta) \to \mathcal{R}_k^{(z,\beta)}$ is entire in β for all z such that $|z| \leq 1, z \neq 1$. Finally, the map $(z,\beta) \to \sum_{k=0}^{N}\mathcal{R}_k^{(z,\beta)}$ is also entire in β for the same values of z.

- Case 2: $z = 1$

Here, the Lerch transcendent reduces to a simpler generalization of Riemann's zeta function, the Hurwitz zeta function [7].

$$\Phi(1, s, \nu) = \zeta(s, \nu) = \sum_{n=0}^{\infty} \frac{1}{(\nu + n)^s} \tag{3.80}$$

Hermite's representation of $\zeta(s, \nu)$ gives the analytic continuation for $\Re(\nu) > 0$:

$$\zeta(s, \nu) = \frac{1}{2\nu^s} + \frac{\nu^{1-s}}{s-1} + 2\int_0^{\infty} \frac{\sin[s \tan^{-1}(t/\nu)]}{(\nu^2 + t^2)^{\frac{s}{2}}} \frac{dt}{e^{2\pi t} - 1} \tag{3.81}$$

The above indicates that the $\Phi(1, 2\beta + k, \frac{1}{\xi} + 1)$ is a meromorphic function of β in the whole of the β-plane with one simple pole at $\beta = (1-k)/2$ with residue 1. The map $(z, \beta) \to \mathcal{R}_k^{(z,\beta)}$ for $z = 1$ is in turn a meromorphic function of β in the whole of the β-plane with a simple pole at $\beta = 1$ with residue the operator $\mathcal{T}_k : H_\infty(D) \to H_\infty(D)$ given by

$$\mathcal{T}_k \circ \phi(\xi) = \frac{(-1)^k \phi^{(k)}(1)}{2k!} \tag{3.82}$$

Note that $0 \equiv \mathcal{T}_k \circ \mathcal{T}_\updownarrow \circ \phi$ for all $k \geq 1, l \geq 0 \Rightarrow$ i.e., the $\mathcal{T}_k$'s are nilpotent for $k \geq 1$. In total, the map $(z, \beta) \to \sum_{k=0}^{\infty} \mathcal{R}_k^{(z,\beta)}$ for $z = 1$ is a meromorphic function of β in the whole of the β-plane with simple poles at $\beta = (1-k)/2, k \in \{0, 1, \ldots, N\}$ having residue the operators $\mathcal{T}_k$ defined above.

Note that, trivially, the operator $\sum_{k=0}^{N} \mathcal{R}_k^{(z,\beta)}$ is nuclear of order 0 since it is of finite rank. Of course, this only applies for β away from the above points when $z = 1$.

Now consider the map $(z, \beta) \to \mathcal{M}^{(z,\beta)} \circ \mathcal{P}_N$. The reason for introducing the disc D' is that $\mathcal{M}^{(z,\beta)} \circ \Phi(\xi)$ is a sum of weightings of $\Phi \circ G_n(\xi)$ and, as mentioned before, the G_n are contraction mappings with $G_n(D) \subset D_0 = \{\xi : |\xi - 1| \leq \frac{2}{3}\} \subset D' \ \forall n$. The bound on $\phi_N(\xi)$ given in equation (3.69) for $\xi \in D' \subset D_0$ leads to the result that $\mathcal{M}^{(z,\beta)} \circ \phi_N$ is bounded for $\beta > -N/2$ (The choice of $\frac{2}{3}$ is rather arbitrary, needing only to be between $\frac{4}{11}$ and $\frac{3}{4}$). Recalling that this is for $|z| = 1$, this result may be demonstrated as follows:

$$\begin{aligned}\left\|\mathcal{M}^{(z,\beta)} \circ \mathcal{P}_N \circ \phi(\xi)\right\|_{H_\infty(D)} &= \sup_{\xi \in H_\infty(D)} \left| \sum_{n=1}^{\infty} \frac{z^n}{(1+n\xi)^{2\beta}} \phi(1 - \frac{\xi}{1+n\xi}) \right| \\ &\leq \sup_{\xi \in H_\infty(D)} \sum_{n=1}^{\infty} \left| \frac{z^n}{(1+n\xi)^{2\beta}} \right| C \left| \frac{-\xi}{1+n\xi} \right|^{N+1} \\ &= \sup_{\xi \in H_\infty(D)} \frac{C}{|\xi^{2\beta}|} \sum_{n=1}^{\infty} \left| \frac{1}{1/\xi + n} \right|^{2\beta+N+1} \end{aligned} \tag{3.83}$$

For fixed $\xi \in D$, the terms in the series behave like $(n\xi)^{-2\beta-N-1} = \text{constant} \times n^{-2\beta-N-1}$ and therefore converges providing $2\beta + N + 1 > 1$, i.e. if $\beta > -N/2$. Thus, the map $(z, \beta) \to \mathcal{M}^{(z,\beta)} \circ \mathcal{P}_N$ is a holomorphic function of β for all $|\beta| < N/2$ and for fixed z, $|z| = 1$.

Using the same arguments as lemma (3.10) and corollary (3.11) showed for $\mathcal{M}^{(z,\beta)}$, since $\mathcal{M}^{(z,\beta)}$: is a bounded operator, it is then a nuclear operator of order 0. The composition of it with $\mathcal{P}_N$, another bounded operator acting on a Banach space, gives that $\mathcal{M}^{(z,\beta)} \circ \mathcal{P}_N$ is a nuclear operator of order 0 by lemma (3.5).

Since all of the above is true for any N, $\mathcal{M}^{(z,\beta)}$ is a meromorphic function of β for $z = 1$. It has simple poles at $\beta = (1-k)/2$ with residues the operators $\mathcal{T}$ defined above. For all other z such that $|z| \leq 1$, $\mathcal{M}^{(z,\beta)}$ is entire in β. As was shown in the first part of the proof, $\mathcal{M}^{(z,\beta)}$ is also a holomorphic function of z for all $\beta \in C$ and z, $|z| < 1$. The preceding has also demonstrated that $\mathcal{M}^{(z,\beta)}$ is a nuclear operator away from these specified singular points. This completes the proof of theorem (3.18). $\square$

It appears feasible that the above theorem may be improved to cater for an analytic extension of the map $(z,\beta) \to \mathcal{M}^{(z,\beta)}$ to the whole of (z,β) space with a cut along the positive z axis beginning at $z=1$. The cut would arise from the presence of such a cut in the analytic continuation of the Lerch transcendent.

All of the above applies to the generalized induced transfer operator since this is basically the induced transfer operator with a linear β shift.

Corollary 3.19 *The map $(z,\beta) \to \mathcal{M}_{(k)}^{(z,\beta)}$ is a holomorphic function in z for all $|z| < 1, \beta$ fixed $\in C$ and a holomorphic function in β for all $\beta \in C, |z|$ fixed $\leq 1, |z| \neq 1$. $\mathcal{M}_{(k)}^{(1\beta)}$ is a meromorphic function of β in the whole of the β plane with simple poles located at $\beta = \frac{1-k-j}{2}, j = 0,1,2,\ldots$ possessing as residues the operator $\mathcal{T}_k$ where $\mathcal{T}_k \circ \phi(\xi) = (-1)^{k+j}\phi^{(k+j)}(1)/2(k+j)!$. $\mathcal{M}_{(k)}^{(z,\beta)}$ is a nuclear operator of order 0 for all $|z| \leq 1, \beta \in C$.*

3.6 Meromorphy of the Trace of $\mathcal{M}^{(z,\beta)}$

An explicit expression for the trace of $\mathcal{M}^{(z,\beta)}$ was determined in section (3.2). This result, equation (3.35), is reproduced here:

$$\text{trace}\mathcal{M}^{(z,\beta)} = \sum_{n=1}^{\infty} \text{trace}\mathcal{M}_n^{(z,\beta)} = \sum_{n=1}^{\infty} \frac{z^n([\overline{n}])^{-2\beta}}{1+([\overline{n}])^2} \tag{3.84}$$

To uncover the behaviour of the analytic continuation of this function will require recourse to the techniques of the previous section.

Theorem 3.20 *The function $(z,\beta) \to$ trace$\mathcal{M}^{(z,\beta)}$ is, in its analytic continuation, a* *holomorphic function of z for $|z| < 1$ and for fixed $\beta \in C$ and is also a holomorphic function of β for $\beta \in C$ and for fixed z, $|z| \leq 1, z \neq 1$. It is a meromorphic function of β when $z = 1$ and has one simple pole at $beta = \frac{1}{2}$ with residue $\frac{1}{2}$.*

Proof Using the decomposition of $\mathcal{M}^{(z,\beta)}$ from the previous section, the trace of $\mathcal{M}^{(z,\beta)}$ may be written as follows:

$$\mathrm{trace}\mathcal{M}^{(z,\beta)} = \sum_{k=0}^{n} \mathrm{trace}\mathcal{R}_k^{(z,\beta)} + \mathrm{trace}\mathcal{M}^{(z,\beta)} \circ \mathcal{P}_N \tag{3.85}$$

This holds for $|z| \leq 1$ and $\beta > \frac{1}{2}$ since the expression for these operators are well defined there.

The traces of the $\mathcal{R}_k^{(z,\beta)}$ are simple to compute since these operators map $H_\infty(D)$ to a one dimensional subspace of of itself. Therefore, each $\mathcal{R}_k^{(z,\beta)}$ has precisely one eigenvalue. By inspection of equation (3.74), the definition of $\mathcal{R}_k^{(z,\beta)}$, the corresponding eigenfunction to the lone eigenvalue is given by

$$\phi_k(\xi) = \xi^{-2\beta}\Phi(z, 2\beta + k, \frac{1}{\xi} + 1) \tag{3.86}$$

The eigenvalue is then seen to be

$$\lambda_k = \frac{(-1)^k z}{k!}\phi_k^{(k)}(1) = \frac{(-1)^k z}{k!}\frac{d^k}{d\xi^k}\, \xi^{-2\beta}\Phi(z, 2\beta + k, \frac{1}{\xi} + 1)\bigg|_{\xi=1} \tag{3.87}$$

This is not a straightforward calculation being complicated by the factor $\xi^{-2\beta}$ and the unhelpfulness of the argument $(\frac{1}{\xi} + 1)$ in the Lerch transcendent function. However, it will be possible to demonstrate the alleged analyticity properties of the trace without a completely explicit formula for the eigenvalues. However, an attempt to present a more detailed calculation will be reserved for the "Miscellaneous" section of the appendices, see section (B.2).

The product rule, $\frac{d}{ds}u(s)v(s) = u'(s)v(s) + u(s)v'(s)$, generalises easily (in analogy to a binomial expansion or Pascal's triangle) to the following result :

$$\frac{d^k}{ds^k}u(s)v(s) = \sum_{l=0}^{k} \binom{k}{l} \frac{d^{(k-l)}}{ds^{(k-l)}}u(s)\frac{d^l}{ds^l}v(s) \tag{3.88}$$

Using this in the expression for λ_k, equation (3.87) becomes

$$\begin{aligned} \lambda_k &= \frac{(-1)^k z}{k!}\frac{d^k}{d\xi^k}\,\xi^{-2\beta}\Phi(z, 2\beta + k, \frac{1}{\xi} + 1)\Big|_{\xi=1} \\ &= \frac{(-1)^k z}{k!}\sum_{l=0}^{k}\binom{k}{l}\frac{d^{(k-l)}}{d\xi^{(k-l)}}\xi^{-2\beta}\frac{d^l}{d\xi^l}\Phi(z, 2\beta + k, \frac{1}{\xi} + 1)\Big|_{\xi=1} \end{aligned} \tag{3.89}$$

The first part of the product in the summand may be evaluated:

$$\begin{aligned} \frac{d^{(k-l)}}{d\xi^{(k-l)}}\xi^{-2\beta} &= (-1)^{(k-l)}2\beta(2\beta+1)(2\beta+2)\ldots(2\beta+k-l-1)\,\xi^{-2\beta-k+l}\Big|_{\xi=1} \\ &= (-1)^{k-l}2\beta(2\beta+1)(2\beta+2)\ldots(2\beta+k-l-1) \end{aligned} \tag{3.90}$$

Also, an expression for the second part of the product is based on the differentiation of the Lerch transcendent which follows simply from its definition:

$$\frac{d}{d\nu}\Phi(z, s, \nu) = \frac{d}{d\nu}\sum_{n=0}^{\infty}\frac{z^n}{(n+\nu)^s} = -s\sum_{n=0}^{\infty}\frac{z^n}{(n+\nu)^(s+1)} = -s\Phi(z, s+1, \nu) \tag{3.91}$$

Recall from the previous section, $\Phi(z, s, \nu)$ was found to have the following properties:

- $\Phi(z, s, \nu)$ is holomorphic in z for $|z| < 1$ and $\beta \in C$; $\Re(\nu) > 0$.
- $\Phi(z, s, \nu)$ is holomorphic in β for $\beta \in C$ and $|z| \leq 1$, $z \neq 1$; $\Re(\nu) > 0$.
- $\Phi(z, s, \nu)$ is meromorphic in β for $\beta \in C$ and $z = 1$ and has a simple pole at $s = 1$ with residue 1; $\Re(\nu) > 0$.

Notice that the derivative of the Lerch transcendent in equation (3.90) is therefore a holomorphic function of s for $z = 1$ since $\Phi(1, s+1, \nu)$ has a simple pole at $s = 0$ which is removed by the multiplying factor of s. In summary, $\frac{d}{d\nu}\Phi(z, s, \nu)$ is

- holomorphic in z for $|z| < 1$ and $\beta \in C$; $\Re(\nu) > 0$.

- holomorphic in β for $\beta \in C$ and $|z| \leq 1$; $\Re(\nu) > 0$.

With this observation, it is useful to rewrite the expression, 3.89 for λ_k for $k > 0$ as follows

$$\begin{aligned}\lambda_k &= \frac{(-1)^k z}{k!} \left(\frac{d^k}{d\xi^k} \xi^{-2\beta} \right) \Phi(z, 2\beta + k, \frac{1}{\xi} + 1) \bigg|_{\xi=1} \\ &+ \frac{(-1)^k z}{k!} \sum_{l=1}^{k} \binom{k}{l} \frac{d^{(k-l)}}{d\xi^{(k-l)}} \xi^{-2\beta} \frac{d^l}{d\xi^l} \Phi(z, 2\beta + k, \frac{1}{\xi} + 1) \bigg|_{\xi=1}\end{aligned} \tag{3.92}$$

Note that $k = 0$ is not included in this part of the discussion. It cannot be broken up into two non-zero pieces as above. The second term on the right of equation (3.92) is holomorphic in β and z separately since it contains only differentiations of the Lerch transcendent. A more detailed analysis of this is provided in the appendices under the heading of "Miscellaneous". The first term contains an 'untouched' Lerch transcendent and so is worth more inspection. Using equation (3.90) for $l = 0$, the first term evaluated at $\xi = 1$ may be written as

$$\frac{(-1)^k z}{k!} (-1)^{k-l} 2\beta(2\beta + 1)(2\beta + 2) \dots (2\beta + k - 1) \Phi(z, 2\beta + k, 2) \tag{3.93}$$

Once again this is a holomorphic function of β for $z = 1$ since the $(2\beta + k - 1)$ factor cancels the simple pole of $\Phi(1, 2\beta + k, 2)$ at $\beta = (1 - k)/2$. Note that this is true only for $k > 0$ as no such factor appears for the case $k = 0$ because the $\xi^{-\beta}$ term is not differentiated. Thus, for the mapping $(z, \beta) \to \text{trace}\mathcal{R}_k^{(z,\beta)}$ the following has been shown for $k > 0$:

- The function $(z, \beta) \to \text{trace}\mathcal{R}_k^{(z,\beta)}$ is holomorphic in z for $|z| < 1$ and $\beta \in C$.

- The function $(z,\beta) \to \text{trace}\mathcal{R}_k^{(z,\beta)}$ is holomorphic in β for $\beta \in C$ and $|z| \leq 1$.

For the case $k = 0$: the expression for $\text{trace}\mathcal{R}_0^{(z,\beta)} = \lambda_0$ is given by inserting $k = 0$ into equation (3.87):

$$\text{trace}\mathcal{R}_0^{(z,\beta)} = \frac{z}{\xi^{-2\beta}}\, \Phi(z, 2\beta + k, \frac{1}{\xi} + 1)\bigg|_{\xi=1} = z\Phi(z, 2\beta, 2) \tag{3.94}$$

Here, there are no saving graces in the form of nice factors appearing to cancel the pole of the Lerch transcendent. Recall that for $z = 1$, $\Phi(1, s, \nu)$ is just the Hurwitz ζ-function. But for $\nu = 2$, the Hurwitz ζ-function itself reduces to the Riemann ζ-function, or, more precisely, $\zeta_R(s) - 1$. The meromorphic qualities are still the same as $\zeta(s)$ extends to the entire complex s-plane with one simple pole at $s = 1$ of residue 1, see [7]. Therefore

$$\text{trace}\mathcal{R}_0^{(z,\beta)} = \zeta_R(2\beta) - 1 \tag{3.95}$$

and it now follows that

- The function $(z,\beta) \to \text{trace}\mathcal{R}_0^{(z,\beta)}$ is holomorphic in z for $|z| < 1$ and $\beta \in C$.
- The function $(z,\beta) \to \text{trace}\mathcal{R}_0^{(z,\beta)}$ is holomorphic in β for $\beta \in C$ and $|z| \leq 1$, $z \neq 1$.
- The function $(z,\beta) \to \text{trace}\mathcal{R}_0^{(z,\beta)}$ is meromorphic in β for $\beta \in C$ and $z = 1$. It has a simple pole at $\beta = \frac{1}{2}$ with residue $\frac{1}{2}$.

The trace of $\mathcal{M}^{(z,\beta)}$ composed with the projection operator $\mathcal{P}_N$ is now examined. The projection operator has already been shown to be a nuclear operator as well as a holomorphic function of β for $\Re\beta > -N/2$. As this analysis is for arbitrary N and, it must be true for for all N and hence for all $\beta \in C$. Therefore, by theorem (3.4), its trace is well defined and has no singular behaviour. This trace is thus holomorphic for β in the whole of the β-plane for fixed $z \leq 1$ and holomorphic in z for $|z| < 1$ and $\beta \in C$. Since, adding a finite number of holomorphic (meromorphic) functions together gives another holomorphic (meromorphic) function, the properties of the trace of $\mathcal{M}^{(z,\beta)}$ follow readily from those of its decomposition.

In total, the trace of the $\mathcal{M}^{(z,\beta)}$ can be categorized as follows:

- The function $(z, \beta) \to \text{trace}\mathcal{M}^{(z,\beta)}$ is holomorphic in z for $|z| < 1$ and $\beta \in C$.
- The function $(z, \beta) \to \text{trace}\mathcal{M}^{(z,\beta)}$ is holomorphic in β for $\beta \in C$ and $|z| \leq 1$, $z \neq 1$.
- The function $(z, \beta) \to \text{trace}\mathcal{M}^{(z,\beta)}$ is meromorphic in β for $\beta \in C$ and $z = 1$. It has a simple pole at $\beta = \frac{1}{2}$ with residue $\frac{1}{2}$.

This completes the proof. $\square$

As before there is the simple corollary regarding the generalized induced transfer operator:

Corollary 3.21 *The function $(z, \beta) \to trace\mathcal{M}_{(k)}^{(z,\beta)}$ is, in its analytic continuation, a holomorphic function of z for $|z| < 1$ and for fixed $\beta \in C$ and is also a holomorphic function of β for $\beta \in C$ and for fixed z, $|z| \leq 1, z \neq 1$. It is a meromorphic function of β when $z = 1$ and has one simple pole at beta $= \frac{1}{2} - k$ with residue $\frac{1}{2}$.*

3.7 Analyticiy Properties of the Determinant of $(1 - \sigma\mathcal{M}^{(z,\beta)})$

The following theorem provides the information about the analyticity of the map from (z, β)-plane to the Fredholm determinant $\det(1 - \mathcal{M}^{(z,\beta)})$.

Theorem 3.22 *The function $(z, \beta) \to \det(1 - \mathcal{M}^{(z,\beta)})$ is, in its analytic continuation, a holomorphic function of z for $|z| < 1$ and for $\beta \in C$ and it is a holomorphic function of β for $\beta \in C$ and for $|z| \leq 1$, $z \neq 1$. It is a meromorphic function of β in the whole β-plane when $z = 1$ with simple poles at $\beta_k = (1 - k)/2$, $k = 0, 1, 2, \ldots$.*

Proof. Theorem (3.4) due to Grothendieck shows that wherever $\mathcal{M}^{(z,\beta)}$ is holomorphic in either β or z, the function $\det(1 - \mathcal{M}^{(z,\beta)})$ is also holomorphic. It is clear that where $\mathcal{M}^{(z,\beta)}$ is singular, $\det(1 - \mathcal{M}^{(z,\beta)})$ will also be singular. Therefore, this determinant will have the same structure as $\mathcal{M}^{(z,\beta)}$ which is demonstrated in theorem (3.18) and the proof is finished. □

Once again, the same argument applies for the generalized induced zeta function:

Corollary 3.23 *The function $(z, \beta) \to \det(1 - \mathcal{M}^{(z,\beta)}_{(k)})$ is, in its analytic continuation, a holomorphic function of z for $|z| < 1$ and for $\beta \in C$ and it is a holomorphic function of β for $\beta \in C$ and for $|z| \leq 1$, $z \neq 1$. It is a meromorphic function of β in the whole β-plane when $z = 1$ with simple poles at $\beta_j = (1 - j - k)/2$, $j = 0, 1, 2, \ldots$.*

Finally, there is the following theorem regarding the analtyicity properties of the induced zeta function.

Theorem 3.24 *The induced zeta function for the Farey map is a meromorphic function of β for all $\beta \in C$ and for fixed z such that $|z| \leq 1$. It is also a meromorphic* *function of z for all z such that $|z| < 1$ and for fixed $\beta \in C$.*

Proof. The theorem follows directly from theorem (3.17) with theorem (3.22) and corollary (3.23). □

Corollary 3.25 *The zeta function for the Farey map is a meromorphic function of β for all $\beta \in C$ and for fixed z such that $|z| \leq 1$ and $z \neq 1$. It is also a meromorphic function of z for all z such that $|z| < 1$ and for fixed $\beta \in C$.*

Proof. This follows immediately from theorem (3.22) using the relation

$$\zeta(z, \beta) = \frac{\zeta_{\text{ind}}(z, \beta)}{1 - z}$$

□

3.8 Concluding Remarks

This chapter has provided strong results on the overall singularity structure of the analytic continuation of the induced zeta function for the Farey map. A very strong link between the Fredholm determinant of a simple generalization of the induced transfer operator and the induced zeta function has been demonstrated. This elegant connection reinforces the observation that the two techniques provide the same thermodynamics for the Farey system. Indeed, it appears reasonable that this connection would apply in more general settings, just as it has been shown to do for normal hyperbolic systems by Ruelle in [27].

It would be interesting to see if the work could be extended to values of z outside the unit disc. However, it is not entirely useful to do this as the connection between the induced transfer operator and the original transfer operator, theorem (2.3), applies only for values of $|z|$ less than 1.

Chapter 4

Presentation and Analysis of the Induced Zeta Function of the Farey map

4.1 The Induced Zeta Function for the Farey map

The form for the induced zeta function was determined in the previous chapter and is repeated here:

$$\begin{aligned} \log \zeta_{\mathrm{ind}}(z, \beta) &= \sum_{n=1}^{\infty} \frac{1}{m} \sum_{g^m(x)=x} \exp \sum_{k=0}^{m-1} \phi_z(g^k(x)) \\ &= \sum_{m=1}^{\infty} \frac{1}{m} \sum_{g^m(x)=x} \prod_{l=0}^{m-1} z^{n(g^l(x))} |g'(g^k(x))|^{-\beta} \end{aligned} \tag{4.1}$$

The main result of this first section is the following theorem regarding the presentation of an explicit expression for the power series expansion of $\log \zeta_{\mathrm{ind}}(z, \beta)$.

Theorem 4.1 *The power series expansion about $z = 0$ of the logarithm of the induced zeta function of the Farey map is given by*

$$\log \zeta_{ind}(z, \beta) = \sum_{n=1}^{\infty} z^n \sum_{m=1}^{n} \frac{1}{m} \sum_{\{i_k\}_1^m; \sum_{k=1}^m i_k = n} \prod_{l=1}^{m} \left| [\overline{i_l, i_{l+1}, \ldots, i_m, i_1, \ldots, i_{l-1}}] \right|^{2\beta}$$

where $[\overline{i_1, i_2, \ldots, i_m}]$ is the periodic continued fraction[1]

$$\cfrac{1}{i_1 + \cfrac{1}{i_2 + \cfrac{1}{\cdots + \cfrac{1}{i_m + \cfrac{1}{i_1 + \cfrac{1}{1 + \cdots}}}}}}$$

Proof For the Farey map, recall that the induced map $g : [1/2, 1) \to [1/2, 1]$ was given by

$$g(x) = \begin{cases} g_n(x) & \forall\ x \in (\frac{n}{n+1}, \frac{n+1}{n+2}], \quad n = 1, 2, \ldots \\ 1 & \text{for } x = \frac{1}{2} \end{cases} \tag{4.2}$$

where g_n were defined by

$$g_n(x) = \frac{1 - x}{1 - n(1 - x)} \tag{4.3}$$

The inverse of G cut down on the interval $[\frac{1}{2}, 1)$ is the countable union of the inverse branches G_n. I.e., $G \equiv \bigcup_{n=1}^{\infty} G_n \equiv \bigcup_{n=1}^{\infty} g_n^{-1} : [\frac{1}{2}, 1) \to (\frac{n}{n+1}, \frac{n+1}{n+2}]$ was found to be

$$G_n(x) = 1 - \frac{x}{1 + nx} \tag{4.4}$$

[1] A general outline of continued fractions and some relevant results are presented in the appendices. It is suggested that the reader refer to this section for clarification on notations and definitions.

The end point 1 has only one inverse:$G(1) = \frac{1}{2}$. The first task is to determine the position of the fixed points of g^m. One way of doing this is to consider the inverse problem of $G^n(x) = x$. To begin with, consider a branch of the inverse map, G_n. G_n may be rewritten as a kind of operator on continued fractions; this new form will also help to expose the fixed points of the induced map.

$$G_n(x) = \cfrac{1}{1 + \cfrac{1}{n - 1 + \cfrac{1}{x}}} \tag{4.5}$$

The equation $G_n(x) = x$ has two solutions as this is really just a quadratic equation:

$$x = 1 - \frac{x}{1+nx} \Rightarrow x + nx^2 = 1 + nx - x \Rightarrow nx^2 + (2-n)x - 1 = 0 \tag{4.6}$$

The solutions can be expressed as:

$$x_{n\pm} = \frac{n - 2 \pm \sqrt{(n-2)^2 + 4n}}{2n} = \frac{n - 2 \pm \sqrt{n^2+4}}{2n} \tag{4.7}$$

Notice that the solutions are always real and that $x_{n+} > 0$ and $x_{n-} < 0$. Thus, in the domain of functions of relevance, namely $J = [\frac{1}{2}, 1]$ and more generally D, the G_n have only one real fixed point at $x_n^* = \frac{n-2+\sqrt{n^2+4}}{2n}$. Since $\sqrt{n^2+4}$ is never a perfect square, the fixed points $g_{n\pm}$ are 'quadratic surds' — a quadratic surd is a solution to a quadratic equation $ax^2 + bx + c = 0$ such that $\sqrt{b^2 - 4ac}$ is not a perfect square. Moreover, there is the following theorem

Theorem 4.2 *Any periodic continued fraction is a quadratic surd and, conversely, every quadratic surd has a periodic continued fraction expansion. I.e., there is a 1-1 correspondence between the quadratic surds and periodic continued fractions.*

Proof. See [24]. $\square$

Whether by direct manipulation of the expression for x_n^* given in equation (4.7) or more simply by inspection of equation (4.5), it follows that x_n^* can also be written as:

$$x_n^* = [1, \overline{n}] = \cfrac{1}{1 + \cfrac{1}{n + \cfrac{1}{n + \cfrac{1}{n + \cdots}}}} \tag{4.8}$$

which is trivially a periodic continued fraction having period 1.

The above may be generalised to the function $G^m : [\frac{1}{2}, 1] \to [\frac{1}{2}, 1]$. Whereas G was made up of the pieces G_n, G^m is made up of the composition maps $G_{\{i_k\}_{k=1}^m} = G_{i_1} \circ G_{i_2} \circ \cdots \circ G_{i_m}$.

$$G_{\{i_k\}_{k=1}^m}(x) = \cfrac{1}{1 + \cfrac{1}{i_1 + \cfrac{1}{i_2 + \cfrac{1}{\cdots + \cfrac{1}{i_m - 1 + \cfrac{1}{x}}}}}} \tag{4.9}$$

Any continued fraction can be expressed in terms of its k$^{\text{th}}$ complete quotient μ_k, (see appendix). Using relationship (A.5)

$$G_{\{i_k\}_{k=1}^m}(x) = \frac{A_{m+1}\mu_{m+2} + A_m}{B_{m+1}\mu_{m+2} + B_m} = \frac{A_{m+1}x + A_m}{B_{m+1}x + B_m} \tag{4.10}$$

Hence, the equation $G_{\{i_k\}_{k=1}^m}(x^*) = x^*$ reduces to finding the solutions to a quadratic equation:

$$p(x) = B_{m+1}x^2 + (B_m - A_{m+1})x - A_m = 0 \tag{4.11}$$

Notice that $p(0) = -A_m < 0$. Further, $p(-1) = (B_{m+1} - B_m) + (A_{m+1} - A_m) > 0$ since the sequence $\{A_k \ + B_k\}$ has positive terms which are strictly increasing for $k \geq 1$, (see appendix, equations (A.7)). One solution must therefore be real and lie between -1 and 0, outside of D. The other solution is real and positive and lies between 0 and 1. This may be seen on inspection of the fixed point which is the periodic continued fraction easily identified via equation (4.9):

$$x^*_{I_m} = [1, \overline{i_1, i_2, \ldots, i_m}] \tag{4.12}$$

Once again, these maps have only one fixed point in the region of interest, D. Thus, the fixed points for G^m, and therefore of g^m have been found. The derivative of G_n is given by

$$G_n'(x) = \frac{-1}{(1+nx)^2} \tag{4.13}$$

Clearly, for $x \in [\frac{1}{2}, 1]$, $|G_n'(x)| < 1$ always and the same is true for $|G_n'(\xi)|$ for $\xi \in D$. Also of note at this point is that a lemma of Ruelle [27] indicates that $G_{\{i_k\}_{k=1}^m}$ has exactly one fixed point in D and the modulus of its derivative at that point is less than 1 .

Returning to the form of the induced zeta function, the exact values of the fixed point may now be substituted into equation (4.1)

$$\log \zeta_{\text{ind}}(z) = \sum_{m=1}^{\infty} \frac{1}{m} \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{\infty} \cdots \sum_{i_m=1}^{\infty} \prod_{l=0}^{m-1} z^{n(g^l([1,\overline{i_1,i_2,\ldots,i_m}]))} |g'(g^l([1, \overline{i_1, i_2, \ldots, i_m}]))|^{-\beta} \tag{4.14}$$

One useful way of enumerating these composition maps is to consider 'ordered integer partitions' of the numbers $\sum_{k=1}^{k=m} i_k = m, m+1, m+2, \ldots$. Note that ordering of the partition is important. For example, in the case of $m = 3$, the sequences $\{2, 1, 1\}$, $\{1, 2, 1\}$ and $\{1, 1, 2\}$ each represent distinct branches of the mapping g^3, namely $g_2 g_1 g_1$, $g_1 g_2 g_1$ and $g_1 g_1 g_2$ respectively. Evidently, the fixed points of g^n may be enumerated in the same way. This type of partition should not be confused with the 'classical' partition of integers worked on by people such as Hardy and Ramanujan (see [1]), where order is not important.

$$\log \zeta_{\text{ind}}(z) = \sum_{m=1}^{\infty} \frac{1}{m} \sum_{n=m}^{\infty} \sum_{\{i_k\}_1^m; \sum_{k=1}^m i_k = n} \prod_{l=0}^{m-1} z^{n(g^l([1,\overline{i_1,i_2,\ldots,i_m}]))} |g'(g^l([1, \overline{i_1, i_2, \ldots, i_m}]))|^{-\beta} \tag{4.15}$$

where the sum over $g^m(x) = x$ has been replaced by a sum over ordered integer partitions of n with n running from m to ∞. The product on the righthand side of (4.15) may still be considerably simplified. Let

$$x^* = [1, \overline{i_1, i_2, \ldots, i_m}] = \cfrac{1}{1 + \cfrac{1}{i_1 + f}} \tag{4.16}$$

where $0 < f = [\overline{i_2, i_3, \ldots, i_m, i_1}] < 1$. Using $x^* = \frac{i_1+f}{i_1+1+f}$, the following bounds can be verified:

$$\frac{i_1 + f}{i_1 + 1 + f} - \frac{i_1}{i_1 + 1} = 1 - \frac{1}{i_1 + 1 + f} - 1 + \frac{1}{i_1 + 1} = \frac{f}{(i_1 + 1)(i_1 + 1 + f)} > 0 \tag{4.17}$$

Similarly:

$$\frac{i_1 + f}{i_1 + 1 + f} - \frac{i_1 + 1}{i_1 + 2} = \frac{f - 1}{(i_1 + 2)(i_1 + 1 + f)} < 0 \tag{4.18}$$

This means that $\frac{i_1}{i_1+1} < x^* < \frac{i_1+1}{i_1+2}$ and therefore $g(x^*) = g_{i_1}(x^*)$. Also, $n(x) = n$ for $x \in (\frac{n}{n+1}, \frac{n+1}{n+2}]$ so $n(x^*) = i_1$. The action of g on each of the fixed points is just to remove the first integer of the periodic part of the continued fraction expansion. This can be seen by noting that g 'undoes' whatever the inverse does to a point in equation (4.5) or simply via the continued fraction representation of g_n:

$$g_n(x) = \cfrac{1}{1 - n + \cfrac{1}{-1 + \cfrac{1}{x}}} \tag{4.19}$$

All this translates to the fact that

$$g^l(x^*) = [1, \overline{i_{l+1}, i_{l+2}, \ldots, i_m, i_1, \ldots, i_l}] \tag{4.20}$$

and

$$n(g^l(x^*)) = i_{l+1} \tag{4.21}$$

The expression (4.15) for the induced zeta function, may now be rewritten as

$$\log \zeta_{\mathrm{ind}}(z) = \sum_{m=1}^{\infty} \frac{1}{m} \sum_{n=m}^{\infty} \sum_{\{i_k\}_1^m; \sum_{k=1}^m i_k = n} \prod_{l=0}^{m-1} z^{i_{l+1}} |g'_{l+1}([1, \overline{i_{l+1}, i_{l+2}, \ldots, i_m, i_1, \ldots, i_l}])|^{-\beta} \tag{4.22}$$

This representation also simplifies the exponent of the product of $z^{i_{l+1}}$'s:

$$\prod_{l=0}^{m-1} z^{i_{l+1}} = z^{\sum_{l=0}^{m-1} i_{l+1}} = z^{\sum_{l=1}^{m} i_l} = z^n \tag{4.23}$$

The term $\prod_{l=0}^{m-1} |g'_{l+1}([1, \overline{i_{l+1}, i_{l+2}, \ldots, i_m, i_1, \ldots, i_l}])|^{-\beta}$ is now examined in more detail. Firstly, from definition (4.3), $g'_n : (\frac{n}{n+1}, \frac{n+1}{n+2}] \to [\frac{1}{2}, 1)$ is given by

$$g'_n(x) = \frac{-1}{(1 - n + nx)^2} \tag{4.24}$$

Thus, the product can be written with $l \to l-1$

$$\begin{aligned}
&\prod_{l=1}^{m} |g'_{i_l}([1, \overline{i_l, i_{l+1}, \ldots, i_m, i_1, \ldots, i_{l-1}}])|^{-\beta} \\
&\quad = \prod_{l=1}^{m} (1 - i_l + i_l \cdot [1, \overline{i_l, i_{l+1}, \ldots, i_m, i_1, \ldots, i_{l-1}}])^{2\beta} \qquad (4.25)
\end{aligned}$$

The above is further simplified via the following result.

Lemma 4.3 *The product*

$$\prod_{l=1}^{m} (1 - i_l + i_l \cdot [1, \overline{i_l, i_{l+1}, \ldots, i_m, i_1, \ldots, i_{l-1}}]) \qquad (4.26)$$

reduces to

$$\prod_{l=1}^{m} [\overline{i_l, i_{l+1}, \ldots, i_m, i_1, \ldots, i_{l-1}}] \qquad (4.27)$$

Proof. The l^{th} complete quotients μ_l of the fixed points provide a nice way of showing this fact; (Note that $\mu_l = \mu_{l+jm}$ for $l = 2, 3, \ldots$ and $j = 0, 1, \ldots$). By definition

$$\mu_0 = x^* = [1, \overline{i_1, i_2, \ldots, i_m}] \qquad (4.28)$$

$$\mu_1 = [1; \overline{i_1, i_2, \ldots, i_m}] \qquad (4.29)$$

$$\mu_l = [\overline{i_{l-1}; i_l, \ldots, i_m, i_1, \ldots, i_{l-2}}] \quad \forall\ l > 1 \qquad (4.30)$$

Each fixed point in the cycle of μ_0 may then be expressed in terms of the μ_l in the following way:

$$[1, \overline{i_l, i_{l+1}, \ldots, i_m, i_1, \ldots, i_{l-1}}] = \cfrac{1}{1 + \cfrac{1}{i_l + \cfrac{1}{\mu_{l+2}}}} = \frac{i_l + \cfrac{1}{\mu_{l+2}}}{1 + i_l + \cfrac{1}{\mu_{l+2}}} \qquad (4.31)$$

Now consider the quantity on the righthand side of relation (4.25), ignoring the power 2β. This quantity can be written as in terms of the μ_l using (4.31)

$$
\begin{aligned}
&\prod_{l=1}^{m}\left(1 - i_l + i_l \cdot \frac{i_l + \frac{1}{\mu_{l+2}}}{1 + i_l + \frac{1}{\mu_{l+2}}}\right) \\
&= \prod_{l=1}^{m} \frac{(1-i_l)(1+i_l+\frac{1}{\mu_{l+2}}) + i_l(i_l + \frac{1}{\mu_{l+2}})}{1 + i_l + \frac{1}{\mu_{l+1}}} \\
&= \prod_{l=1}^{m} \frac{(1+i_l+\frac{1}{\mu_{l+2}}) - i_l - i_l(i_l + \frac{1}{\mu_{l+2}}) + i_l(i_l + \frac{1}{\mu_{l+2}})}{1 + i_l + \frac{1}{\mu_{l+2}}} \\
&= \prod_{l=1}^{m} \frac{1 + \frac{1}{\mu_{l+2}}}{1 + i_l + \frac{1}{\mu_{l+2}}} = \prod_{l=1}^{m} \frac{1 + \frac{1}{\mu_{l+2}}}{1 + \mu_{l+1}} = \prod_{l=1}^{m} \frac{1 + \frac{1}{\mu_{l+2}}}{1 + \frac{1}{\mu_{l+1}}} \prod_{l=1}^{m} \frac{1}{\mu_{l+1}} \\
&= \frac{(1+\frac{1}{\mu_3})}{(1+\frac{1}{\mu_2})} \frac{(1+\frac{1}{\mu_4})}{(1+\frac{1}{\mu_3})} \frac{(1+\frac{1}{\mu_5})}{(1+\frac{1}{\mu_4})} \cdots \frac{(1+\frac{1}{\mu_m})}{(1+\frac{1}{\mu_{m-1}})} \frac{(1+\frac{1}{\mu_1})}{(1+\frac{1}{\mu_m})} \frac{(1+\frac{1}{\mu_2})}{(1+\frac{1}{\mu_1})} \prod_{l=1}^{m} \frac{1}{\mu_{l+1}} \\
&= \prod_{l=1}^{m} \frac{1}{\mu_{l+1}} = \prod_{l=1}^{m} [\overline{i_l, i_{l+1}, \ldots, i_m, i_1, \ldots, i_{l-1}}]
\end{aligned}
\tag{4.32}
$$

where the fact that $\mu_{m+1} = \mu_1$ and $\mu_{m+2} = \mu_2$ have been used. $\square$

Substituting this result into equation (4.22) along with the identification (4.23), the expression for the logarithm of the induced zeta function is now

$$
\log \zeta_{\text{ind}}(z) = \sum_{m=1}^{\infty} \frac{1}{m} \sum_{n=m}^{\infty} z^n \sum_{\{i_k\}_1^m ; \sum_{k=1}^{m} i_k = n} \prod_{l=1}^{m} \left| [\overline{i_l, i_{l+1}, \ldots, i_m, i_1, \ldots, i_{l-1}}] \right|^{2\beta} \tag{4.33}
$$

Interchanging the summands, $\sum_{m=1}^{\infty}\sum_{n=m}^{\infty} \equiv \sum_{n=1}^{\infty}\sum_{m=1}^{n}$, the zeta function reduces to the elegant form

$$
\log \zeta_{\text{ind}}(z, \beta) = \sum_{n=1}^{\infty} z^n \sum_{m=1}^{n} \frac{1}{m} \sum_{\{i_k\}_1^m ; \sum_{k=1}^{m} i_k = n} \prod_{l=1}^{m} \left| [\overline{i_l, i_{l+1}, \ldots, i_m, i_1, \ldots, i_{l-1}}] \right|^{2\beta} \tag{4.34}
$$

which completes the proof of theorem (4.1). $\square$

Note that this is now a power series in z. Recall that the radius of convergence of the power series for $\zeta(z, \beta)$, which is not necessarily the same as that of the power series for $\log(\zeta(z, \beta)$, gives the pressure function for the Farey map. The focus of this work is next turned to examining the radius of convergence of $\log(\zeta(z,\beta)$ directly via equation (4.34) with conclusions about the pressure function being drawn from these observations.

4.2 Bounds on the Pressure function of the Farey map

Bounds on the pressure function for the Farey map, -$\beta F(\beta)$, may now be obtained by an analysis of the radius of convergence of the induced zeta function. A good picture of the pressure function can be developed this way, in the absence of an explicit expression, including the identification of a phase transition and the scaling behaviour near the critical point.

To begin with, an important ingredient for this work is an answer to the question: how many ordered partitions of n into m parts are there?

Lemma 4.4 *The number of ordered partitions of an integer n into m integers is equal to $\binom{n-1}{m-1}$.*

Proof. One way to think of this question is to imagine a plank of wood that is n units in length with the numbers 1 to n written evenly on one side and rulings made between, see figure (4.1). The number of ordered partitions is equivalent to how many

1	2	3	4	$\cdots\cdots$	n-2	n-1	n

Figure 4.1: A piece of wood

physically different ways the length of wood can be chopped up into m pieces of integral lengths. Two partitions are 'physically different' if and only if for one partition the wood is cut between one pair of the inscribed integers but is not cut between the same pair for the other partition. The number of cuts required is $m-1$ and there are $n-1$ places to make these cuts. Thus, there are $n-1$ places from which to choose $m-1$.

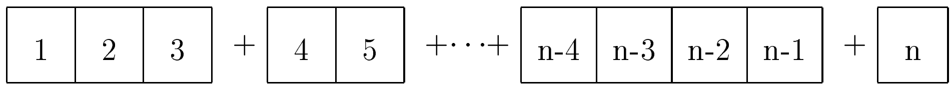


Figure 4.2: A chopped up piece of wood

Mathematically, this means there are $\binom{n-1}{m-1}$ ordered partitions of n into m integers.

$\square$

This result will be very useful in the estimates that will be developed in the rest of this chapter. The simplest place to start with is the case $\beta = 0$. Here, the pressure function is just the topological entropy. Since the Farey map has two branches, the pressure function is simply $\log 2$ when $\beta = 0$. Nevertheless, for reasons of pedagogy, it is a useful exercise to calculate the pressure function from the zeta function and the induced zeta function for $\beta = 0$ as similar methods will be employed in obtaining other bounds later on.

4.2.1 The pressure function at $\beta = 0$

Proposition 4.5 *The pressure function for the Farey map evaluated at* $\beta = 0$ *is equal to* $\log 2$. *I.e.,*

$$-\beta F(\beta)|_{\beta=0} \equiv \log 2$$

Proof. Consider the zeta function for the Farey map for $\beta = 0$, (refer to definition (1.2)). Recalling that $\phi(x) = -\beta \log |f'(x)|$ is the definition of the interaction, the zeta function reduces as follows:

$$\begin{aligned}\zeta(z,0) &= \exp\sum_{n=1}^{\infty}\frac{z^n}{n}\sum_{f^n(x)=x}\exp\sum_{k=0}^{n-1}\phi(f^kx)\,|_{\beta=0}\\ &= \exp\sum_{n=1}^{\infty}\frac{z^n}{n}\sum_{f^n(x)=x}\exp\sum_{k=0}^{n-1}-\beta\log|f'(f^kx)|\,|_{\beta=0}\\ &= \exp\sum_{n=1}^{\infty}\frac{z^n}{n}\sum_{f^n(x)=x}1 \qquad (4.35)\end{aligned}$$

The sum on the right is equal to the number of fixed points of f^n. Since the Farey map has 2 branches, f^2 will have 4 branches, f^3 will have 8 and so on. Thus, the number of fixed points is 2^n. So, continuing on from the end of equation (4.35):

$$\zeta(z,0)=\exp\sum_{n=1}^{\infty}\frac{z^n}{n}2^n=\exp\log\frac{1}{1-2z}=\frac{1}{1-2z} \qquad (4.36)$$

where the standard Taylor series expansion, $-\log(1-x)=\sum_{n=1}^{\infty}\frac{x^n}{n}$, has been used in obtaining the last line. It is clear that the ζ-function has only a simple pole at $z=\frac{1}{2}$. Therefore the radius of convergence is $\frac{1}{2}$. This corresponds to $\exp\beta F(\beta)$ and so the pressure function at $\beta=0$ takes on the value $-\beta F(\beta)=\log 2$. Note that the radius of convergence of a power series $\sum_{n=0}^{\infty}a_nz^n$, is given directly by the calculation $\frac{1}{\rho_c}=\lim_{n\to\infty}\sup|a_n|^{\frac{1}{n}}$. This method of calculation will be most useful when an explicit expression for the ζ-function cannot be found (i.e. when $\beta\neq 0$!).

The other way mentioned of finding the pressure function for $\beta=0$ was to tackle the induced zeta function. Using $\beta=0$ in equation (4.34) the induced zeta function becomes

$$\zeta_{\text{ind}}(z)=\exp\sum_{n=1}^{\infty}z^n\sum_{m=1}^{n}\frac{1}{m}\sum_{\{i_k\}_1^m;\sum_{k=1}^m i_k=n}1 \qquad (4.37)$$

The sum on the rightmost of equation (4.37) is equal to the number of ordered partitions of n into m parts. Thus, using lemma (4.4), the expression for the induced zeta function simplifies to

$$\zeta_{\text{ind}}(z) = \exp \sum_{n=1}^{\infty} z^n \sum_{m=1}^{n} \frac{1}{m} \binom{n-1}{m-1} \tag{4.38}$$

This calculation may now be seen to revolve around the problem of determining the quantity $\sum_{m=1}^{n} \frac{1}{m}\binom{n-1}{m-1}$. One way of evaluating sums like $\sum_{m=0}^{n}\binom{n}{m} m^k$, where k is an integer, is to replace m^k by $\frac{d^k}{d\lambda^k} e^{m\lambda} \mid_{\lambda=0}$ (or a multiple integral if k is negative and $m \neq 0$). Pulling the differential(s) (integral(s)) out of the sum, leaves the form of a simple binomial expansion of $(1+e^{\lambda})^n$. For example, when $k > 0$,

$$\sum_{m=0}^{n} \binom{n}{m} m^k = \frac{d^k}{d\lambda^k} \sum_{m=0}^{n} \binom{n}{m} e^{m\lambda} \bigg|_{\lambda=0} = \frac{d^k}{d\lambda^k} (1+e^{\lambda})^n \bigg|_{\lambda=0} \tag{4.39}$$

However, there is a much simpler way for the particular case $k = -1$

Lemma 4.6

$$\sum_{m=1}^{n} \frac{1}{m} \binom{n-1}{m-1} = \frac{1}{n}(2^n - 1) \tag{4.40}$$

Proof.

$$\sum_{m=1}^{n} \frac{1}{m} \binom{n-1}{m-1} = \frac{1}{n} \sum_{m=1}^{n} \binom{n}{m} = \frac{1}{n}(2^n - 1) \tag{4.41}$$

□

Using this small lemma in equation (4.38), the induced zeta function evaluated at $\beta = 0$ can be found as follows:

$$\begin{aligned}
\zeta_{\text{ind}}(z) &= \exp \sum_{n=1}^{\infty} z^n \sum_{m=1}^{n} \frac{1}{m} \binom{n-1}{m-1} \\
&= \exp \sum_{n=1}^{\infty} z^n \frac{1}{n}(2^n - 1) \\
&= \exp \left(\sum_{n=1}^{\infty} z^n \frac{2^n}{n} - \sum_{n=1}^{\infty} z^n \frac{1}{n} \right)
\end{aligned}$$

$$
\begin{aligned}
&= \exp\left(\log\frac{1}{1-2z} - \log\frac{1}{1-z}\right) \\
&= \frac{1-z}{1-2z} \qquad (4.42)
\end{aligned}
$$

The actual zeta function is the product of the induced zeta function and the complementary zeta function, ζ_{Y^c}. Recall that ζ_{Y^c} for the Farey map was found to be $(1-z)^{-1}$, see equation (2.5). Thus

$$
\zeta(z,0) = \zeta_{Y^c}(z,0)\cdot\zeta_{\text{ind}}(z,0) = \frac{1}{1-z}\frac{1-z}{1-2z} = \frac{1}{1-2z} \qquad (4.43)
$$

and therefore this second method agrees with the first.

□

4.2.2 The pressure function for $\beta \geq 1$

Proposition 4.7 *The pressure function for the Farey map is monotonically decreasing for all β and is identically 0 for all $\beta \geq 1$.*

Proof. The first step will be to show that the pressure function is always greater than or equal to zero. It will then be shown to be monotonically decreasing with β and actually equal to zero for $\beta = 1$. It is then immediately inferred that -$\beta F(\beta)$ must be equal to 0 for all $\beta \geq 1$.

In what follows, bounds on the pressure function will naturally come from bounds on the coefficients of z in the induced zeta function. These coefficients will be denoted by a_n; i.e. $\log\zeta_{\text{ind}}(z,\beta) = \sum_{n=1}^{\infty} a_n z^n$. Explicitly, the a_n are given by equation (4.34):

$$
a_n = \sum_{m=1}^{n} b_{(n,m)} = \sum_{m=1}^{n}\frac{1}{m}\sum_{\{i_k\}_1^m;\sum_{k=1}^m i_k=n}\prod_{l=1}^{m}\left([\overline{i_l, i_{l+1},\ldots,i_m,i_1,\ldots,i_{l-1}}]\right)^{2\beta} \qquad (4.44)
$$

where the $b_{(n,m)}$ have been introduced to represent the terms of the sum over m. Note that each a_n is a sum of positive terms. Two of the $b_{(n,m)}$ are immediately calculable for all n: these are the terms for $m = 1$ and $m = n$.

First, consider the term $b_{(n,1)}$. Trivially, there is only one ordered partition of an integer n into 1 part. The part must, of course, be itself n. Therefore, the period one continued fraction produced by this sequence is $[\overline{n}]$ and the expression for $b_{(n,1)}$ becomes[2]

$$b_{(n,1)} = ([\overline{n}])^{2\beta} \tag{4.45}$$

The behaviour of $b_{(n,1)}$ as n approaches infinity is also clear.

$$b_{(n,1)} \sim \left(\frac{1}{n}\right)^{2\beta} \quad \text{as} \quad n \to \infty \tag{4.46}$$

In order to calculate the quantity $b_{(n,n)}$, note that there is only one ordered partition of n into n parts. So, as for the previous partition, only one sequence is summed over, this time being $\{\overbrace{1, 1, \ldots, 1}^{n\ 1\text{'s}}\}$. The corresponding continued fraction is the reciprocal of the golden ratio[3], $\rho_g = \frac{1+\sqrt{5}}{2}$. This sequence of ones is unchanged by cyclic permutation, so the product over $l = 1$ to n produces the number $(\rho_g - 1)^n = \rho^{-n}$. All this implies

$$b_{(n,n)} = \frac{1}{n}\rho_g^{-2n\beta} \tag{4.47}$$

[2] $[\overline{n}]$ is the positive solution to the quadratic equation $x^2 + nx - 1 = 0$ and it represents the quadratic surd $\frac{1}{2}(-n + \sqrt{n^2+4})$.

[3] The golden ratio is, in some sense, the most essential periodic continued fraction. It satisfies the quadratic relation, $x^2 - x - 1 = 0$ and may be expressed in quadratic surd form as $\frac{1+\sqrt{5}}{2}$. From the quadratic equation, it follows that ρ_g satisfies several other nice relationships such as $\rho_g^2 = \rho_g + 1$ and $\rho_g = 1 + \frac{1}{\rho_g}$. There is also a fundamental connection to the Fibonacci series which will be utilised later on.

As a lower bound for a_n, consider the term $b_{(n,1)}$ and the following:

$$a_n = \sum_{m=1}^{n} b_{n,m} > b_{n,1} = ([\overline{n}])^{2\beta} \tag{4.48}$$

Note that $\frac{1}{n} > [\overline{n}] = \frac{1}{n+[\overline{n}]} > \frac{1}{2n}$ and therefore for $\beta > 0$

$$\lim_{n\to\infty} |a_n|^{\frac{1}{n}} \geq \lim_{n\to\infty} (2n)^{\frac{-2\beta}{n}} = 1 \tag{4.49}$$

and similarly for $\beta < 0$,

$$\lim_{n\to\infty} |a_n|^{\frac{1}{n}} \geq \lim_{n\to\infty} (n)^{\frac{-2\beta}{n}} = 1 \tag{4.50}$$

Therefore a lower bound on $\frac{1}{\rho_c}$ is 1. The first part of this section is then complete as this implies for the pressure function that $-\beta F(\beta) \geq \log 1 = 0$ for all β.

Consider the form of the continued fractions in the expression for the induced zeta function (equation (4.34)). They are all strictly less than one and raised to a power 2β. Thus, as β increases, each of these terms decrease. Hence, the $b_{(n,m)}$ and finally the coefficients a_n must also decrease with increasing β. This means that the radius of convergence of the induced zeta function may not decrease and conversely the pressure function may not increase with β.

It is clear then that the pressure function is a monotonically decreasing function of β. Since the pressure function is always greater than or equal to 0, if it can be shown to be 0 for any finite β then it must be 0 for all β to the right of this point as well. Consider then the case for $\beta = 1$. Here the transfer operator reduces to the Perron-Frobenius operator which was mentioned in the introduction. The invariant density ψ_I of a mapping f is known to be a solution to the Perron Frobenius equation which is defined as

$$\psi_I(x) = \sum_{f^n y = x} \frac{\psi_I(y)}{|f'(y)|} \tag{4.51}$$

The existence of ψ_I would show that 1 is an eigenvalue of the transfer operator when $\beta = 1$. However, for the Farey map, $\psi_I(x) = 1/x$ which cannot be normalised and is therefore not an eigenfunction of $\mathcal{L}^{(\beta)}$. However, it is still observed that $r(\mathcal{L}^{(1)}) = 1$; see, for example, [8] and [22]. Since the spectral radius of the transfer operator corresponds to $\exp -\beta F(\beta)$, [28], it can then be inferred that $-\beta F(\beta)|_{\beta=0} = 0$. Therefore, using the reasoning above, the pressure function is equal to 0 for all $\beta \geq 1$. This does not preclude that values of β to the left of 1 might also be 0 and this will be discussed later on.

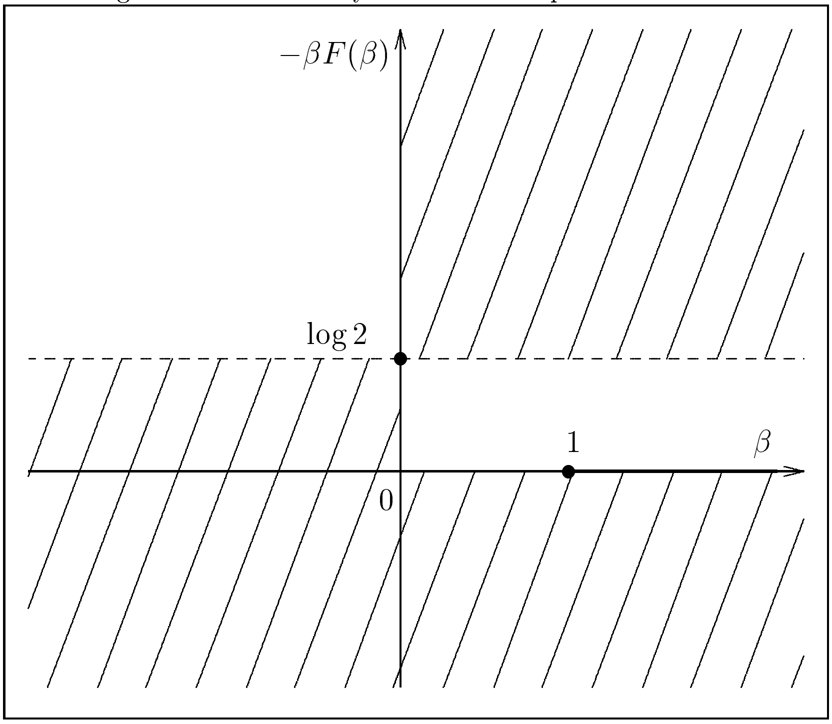
$\square$

Graphically, the constraints found so far are represented in figure (4.3). The clear regions represent the possible region where the pressure function may lie. The thick line and the two dots are the actual values of -$\beta F(\beta)$. Since the pressure function is decreasing, it cannot be smaller than $\log 2$ for $\beta < 0$ and conversely, it cannot be larger that $\log 2$ for $\beta > 0$. (Note that bounds found later on will be displayed separately and then all together in a final figure.)

Corollary 4.8 *The pressure function for the Farey map exhibits a phase transition for β somewhere in* $(0, 1]$.

Proof. The analytic continuation of a constant function is itself the same constant function. Since the pressure function is not 0 at least when $\beta = 0$ by proposition (4.5), $\beta F(\beta)$ cannot be analytic everywhere on the real line and, in particular, must express

Figure 4.3: Preliminary bounds on the pressure function



non-analytic behaviour between $\beta = 0$ and $\beta = 1$. The preceding work is therefore enough to provide conclusive evidence for the existence of a phase transition in the pressure function.[4] □

4.2.3 Some general bounds on the pressure function

Some more precise bounds are developed in this section, further constraining the shape of the pressure function. Recall that $b_{(n,n)} = \frac{1}{n}\rho_g^{-2n\beta}$ was the final term in the sum for the a_n. In fact, ρ_g^n is a bound on all the other possible periodic continued fractions for a given n, i.e. all of those periodic continued fractions whose elements in their repeating sequence of entries sum to n. This is proven in the following lemma:

[4]The appearance of non-analytic behaviour in some quantity with respect to a given parameter is equivalent to the physical notion of the existence of a phase transition. For a formal definition of a phase transition, see Huang [13].

Lemma 4.9

$$\frac{1}{\rho_g^n} < \prod_{k=1}^{m} [\overline{i_k, \ldots, i_m, i_1, \ldots, i_{k+1}}] \;\; \forall \;\; \textit{sequences} \;\; \{i_k\}_1^m \;\; \textit{such that} \;\; \sum_{k=1}^{m} i_k = n \tag{4.52}$$

for all $m = 1, 2, \ldots, n-1$*. Trivially, for* $m = n$*, the inequality becomes an equality.*

Proof. Consider a periodic continued fraction whose entries of the generating sequence sum to n written as $\mu = \mu_0 = [\overline{i_1, \ldots, i_m}]$. The overall idea of the proof is to show something along the lines of the statement:

$$\frac{1}{\rho_g^{i_1}} < [\overline{i_1, \ldots, i_m}] \tag{4.53}$$

To do this, it is helpful to obtain an estimate of the integer part of ρ_g^k. An examination of the first few k with the help of the relation $\rho_g = 1 + \frac{1}{\rho_g}$ shows that

$$\begin{aligned}
\rho_g &= 1 + \frac{1}{\rho_g} \\
\rho_g^2 &= \rho_g \left(1 + \frac{1}{\rho_g}\right) = \rho_g + 1 = 1 + \frac{1}{\rho_g} + 1 = 2 + \frac{1}{\rho_g} \\
\rho_g^3 &= \rho_g \left(2 + \frac{1}{\rho_g}\right) = 2\rho_g + 1 = 2\left(1 + \frac{1}{\rho_g}\right) + 1 = 3 + \frac{2}{\rho_g}
\end{aligned} \tag{4.54}$$

It is clear from the above that ρ_g^k may be expressed in the form $a_k + \frac{b_k}{\rho_g}$. Therefore:

$$\begin{aligned}
\rho_g^{k+1} &= \rho_g \cdot \rho_g^k = \rho_g \left(a_k + \frac{b_k}{\rho_g}\right) = a_k \rho_g + b_k = a_k \left(1 + \frac{1}{\rho_g}\right) + b_k \\
&= a_k + b_k + \frac{a_k}{\rho_g} = a_{k+1} + \frac{b_{k+1}}{\rho_g}
\end{aligned} \tag{4.55}$$

The recursion relations are then $a_{k+1} = a_k + b_k$ and $b_{k+1} = a_k$. The combining of these two expressions leads to the familiar Fibonacci formula: $a_{k+1} = a_k + a_{k-1}$. Noting also that $a_0 = a_1 = 1$, it is clear that the a_k sequence is indeed the Fibonacci series $1, 1, 2, 3, 5, 8, 13, \ldots$, the elements of which will be denoted by f_k, $k = 0, 1, \ldots$.

Therefore, $\rho_g^{i_1}$ may be expressed in the following form:

$$\rho_g^{i_1} = f_{i_1} + \frac{f_{(i_1-1)}}{\rho_g} \tag{4.56}$$

Now, $f_n > n$ for all $n > 3$ while $f_n = n$ for $n = 1, 2$ and 3. Note that because $\rho_g = \frac{1+\sqrt{5}}{2} \approx 1.618$, the second term in relation (4.56) is strictly greater than one for all $i_1 \geq 3$. Returning then to the conjecture of equation (4.53), it is seen to be true for $i_1 \geq 3$ since, using relation (4.56) and the above inequalities:

$$\begin{aligned} \rho_g^{i_1} &= f_{i_1} + \frac{f_{(i_1-1)}}{\rho_g} \geq i_1 + \frac{f_{(i_1-1)}}{\rho_g} > i_1 + 1 \\ &> i_1 + [\overline{i_2, \ldots, i_m, i_1}] = [\overline{i_1, \ldots, i_m}]^{-1} \end{aligned} \tag{4.57}$$

where the fact has been used that the periodic continued fractions being considered are all strictly less than 1. Equation (4.53) is therefore true for $i_k \geq 3$. The two cases left will be treated separately.

$\boxed{i_k = 1}$: assume that $\frac{1}{\rho_g} < [\overline{1, i_2, \ldots, i_m}]$. Writing the continued fraction in the form

$$[\overline{1, i_2, \ldots, i_m}] = \cfrac{1}{1 + \cfrac{1}{\mu_2}} \tag{4.58}$$

where μ_k represents the k^{th}-complete quotient and $\mu_2 = [\overline{i_2; i_3, \ldots, i_m, 1}]$. It follows that

$$\frac{1}{\rho_g} < \cfrac{1}{1 + \cfrac{1}{i_2 + \mu_3^{-1}}} \Rightarrow \rho_g < \mu_2 = i_2 + \frac{1}{\mu_3} \tag{4.59}$$

This is always true if $i_2 \geq 2$ since $\rho_g < 2$ but it may break down if $i_2 = 1$. However, all is not lost. Assume that indeed

$$\frac{1}{\rho_g} > [\overline{1, 1, i_3, \ldots, i_m}] \tag{4.60}$$

It is simple to show that this implies

$$\rho_g < [\overline{i_3; \ldots, i_m, 1, 1}] = \mu_3 \tag{4.61}$$

Then the following inequality still holds:

$$\frac{1}{\rho_g^2} < [\overline{1, 1, i_3, \ldots, i_m}] \cdot [\overline{1, i_3, \ldots, i_m, 1}] \tag{4.62}$$

This is proven as follows:

$$\begin{aligned} [\overline{1, 1, i_3, \ldots, i_m}] \cdot [\overline{1, i_3, \ldots, i_m, 1}] &= \frac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\mu_3}}} \cdot \frac{1}{1 + \cfrac{1}{\mu_3}} = \frac{\mu_3}{1 + 2\mu_3} \\ &= \frac{1}{\cfrac{1}{\mu_3} + 2} > \frac{1}{\cfrac{1}{\rho_g} + 2} = \frac{1}{\rho_g^2} \end{aligned} \tag{4.63}$$

where equation (4.61) has been used to bring about the inequality. So while the idea of equation (4.53) is not exactly true for all i_1, an inequality still exists if more terms of the product are involved. The important fact is that the power of the inverse of the golden mean (i.e., 2) is the sum of the leading entries in the two continued fractions (both 1).

$\boxed{i_k = 2}$

A similar approach to the above is employed in the case $i_1 = 2$. Assume that $\frac{1}{\rho_g^2} < [\overline{2, i_2, \ldots, i_m}]$. The implication of this is given as follows:

$$\frac{1}{\rho_g^2} < \frac{1}{2 + \frac{1}{\mu_2}} \Rightarrow \rho_g < \mu_2 = i_2 + \frac{1}{\mu_3} \tag{4.64}$$

which is always true if $i_2 \geq 2$ since $\rho_g < 2$. Once again, the case $i_2 = 1$ must be considered further. Assume that the desired result is not true, i.e.

$$\frac{1}{\rho_g^2} > [\overline{2, 1, i_3 \ldots, i_m}] = \frac{1}{2 + \frac{1}{1 + \frac{1}{\mu_3}}} \tag{4.65}$$

It easily follows that

$$\rho_g < \mu_3 = i_3 + \mu_4^{-1} \tag{4.66}$$

Then the following inequality still holds:

$$\frac{1}{\rho_g^3} < [\overline{2, 1, i_3, \ldots, i_m}] \cdot [\overline{1, i_3, \ldots, i_m, 2}] \tag{4.67}$$

since

$$\begin{aligned} [\overline{2, 1, i_3, \ldots, i_m}] \cdot [\overline{1, i_3, \ldots, i_m, 2}] &= \frac{1}{2 + \frac{1}{1 + \frac{1}{\mu_3}}} \cdot \frac{1}{1 + \frac{1}{\mu_3}} \\ &= \frac{1}{\frac{2}{\mu_3} + 3} > \frac{1}{\frac{2}{\rho_g} + 3} = \cdots = \frac{1}{\rho_g^3} \end{aligned} \tag{4.68}$$

Once again, an inequality is found when the power of the inverse of the golden ratio (which is 3) is equal to the sum of the leading entries in the two continued fractions (2 and 1 respectively). Finally, the question may be asked as to what power N of ρ_g^{-1} is required for the following inequality to be true:

$$\frac{1}{\rho_g^N} < \prod_{k=1}^{m} [\overline{i_k, \ldots, i_m, i_1, \ldots, i_{k+1}}] \tag{4.69}$$

Since for all continued fractions with a leading entry of $i_1 \geq 3$ the least power ρ_g^{-1} required for the inequality to hold is precisely i_1, and the special case inequalities are satisfied when the sum of the leading entries equals the power of ρ_g^{-1}, the inequality for the entire product holds if ρ_g^{-1} is raised to the sum of all the leading entries which, of course, is n. This rather long sentence completes the proof.

□

The result of this lemma gives a nice bound on the pressure function. Firstly, note the following remark:

Remark 4.10 *Lemma (4.9) immediately implies that*

$$\left(\frac{1}{\rho_g^n}\right)^{2\beta} < \prod_{k=1}^{m} [\overline{i_k, \ldots, i_m, i_1, \ldots, i_{k+1}}]^{2\beta} \quad \textit{if} \quad \beta > 0 \tag{4.70}$$

and

$$\left(\frac{1}{\rho_g^n}\right)^{2\beta} > \prod_{k=1}^{m} [\overline{i_k, \ldots, i_m, i_1, \ldots, i_{k+1}}]^{2\beta} \quad \textit{if} \quad \beta < 0 \tag{4.71}$$

Proposition 4.11 *The pressure function for the Farey map is bounded in the following way:*

$$-\beta F(\beta) \geq \log 2 - 2\beta \log \rho_g \ \textit{ when } \beta > 0$$

and

$$-\beta F(\beta) \leq \log 2 - 2\beta \log \rho_g \textit{ when } \beta < 0$$

Proof. It will be useful to invoke the following theorem due to Alfred Pringsheim [12].

Theorem 4.12 *If the coefficients of a power series* $\sum_{n=0}^{\infty} a_n z^n$ *are all positive then the power series has a singularity on the positive real axis lying on its circle of convergence. I.e., the power series has a singularity at* $z = \rho_c$.

It can thus be assumed that z is positive and real in the search for the radius of convergence of the induced zeta function since finding bounds on where its first singularity lies on the positive z-axis is equivalent to finding bounds on the radius of convergence. Therefore, for $\beta > 0$, the induced zeta function, equation (4.34) can be estimated as follows using remark (4.10):

$$\begin{aligned}
\zeta_{\text{ind}}(z, \beta) &= \exp \sum_{n=1}^{\infty} z^n \sum_{m=1}^{n} \frac{1}{m} \sum_{\{i_k\}_1^m; \sum_{k=1}^m i_k = n} \prod_{l=1}^{m} \left| \overline{[i_l, i_{l+1}, \ldots, i_m, i_1, \ldots, i_{l-1}]} \right|^{2\beta} \\
&> \exp \sum_{n=1}^{\infty} z^n \sum_{m=1}^{n} \frac{1}{m} \binom{n-1}{m-1} \rho_g^{2n\beta} \\
&= \exp \sum_{n=1}^{\infty} z^n \frac{1}{n} (2^n - 1) \rho_g^{-2n\beta} \\
&= \exp \sum_{n=1}^{\infty} z^n \frac{(2\rho_g^{-2\beta})^n}{n} - \sum_{n=1}^{\infty} z^n \frac{\rho_g^{-2n\beta}}{n} \\
&= \frac{1 - z\rho_g^{-2\beta}}{1 - 2z\rho^{-2\beta}} \qquad (4.72)
\end{aligned}$$

Clearly, the radius of convergence for this function is $\rho_c = \frac{1}{2}\rho^{2\beta}$. This implies that the radius of convergence of the induced zeta function must be less than or equal to $\frac{1}{2}\rho^{2\beta}$. Therefore, the above demonstrates that $-\beta F(\beta) \geq \log 2 - 2\beta \log \rho_g$ for $\beta > 0$.

A similar argument shows also that $-\beta F(\beta) \leq \log 2 - 2\beta \log \rho_g$ for $\beta < 0$. Note that this line of constraint passes through $\log 2$ when $\beta = 0$ as would be expected. □

A second general bound on the pressure function is presented in the following proposition:

Proposition 4.13

$$-\beta F(\beta) \geq -2\beta \log \rho_g \quad \forall \quad \beta$$

Proof. Recall that the coefficients of the power series for $\log \zeta_{\text{ind}}(z,\beta)$ were given by $a_n = \sum_{m=1}^{n} b_{(n,m)}$. Clearly then

$$a_n > b_{(n,n)} = \frac{1}{n} \rho_g^{-2n\beta} \tag{4.73}$$

This implies for real and positive z that

$$\zeta_{\text{ind}}(z,\beta) = \exp \sum_{n=1}^{\infty} a_n z^n > \exp \sum_{n=1}^{\infty} \frac{1}{n} \rho_g^{-2n\beta} z^n = \frac{1}{1 - z\rho_g^{-2\beta}} \tag{4.74}$$

The function on the final line of the above has a simple pole at $z = \rho_g^{2\beta}$ which is therefore its radius of convergence. Since the induced zeta function is greater than this function it must have a radius of convergence less than or equal to $\rho_c = \rho_g^{2\beta}$. □

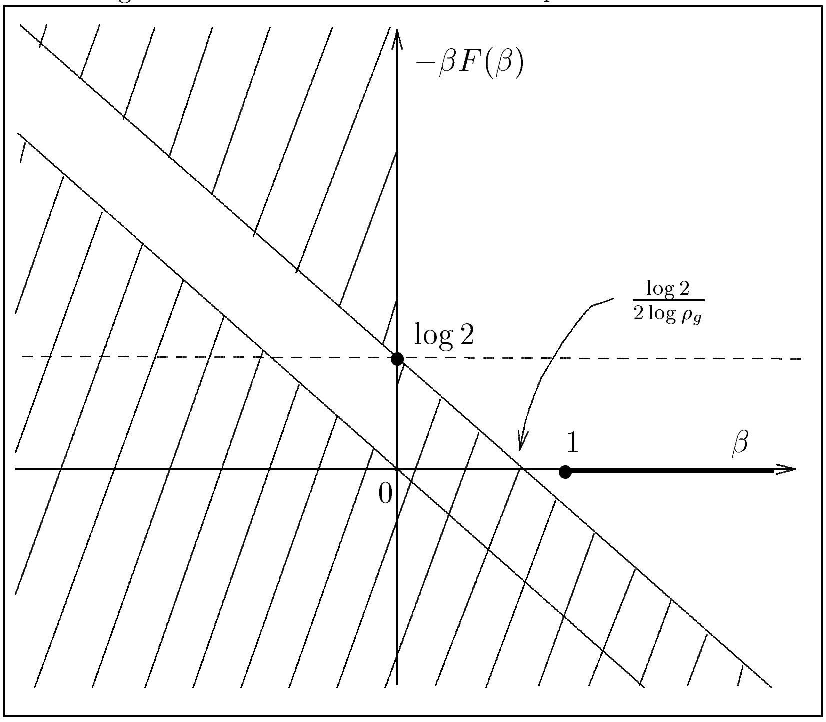
The information provided by both of these bounds is displayed in figure (4.4).

One more bound on the pressure function is presented in this section.

Proposition 4.14 *For $\beta > 0$, a upper bound on the pressure function for the Farey map is given by*

$$-\beta F(\beta) \leq \log \left(1 + \left(\frac{2}{3}\right)^{2\beta}\right)$$

Figure 4.4: Some more bounds on the pressure function



and for $\beta < 0$, the corresponding lower bound is

$$-\beta F(\beta) \geq \log\left(1+\left(\frac{2}{3}\right)^{2\beta}\right)$$

Proof. The proof requires an examination of the following representation of the induced zeta function obtained from examination of the representations given in equations (4.22), (4.33) and (4.34):

$$\log\zeta_{\text{ind}}(z,\beta) = \sum_{n=1}^{\infty} z^n \sum_{m=1}^{n} \frac{1}{m} \sum_{\{i_k\}_1^m;\sum_{k=1}^m i_k=n} \prod_{l=1}^{m} |G_l'([1,\overline{i_{l+1},i_{l+2},\ldots,i_m,i_1,\ldots,i_l}])|^{\beta} \tag{4.75}$$

Now, from equation (4.13), $G_n'(x) = \frac{-1}{(1+nx)^2}$. Since $x \in [\frac{1}{2},1]$ it follows that a bound on the maximum possible value of $|G_n'(x)|$ would be its value for $n=1$ and $x=\frac{1}{2}$. Therefore, $|G_n'(x)| < \frac{1}{(1+1.\frac{1}{2})^2} = \left(\frac{2}{3}\right)^2$. So, for real and positive z and for $\beta > 0$, it follows that

$$
\begin{aligned}
\zeta_{\text{ind}}(z,\beta) &< \exp\sum_{n=1}^{\infty} z^n \sum_{m=1}^{n} \frac{1}{m} \sum_{\{i_k\}_1^m;\sum_{k=1}^m i_k=n} \prod_{l=1}^{m} \left(\frac{2}{3}\right)^{2m\beta} \\
&= \exp\sum_{n=1}^{\infty} z^n \sum_{m=1}^{n} \frac{1}{m} \left(\frac{2}{3}\right)^{2m\beta} \sum_{\{i_k\}_1^m;\sum_{k=1}^m i_k=n} 1 \\
&= \exp\sum_{n=1}^{\infty} z^n \sum_{m=1}^{n} \frac{1}{m} \left(\frac{2}{3}\right)^{2m\beta} \binom{n-1}{m-1} \\
&= \exp\sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{m=1}^{n} \binom{n}{m} \left(\left(\frac{2}{3}\right)^{2\beta}\right)^m \\
&= \exp\sum_{n=1}^{\infty} \frac{z^n}{n} \left(\left(1+\left(\frac{2}{3}\right)^{2\beta}\right)^n - 1\right) \\
&= \frac{1-z}{1-z\left(1+\left(\frac{2}{3}\right)^{2\beta}\right)}
\end{aligned} \tag{4.76}
$$

The final expression shows a function with a radius of convergence $\rho_c = \left(1+\left(\frac{2}{3}\right)^{2\beta}\right)^{-1}$. Since the induced zeta function is smaller than this function its radius of convergence must be at least ρ_c. As the pressure function is the logarithm of the inverse of the radius of convergence of the zeta function, it is bounded in the following way for $\beta > 0$:

$$
-\beta F(\beta) \leq \log\left(1+\left(\frac{2}{3}\right)^{2\beta}\right) \tag{4.77}
$$

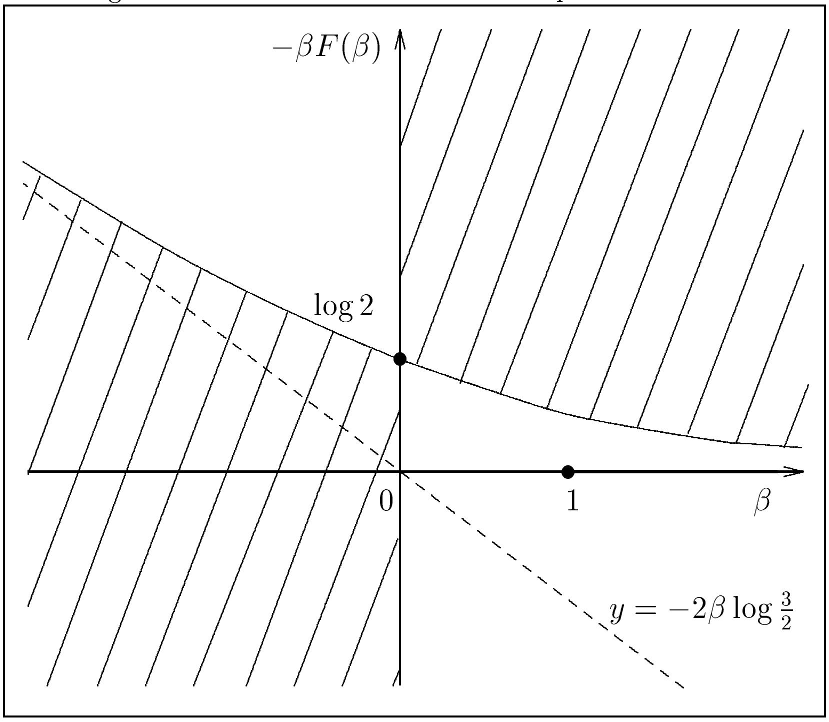
A similar argument for $\beta < 0$ shows that there

$$
-\beta F(\beta) \geq \log\left(1+\left(\frac{2}{3}\right)^{2\beta}\right) \tag{4.78}
$$

$\square$

Note that, as expected, $\log\left(1+\left(\frac{2}{3}\right)^{2\beta}\right)\Big|_{\beta=0} = \log 2$. Also, this function asymptotes to the the β axis as $\beta \to \infty$ and, on the other hand, it asymptotes to $-2\beta\log\frac{3}{2}$ as $\beta \to -\infty$. A plot of this bound is given separately in figure 4.5.

Figure 4.5: ...and another bound on the pressure function



4.2.4 The pressure function for $\beta \to -\infty$

The previous section has provided some strong constraints on the behaviour of the pressure function for negative β. In fact, it has been shown that $-\beta F(\beta) = -2\beta \log \rho_g + O(1)$. The work here will improve on these bounds with the main result being the following theorem regarding the asymptotic behaviour of the pressure function for large negative β:

Theorem 4.15 *For every $\delta > 0$, there exists a $\beta(\delta)$ such that for all $\beta < \beta(\delta)$*

$$-\beta F(\beta) - -2\beta \log \rho_g < \delta$$

Proof.

Motivation

Recall that the coefficients of the induced zeta function, a_n, are ultimately a sum over the continued fraction terms $\frac{1}{m}\prod_{l=1}^{m}([\overline{i_l, i_{l+1}, \ldots, i_m, i_1, \ldots, i_{l-1}}])^{2\beta}$, see equation (4.44). In the limit β approaches negative infinity, the second part of remark (4.10) appears to indicate that $\frac{1}{n}{\rho_g}^{-2n\beta} = b_{(n,n)}$ will be the dominant term for any given n. However, this is not necessarily true since the factor $\frac{1}{m}$ has not been considered. Also, and much more importantly, in the limit $n \to \infty$, terms containing continued fractions which approximate the golden mean may be of the order of $b_{(n,n)}$.

In reference to this last observation, consider $b_{(n,n-1)}$ for $n > 2$. This term will be evaluated explicitly as a motivation for the final proof of the main theorem.

Proposition 4.16

$$b_{(n,n-1)} = (f_{n-1} + \sqrt{f_n f_{n-2}})^{-2\beta} \tag{4.79}$$

where f_n is the n^{th} Fibonacci number.

Proof. From lemma (4.4), it is known that there are $\binom{n-1}{n-2} = n-1$ ordered partitions of n into $n-1$ parts. Clearly, these partitions are those that consist of 1 'two' and $n-2$ 'ones' and may be represented by the $n-1$ sequences:

$$\{\overbrace{1, 1, \ldots, 1}^{k \text{ ones}}, 2, \overbrace{1, \ldots, 1, 1}^{n-k-2 \text{ ones}}\} \tag{4.80}$$

where $0 \le k \le n-2$. Note that these sequences have no periodic structures of length less than $n-1$ and are merely cyclic rotations of each other. Therefore, they each generate the same product of continued fractions and the factor $\frac{1}{m}$ is cancelled. That is,

$$b_{(n,n-1)} = \prod_{k=0}^{n-2} ([\underbrace{1,1,\ldots,1}_{k \text{ ones}},2,\underbrace{1,\ldots,1,1}_{n-k-2 \text{ ones}}])^{2\beta} \tag{4.81}$$

More generally, if a sequence with m elements contains no smaller periodic blocks than itself then the $m-1$ cyclic rotations of it will also be summed over in the expression for the induced zeta function (4.34) thereby cancelling the factor of $\frac{1}{m}$.

Lemma 4.17 *Any continued fraction $C(n,x)$ of the form*

$$[\underbrace{1,1,\ldots,1}_{n \text{ ones}},x] \tag{4.82}$$

where x is any real number and $n = 0,1,2,\ldots$, is equal to the fraction

$$\frac{f_{n-1}x + f_{n-2}}{f_n x + f_{n-1}} \tag{4.83}$$

where $f_0, f_1, f_2, f_3, f_4 \ldots = 1,1,2,3,5,\ldots$ is the Fibonacci series which satisfies the recursion relation $f_{n+1} = f_n + f_{n-1}$.

Proof. Assume the assertion of the lemma is true for k some positive integer. That is,

$$C(k,x) = [\overbrace{1,1,\ldots,1}^{}_{k \text{ ones}},x] = \frac{f_{k-1}x + f_{k-2}}{f_k x + f_{k-1}} \tag{4.84}$$

Then,

$$\begin{aligned} C(k+1,x) &= [\overbrace{1,1,\ldots,1}^{}_{k+1 \text{ ones}},x] = \frac{1}{1+C(k,x)} \\ &= \frac{1}{1+\dfrac{f_{k-1}x+f_{k-2}}{f_k x + f_{k-1}}} = \frac{f_k x + f_{k-1}}{f_k x + f_{k-1} + f_{k-1}x + f_{k-2}} = \frac{f_k x + f_{k-1}}{f_{k+1}x + f_k} \end{aligned} \tag{4.85}$$

Therefore, if the assertion is true for $n = k$ then it is true for $n = k + 1$. Now consider the case $n = 0$. The Fibonacci series may be extended to negative values of n by application of the recursion relation rewritten as $f_n = f_{n+2} - f_{n+1}$. This gives $f_{-1} = 0, f_{-2} = 1, f_{-3} = -1$ and so on. Using this when there are no 1's preceding x in the continued fraction:

$$\frac{1}{x} = \frac{0x+1}{x+0} = \frac{f_{-1}x + f_{-2}}{f_0 x + f_{-1}} \tag{4.86}$$

Thus the statement is true for the case $n = 0$ and by induction must be therefore be true for all $n = 0, 1, 2, \ldots$. □

Two results needed to calculate $b_{(n,n-1)}$ now follow from this lemma.

Corollary 4.18 *Let* $t(n) = [\overline{2, \underbrace{1, 1, \ldots, 1}_{n-2 \text{ ones}}}]$. *Then* $b_{(n,n-1)}$ *simplifies to the following:*

$$b_{n,n-1} = \frac{t(n)}{f_{n-2} + f_{n-3}t(n)} \tag{4.87}$$

Proof. Returning to the definition of $b_{(n,n-1)}$ in equation (4.81), it can be rewritten using $t(n)$ as follows

$$b_{(n,n-1)} = \prod_{k=0}^{n-2} ([\underbrace{1, 1, \ldots, 1}_{k \text{ ones}}, t(n)^{-1}])^{2\beta} \tag{4.88}$$

Using lemma (4.17) this simplifies as follows:

$$b_{(n,n-1)} = \prod_{k=0}^{n-2} \left(\frac{f_{k-1}t(n)^{-1} + f_{k-2}}{f_k t(n)^{-1} + f_{k-1}} \right)^{2\beta}$$

$$= \left(\frac{f_{-1}t(n)^{-1} + f_{-2}}{f_0 t(n)^{-1} + f_{-1}} \cdot \frac{f_0 t(n)^{-1} + f_{-1}}{f_1 t(n)^{-1} + f_0} \cdots \frac{f_{n-4}t(n)^{-1} + f_{n-3}}{f_{n-3}t(n)^{-1} + f_{n-4}} \cdot \frac{f_{n-3}t(n)^{-1} + f_{n-4}}{f_{n-2}t(n)^{-1} + f_{n-3}} \right)^{2\beta}$$

$$= \left(\frac{f_{-1} t(n)^{-1} + f_{-2}}{f_{n-2} t(n)^{-1} + f_{n-3}} \right)^{2\beta} = \left(\frac{1}{f_{n-2} t(n)^{-1} + f_{n-3}} \right)^{2\beta} = \left(\frac{t(n)}{f_{n-2} + f_{n-3} t(n)} \right)^{2\beta} \quad (4.89)$$

□

Corollary 4.19 *The periodic continued fraction $t(n)$ defined in corollary (4.18) is given in quadratic surd form by*

$$t(n) = \frac{-f_{n-2} + \sqrt{f_n f_{n-2}}}{f_{n-1}} \quad (4.90)$$

Proof. Since $t(n)$ is a periodic continued fraction it can be written in terms of itself as follows:

$$t(n) = [2, \underbrace{1, 1, \ldots, 1}_{n-2 \text{ ones}}, t(n)^{-1}] \quad (4.91)$$

Using lemma (4.17) to simplify this:

$$\begin{aligned} t(n) &= \frac{1}{2 + \dfrac{f_{n-3} t(n)^{-1} + f_{n-4}}{f_{n-2} t(n)^{-1} + f_{n-3}}} = \frac{f_{n-2} + f_{n-3} t(n)}{2(f_{n-2} + f_{n-3} t(n)) + f_{n-3} + f_{n-4} t(n)} \\ &= \frac{f_{n-2} + f_{n-3} t(n)}{2f_{n-2} + f_{n-3} + (2f_{n-3} + f_{n-4}) t(n)} = \frac{f_{n-2} + f_{n-3} t(n)}{f_n + f_{n-1} t(n)} \end{aligned} \quad (4.92)$$

where the fact that $2f_{k-1} + f_{k-2} = f_{k-1} + (f_{k-1} + f_{k-2}) = f_{k-1} + f_k = f_{k+1}$ has been used to obtain the final expression. A rearrangement of equation (4.92) yields the appropriate quadratic equation of which $t(n)$ is a solution:

$$f_{n-1} t(n)^2 + (f_n - f_{n-3}) t(n) - f_{n-2} = 0 \quad (4.93)$$

Note that $f_n - f_{n-3} = (f_{n-1} + f_{n-2}) + (f_{n-2} - f_{n-1}) = 2f_{n-2}$. Equation (4.93) has only one positive solution which must be $t(n)$:

$$t(n) = \frac{-f_{n-2} + \sqrt{f_{n-2}^2 + f_{n-1}f_{n-2}}}{f_{n-1}} = \frac{-f_{n-2} + \sqrt{f_{n-2}f_n}}{f_{n-1}} \tag{4.94}$$

□

Substituting the result of corollary (4.19) into that of corollary (4.18) and proceeding through much algebra, the result for $b_{(n,n-1)}$ is obtained as follows:

$$\begin{aligned} b_{(n,n-1)} &= \left(\frac{t(n)}{f_{n-2} + f_{n-3}t(n)}\right)^{2\beta} \\ &= \left(\frac{-f_{n-2} + \sqrt{f_{n-2}f_n}}{f_{n-2}f_{n-1} - f_{n-3}f_{n-2} + f_{n-3}\sqrt{f_{n-2}f_n}}\right)^{2\beta} \\ &= \left(\frac{-f_{n-2} + \sqrt{f_{n-2}f_n}}{f_{n-2}f_{n-1} - f_{n-3}f_{n-2} + f_{n-3}\sqrt{f_{n-2}f_n}} \cdot \frac{f_{n-2} + \sqrt{f_{n-2}f_n}}{f_{n-2} + \sqrt{f_{n-2}f_n}}\right)^{2\beta} \\ &= \left(\frac{f_{n-2}f_{n-1}}{f_{n-2}f_{n-1}^2 + f_{n-2}f_{n-1}\sqrt{f_{n-2}f_n}}\right)^{2\beta} \\ &= \left(\frac{f_{n-2}f_{n-1}}{f_{n-2}f_{n-1}(f_{n-1} + \sqrt{f_{n-2}f_n})}\right)^{2\beta} \\ &= \left(f_{n-1} + \sqrt{f_{n-2}f_n}\right)^{-2\beta} \end{aligned} \tag{4.95}$$

This completes the proof of proposition (4.16). □

It is now of interest to examine the behaviour of $b_{(n,n-1)}$ as n approaches infinity. This requires knowledge of how Fibonacci numbers behave for large n. The appendices contain a derivation of the following expression for f_n:

$$f_n = \frac{1}{1+\rho_g^2}\left(\rho_g^{n+2} + \frac{(-1)^n}{\rho_g^n}\right) \tag{4.96}$$

Since $\rho_g > 1$ it follows that

$$f_n \sim \frac{\rho_g^{n+2}}{1+\rho_g^2} \quad \text{as} \quad n \to \infty \tag{4.97}$$

So, using equation (4.96) and proposition (4.16), the asymptotic behaviour of $b_{(n,n-1)}$ is found as follows:

$$\begin{aligned} b_{(n,n-1)} &= \left(f_{n-1} + \sqrt{f_{n-2}f_n}\right)^{-2\beta} \\ &\sim \left(\frac{\rho_g^{n+1}}{1+\rho_g^2} + \sqrt{\frac{\rho_g^n}{1+\rho_g^2}\frac{\rho_g^{n+2}}{1+\rho_g^2}}\right)^{-2\beta} \\ &= \left(2\frac{\rho_g^{n+1}}{1+\rho_g^2}\right)^{-2\beta} \quad \text{as} \quad n \to \infty \end{aligned} \tag{4.98}$$

Recalling that $b_{(n,n)} = \frac{1}{n}\rho_g^{-2n\beta}$ it is now observed that

$$b_{(n,n-1)} \sim n\left(\frac{1+\rho_g^2}{2\rho_g}\right)^{2\beta} b_{(n,n)} \tag{4.99}$$

in the limit of large n. Thus, as was mooted earlier, $b_{(n,n)}$ is not the sole dominant term of a_n in the limit $n \to \infty$. Indeed, equation (4.99) shows that it is negligible in comparison with $b_{(n,n-1)}$. The insight gained here is that any sequence with enough 'ones' will contribute to a_n in the limit.

The Main Proof

It is useful to divide positive integer sequences whose entries sum to n into two main groups. These categories will be referred to as Type I and Type II. Type I sequences are those which have at least $[(1-\epsilon)n]$ entries being equal to 1 where $0 < \epsilon < 1$ and the expression $[i]$ denotes the integer part of i. Type II sequences will be all the rest. Note that $n \leq [(1-\epsilon)n] + [\epsilon n] + 1 \leq n+1$.

Each a_n will then be estimated by considering upper bounds on the number of each type of sequence, $N(I)$ and $N(II)$ respectively, and upper bounds on the continued fraction products corresponding to these sequences.

Firstly, consider type I sequences and assume that $\beta < 0$. Lemma (4.9) and remark (4.10) show that the simplest bound on each Type I continued fraction product is $\rho_g^{2n|\beta|}$. A sufficiently useful bound on the number of these sequences is not so trivial and will require some work. A typical Type I sequence may be represented as follows[5]

$$\{i_1, \underbrace{1, 1, \ldots, 1}_{N_1}, i_2, \underbrace{1, 1, \ldots, 1}_{N_2}, i_3, 1, \ldots, 1, i_k, \underbrace{1, 1, \ldots, 1}_{N_k}\} \tag{4.100}$$

where $\sum_{j=1}^{k} N_k = [(1-\epsilon)n]$ and $\sum_{j=1}^{k} i_k = n - [(1-\epsilon)n]$. Using lemma (4.4), there are $\binom{n-[(1-\epsilon)n]-1}{k-1}$ different ways of partitioning the $\{i_j\}$. Since the N_i represent lengths of strings of 1's, they may be equal to 0. A simple extension of lemma (4.4) shows that the number of different partitions of $[(1-\epsilon)n]$ into the N_j is $\binom{[(1-\epsilon)n]+k-1}{k-1}$. Note that overcounting has occurred as cyclically equivalent sequences have been separately counted in this process. Also, the number of entries of each sequence must be at least $[(1-\epsilon)n]+1$ and at most n when all are equal to 1, i.e. $1 \leq k \leq n - [(1-\epsilon)n]$. This leads to the following estimate:

$$N(I) \quad < \quad \sum_{k=1}^{n-[(1-\epsilon)n]} \binom{[(1-\epsilon)n]+k-1}{k-1} \binom{n-[(1-\epsilon)n]-1}{k-1}$$

[5] Note that there is no loss of generality caused by assuming that the first entry is i_1 as all sequences starting with a 1 may be cyclically permuted to obtain a first entry greater than 1. This equivalence means the two cycles give the same continued fraction product term.

$$< \sum_{k=1}^{n-[(1-\epsilon)n]} \binom{[(1-\epsilon)n]+k-1}{k-1} \cdot \sum_{k=1}^{n-[(1-\epsilon)n]} \binom{n-[(1-\epsilon)n]-1}{k-1} \quad (4.101)$$

Relabeling the index $k \to k+1$, using $n - [(1-\epsilon)n] \leq [\epsilon n] + 1$ and noting that $\binom{n+j-[(1-\epsilon)n]}{j-1} < \binom{n+k-[(1-\epsilon)n]}{k-1}$ for all $j < k$, it follows from equation (4.101) that

$$\begin{aligned} N(I) &< ([\epsilon n]+1)\binom{n-1}{n-[(1-\epsilon)n]-1} \cdot \sum_{k=0}^{n-[(1-\epsilon)n]} \binom{n-[(1-\epsilon)n]}{k} \\ &= ([\epsilon n]+1)\binom{n-1}{n-[(1-\epsilon)n]-1} 2^{n-[(1-\epsilon)n]} \end{aligned} \quad (4.102)$$

For large ϵn, $[\epsilon n]$ may be approximated by ϵn (note that ϵ is fixed). Using Stirling's approximation that $n! \sim n^n e^n$, a final estimate is obtained:

$$N(I) \lesssim \frac{\epsilon n 2^{\epsilon n}}{\epsilon^{\epsilon n}(1-\epsilon)^{(1-\epsilon)n}} \text{ for large } n. \quad (4.103)$$

The number of Type II sequences will be estimated by the total number of sequences, 2^n. The main work of this section is therefore to calculate a suitable upper bound for Type II orbit products. This will be done by showing that the orbit product of any sequences with less that $[(1-\epsilon)n]$ 'ones' is bounded by the orbit product of some sequence containing $[(1-\epsilon)n]$ 'ones' and $[\frac{\epsilon n}{2}]$ 'twos'. An appropriate bound on the orbit product of any sequence of the latter type will then be found.

Firstly, the orbit product of a purely periodic continued fraction of period m is found to have the following representation. This is developed in the appendices in section (A.2).

Lemma 4.20 *The orbit product of a purely periodic continued fraction*

$$C = [\overline{i_0; i_1, \ldots, i_{m-1}}]$$

is given by

$$\prod_{j=0}^{m-1} \mu_j = \frac{(A_{m-1}+B_{m-2}) + \sqrt{(A_{m-1}+B_{m-2}) + 4(-1)^{m-1}}}{2}$$

Proof. See lemma (A.1) and corollary (A.2) in the appendices. □

Note that by derivation the A_j and B_j are calculated from μ_0. The first step is to show a form of ordering for these orbit products:

Lemma 4.21 *Consider two sequences* $\{i_j\}_{k=0}^{m-1}$ *and* $\{i'_j\}_{k=0}^{m}$ *where* $i_j = i'_j$ *for all* $k = 0, \ldots, m-2$ *and* $i_{m-1} = i'_{m-1} + i'_m$*. Note that all entries must be greater than or equal to 1 so* i_{m-1} *must be at least 2. Then*

$$\prod_{j=0}^{m-1} [\overline{i_j; i_{j+1}, \ldots, i_{m-1}, i_0, \ldots, i_{j-1}}] < \prod_{j=0}^{m} [\overline{i'_j; i'_{j+1}, \ldots, i'_m, i'_0, \ldots, i'_{j-1}}]$$

i.e.,

$$\prod_{j=0}^{m-1} \mu_j < \prod_{j=0}^{m} \mu'_j$$

Proof. Let $i'_m = k$ and $i'_{m-1} = i_{m-1} - k$ and so $1 \leq k \leq i_{m-1} - 1$. The aim of the proof is to show that the quantity $p(k) = \prod_{j=0}^{m} \mu'_j - \prod_{k=0}^{m-1} \mu_j$ is strictly greater than 0. Since the first $m-2$ entries for the continued fractions μ_0 and μ'_0 are the same, it is clear that $B_k = B'_k$ and $A_k = A'_k$ for $0 \leq k \leq m-2$. Using the recursion relations for the A_k and B_k it is then seen that $B'_{m-1} = (i_{m-1} - k)B_{m-2} + B_{m-3}$, $A'_{m-1} = (i_{m-1} - k)A_{m-2} + A_{m-3}$ and $A'_m = (k(i_{m-1} - k) + 1)A_{m-2} + kA_{m-3}$. Therefore,

$$A'_m + B'_{m-1} \quad = \quad (a_{m-1} - k)B_{m-2} + B_{m-3} + (k(i_{m-1} - k) + 1)A_{m-2} + kA_{m-3}$$

$$= \quad B_{m-1} + A_{m-2} + k(A_{m-1} - B_{m-2}) - k^2 A_{m-2} \equiv q(k) \qquad (4.104)$$

These observations together with lemma (4.20) provide the following expression

$$\begin{aligned} 2p(k) &= 2\prod_{j=0}^{m} \mu'_j - 2\prod_{k=0}^{m-1} \mu_j \\ &= q(k) - (A_{m-1} + B_{m-2}) \\ &+ \sqrt{q(k)^2 + 4(-1)^m} - \sqrt{(A_{m-1} + B_{m-2}) + 4(-1)^{m-1}} \end{aligned} \qquad (4.105)$$

Now consider $p(x)$ where $x \in [1, a_{m-1} - 1]$. Note that since $p(1)$ and $p(a_{m-1} - 1)$ must exist with a non-zero surd term and $q(k)$ has a single maximum, the square root term must be well defined and non-zero for x in this range. The derivative of $p(x)$ is given by

$$2p'(x) = q'(x)\left(1 + \frac{q(x)}{\sqrt{q(x)^2 + 4(-1)^m}}\right) \qquad (4.106)$$

where $q'(x) = A_{m-1} - B_{m-2} - 2kA_{m-2}$. The term in brackets can never equal zero and $q'(x)$ has one zero at $x = \frac{A_{m-1} - B_{m-2}}{2A_{m-2}}$. This corresponds to a sole maximum which may or may not be in the interval $[1, a_{m-1} - 1]$. Whatever the case, the minimum value of $p(x)$ on $[1, a_{m-1} - 1]$ must occur at one or both of the end points of the interval. To complete the proof, it remains then to show that $p(1)$ and $p(a_{m-1} - 1)$ are strictly greater than zero.

Now, $q(1) = B_{m-1} - B_{m-2} + A_{m-1}$, so

$$\begin{aligned} 2p(1) &= B_{m-1} - 2B_{m-2} \qquad (4.107) \\ &+ \sqrt{(B_{m-1} - B_{m-2} + A_{m-1})^2 + 4(-1)^m} - \sqrt{(A_{m-1} + B_{m-2})^2 + 4(-1)^{m-1}} \end{aligned}$$

The cases $m = 1$ and $m \geq 2$ will be examined separately. Consider the first term in the above. Since $B_{m-1} = i_{m-1}B_{m-2} + B_{m-3}$ for all $m \geq 2$ and $i_{m-1} \geq 2$ it follows that

$B_{m-1} - 2B_{m-2} \geq B_{m-3} \geq 0$.

For $m \geq 2$ the difference of the square roots in equation (4.108) can be shown to be greater than zero by examining the difference of the arguments of these square roots. This difference simplifies to

$$d = (2A_{m-1} + B_{m-1})(B_{m-1} - 2B_{m-2}) + 8(-1)^m \tag{4.108}$$

Now, $B_{m-1} - 2B_{m-2} \geq 0$ with equality only holding when $m = 2$ and $i_1 = 2$. Clearly, $d \geq 8 > 0$ when m is even. For m odd, note first that the minimum value of $(2A_{m-1} + B_{m-1})(B_{m-1} - 2B_{m-2})$ increases with m. Thus, the case $m = 3$ will provide enough evidence for all odd m. For $m = 3$, equation (4.108) becomes explicitly

$$d = (2i_2 + 2i_0 + 2i_0 i_1 i_2 + i_1 i_2 + 1)(i_1(i_2 - 1) + 1) - 8 \tag{4.109}$$

Noting that $i_2 \geq 2$ and $i_0, i_1 \geq 1$, it is clear that $d > 0$.

Finally, for $m = 1$, equation (4.108) reduces to

$$\begin{aligned} 2p(1) &= B_0 - 2B_{-1} + \sqrt{(B_0 - B_{-1} + A_0)^2 - 4} \\ &- \sqrt{(A_0 + B_{-1})^2 + 4} = 1 + \sqrt{(1 + i_0)^2 - 4} - \sqrt{i_0^2 + 4} \end{aligned} \tag{4.110}$$

By inspection, the minimum of $2p(1)$ for $m = 1$ must occur when i_0 is smallest, i.e. when $i_0 = 2$. This gives $2p(1) = 1 + \sqrt{5} - \sqrt{8} > 0$. Therefore $p(1)$ is always strictly greater than zero. A similar line of reasoning shows the same is true for $p(i_{m-1} - 1)$ completing the proof. □

An immediate generalization is the following

Corollary 4.22 *Given the orbit product corresponding to an ordered partition of n, any further partitioning will give an orbit product strictly larger than the original. I.e.,* *given a partition (sequence)* $\{i_0, i_2, \ldots, i_{m-1}\}$ *and a finer partition*

$$\{i_{01}, i_{02}, \ldots, i_{0n_1}, i_{11}, i_{12}, \ldots, i_{1n_2}, \ldots, i_{m-11}, i_{m-12}, \ldots, i_{m-1n_{m-1}}\}$$

where $\sum_{l=1}^{n_j} i_{jl} = i_j$*, then the latter has a strictly larger orbit product.*

Note that this corollary is a much stronger result than previously shown as it gives lemma (4.9) straight away. Corollary (4.22) also provides that the orbit product of every Type II sequence is bounded by the orbit product of a sequence that contains $2[\frac{(1-\epsilon)n+1}{2}]$ 'ones' and $[\frac{\epsilon n}{2}]$ 'twos'. A bound on the latter quantity is now determined.

Lemma 4.23 *Given* $\{i_j\}_{j=0}^{m-1}$*, a sequence of* m_1 *1's and* m_2 *2's in any order with* $m_1, m_2 > 0$ *and* $m_1 + m_2 = m$*, then the following holds:*

$$\prod_{j=0}^{m-1} \mu_j = \prod_{j=0}^{m-1} [\overline{i_j; \ldots, i_{m-1}, i_0, \ldots, i_{j-1}}] < \rho_g^{m_1} \left(\frac{5}{2}\right)^{m_2}$$

where $\rho_g = [\overline{1};] = [\overline{1}]^{-1}$ *is the golden ratio.*

Proof. The proof proceeds along similar lines to that of lemma (4.9); basic building blocks of inequalities are found which can then be pieced together to provide the overall inequality. Consider any orbit product based on a sequence of 1's and 2's. The indices of this product may be cyclically permuted so that μ_0 has a 1 as its first entry and a 2 as its last entry in the periodic block. The sequence corresponding to this μ_0 can then be broken down into M blocks containing a string of 1's followed by a string of 2's. I.e.

$$\mu_0 = [\overline{\underbrace{1; \ldots, 1, 2, \ldots, 2}_{\text{block 1}}, \underbrace{1, \ldots, 1, 2, \ldots, 2}_{\text{block 2}}, \ldots, \underbrace{1, \ldots, 1, 2, \ldots, 2}_{\text{block } M}}] \tag{4.111}$$

Consider now the continued fraction

$$t = [\underbrace{1; 1, \ldots, 1}_{N_1 \text{ 'ones'}}, \underbrace{2, 2, \ldots, 2}_{N_2 \text{ 'twos'}}, \mu_N] \tag{4.112}$$

where $N = N_1 + N_2$ and the first entry of μ_N is a 1 and also the partial orbit product

$$\prod_{j=0}^{N-1} \mu_j = \prod_{j=1}^{N_1} [\underbrace{1; 1, \ldots, 1}_{N_1 - j \text{ `ones'}}, \mu_{N_1}]$$
$$\times \prod_{j=1}^{N_2} [\underbrace{2; 2, \ldots, 2}_{N_2 - j \text{ `twos'}}, \mu_N] \tag{4.113}$$

Now, a similar result to lemma (4.17) shows that $\mu_{N_1} = \frac{g_{N_2}\mu_N + g_{N_2-1}}{g_{N_2-1}\mu_N + g_{N_2-2}}$. The g_n are related to the 'silver mean', ρ_s, which is equal to $\sqrt{2} + 1 = [\overline{2};]$ and satisfies the equation $x = 2 + \frac{1}{x}$. The g_n also follow the recursion relation $g_n = 2g_{n-1} + g_{n-2}$ and the first few are given by $g_0, g_1, g_2, g_3, \ldots = 1, 2, 5, 12, \ldots$.

In a similar fashion to the proof of corollary (4.18) the product simplifies to

$$\prod_{j=0}^{N-1} \mu_j = f_{N_1} g_{N_2} + f_{N_1-1} g_{N_2-1} + \frac{f_{N_1} g_{N_2-1} + f_{N_1-1} g_{N_2-2}}{\mu_N} \tag{4.114}$$

Note that the smallest continued fraction that can be made out of 1's and 2's is $[\overline{1, 2}] = \frac{1+\sqrt{3}}{2}$. So, $\mu_N \geq \frac{1+\sqrt{3}}{2}$ and therefore $\frac{1}{\mu_N} \leq \sqrt{3} - 1$. The product is then bounded in the following way

$$\prod_{j=0}^{N-1} \mu_j \leq f_{N_1} g_{N_2} + f_{N_1-1} g_{N_2-1} + (\sqrt{3} - 1)(f_{N_1} g_{N_2-1} + f_{N_1-1} g_{N_2-2}) \tag{4.115}$$

The proof now proceeds by the method of induction. Assume that

$$\prod_{j=0}^{N-1} \mu_j < \rho_g^{N_1} \left(\frac{5}{2}\right)^{N_2} \tag{4.116}$$

holds for $N_1 = k$ and $N_1 = k - 1$. Then for $N_1 = k + 1$, it follows easily using the relationship $f_{k+1} = f_k + f_{k-1}$ that

$$f_{k+1} g_{N_2} + f_k g_{N_2-1} + (\sqrt{3} - 1)(f_{k+1} g_{N_2-1} + f_k g_{N_2-2})$$
$$< \rho_g^k \left(\frac{5}{2}\right)^{N_2} + \rho_g^{k-1} \left(\frac{5}{2}\right)^{N_2} = \rho_g^{k+1} \left(\frac{5}{2}\right)^{N_2} \tag{4.117}$$

The same procedure also shows that if the assertion is true for $N_2 = k$ and $N_2 = k-1$, it is true for $N_2 = k+1$.

$$\begin{aligned}
& f_{N_1} g_{k+1} + f_{N_1-1} g_k + (\sqrt{3}-1)(f_{N_1} g_k + f_{N_1-1} g_{k-1}) \\
< \;& 2\rho_g^{N_1} \left(\frac{5}{2}\right)^k + \rho_g^{N_1} \left(\frac{5}{2}\right)^{k-1} \\
= \;& \rho_g^{N_1} \left(\frac{5}{2}\right)^k \left(2 + \frac{1}{2.5}\right) < \rho_g^{N_1} \left(\frac{5}{2}\right)^k \left(2 + \frac{1}{\rho_s}\right) \\
= \;& \rho_g^{N_1} \left(\frac{5}{2}\right)^k \rho_s < \rho_g^{N_1} \left(\frac{5}{2}\right)^{k+1}
\end{aligned} \tag{4.118}$$

Finally, it remains to check that the claim is true for $(N_1, N_2) = (1,1), (1,2), (2,1)$ and $(2,2)$. Substituting the values for the f_n and the g_n into equation (4.115) when $(N_1, N_2) = (1,1)$:

$$\prod_{j=0}^{N-1} \mu_j \leq 3 + (\sqrt{3}-1)(1) = \sqrt{3} + 2 \approx 3.73 < \rho_g^1 \left(\frac{5}{2}\right)^1 \approx 3.91$$

Similarly, $3\sqrt{3} + 4 \approx 9.20 < \rho_g^1 \left(\frac{5}{2}\right)^2 \approx 9.43$, $2\sqrt{3} + 3 \approx 6.46 < \rho_g^2 \left(\frac{5}{2}\right)^1 \approx 6.54$ and $3\sqrt{3} + 4 \approx 15.7 < \rho_g^2 \left(\frac{5}{2}\right)^2 \approx 16.4$. So, by induction, the inequality in equation (4.116) is true for all $N_1, N_2 \geq 1$. By joining the inequalities for each of the blocks together, the lemma is seen to be true. $\square$

A corollary to lemmas (4.21) and (4.23) is the following:

Corollary 4.24 *All Type II periodic continued fraction products are bounded above by the quantity*

$$\rho_g^{2[\frac{(1-\epsilon)n+1}{2}]|\beta|} \left(\frac{5}{2}\right)^{2[\frac{\epsilon n}{2}]|\beta|}$$

For fixed ϵ and β, the a_n may now be estimated by adding the bounds on the number of Type I and Type II sequences multiplied by the bounds on Type I and Type II orbit products respectively. I.e.,

$$\begin{aligned} a_n &< N(I)\rho_g^{2n\beta} + N(II)\rho_g^{2[\frac{(1-\epsilon)n+1}{2}]\beta}\left(\frac{5}{2}\right)^{2[\frac{\epsilon n}{2}]\beta} \\ &= \frac{\epsilon n}{\epsilon^{\epsilon n}(1-\epsilon)^{(1-\epsilon)n}} 2^{\epsilon n}\rho_g^{2n|\beta|} + 2^n \rho_g^{4[\frac{(1-\epsilon)n+1}{2}]|\beta|}\left(\frac{5}{2}\right)^{2[\frac{\epsilon n}{2}]|\beta|} \end{aligned} \tag{4.119}$$

Taking the n^{th} root of this estimate and using the fact that $|a+b|^{\frac{1}{n}} < |a|^{\frac{1}{n}} + |b|^{\frac{1}{n}}$, it is observed that

$$\begin{aligned} |a_n|^{\frac{1}{n}} &< \left(\frac{\epsilon n}{\epsilon^{\epsilon n}(1-\epsilon)^{(1-\epsilon)n}} 2^{\epsilon n}\rho_g^{2n|\beta|}\right)^{\frac{1}{n}} + \left(2^n \rho_g^{4[\frac{(1-\epsilon)n+1}{2}]|\beta|}\left(\frac{5}{2}\right)^{2[\frac{\epsilon n}{2}]|\beta|}\right)^{\frac{1}{n}} \\ &\to \frac{2^{\epsilon}}{\epsilon^{\epsilon}(1-\epsilon)^{(1-\epsilon)}}\rho_g^{2|\beta|} + 2\rho_g^{2(1-\epsilon)|\beta|}\left(\frac{5}{2}\right)^{\epsilon|\beta|} \quad \text{as } n \to \infty \end{aligned} \tag{4.120}$$

Thus, a lower bound for the radius of convergence of the induced zeta function has been obtained. Finally, the logarithm of the above gives the following bound on the pressure function:

$$-\beta F(\beta) - 2|\beta|\log\rho_g < \log\left(\frac{2^{\epsilon}}{\epsilon^{\epsilon}(1-\epsilon)^{(1-\epsilon)}} + \left(\frac{5}{2\rho_g^2}\right)^{\epsilon|\beta|}\right) \tag{4.121}$$

Let $r(\epsilon) = \frac{2^{\epsilon}}{\epsilon^{\epsilon}(1-\epsilon)^{(1-\epsilon)}}$. Note that $r(\epsilon) \to 1$ as $\epsilon \to 0$. In particular, $r(\epsilon) > 1$ for all $\epsilon \in (0,1]$. Therefore, for any $\delta > 0$, there exists a value $\epsilon(\delta)$ such that for all $\epsilon < \epsilon(\delta)$, $r(\epsilon) < 1 + \delta/2$. Now set $\epsilon = \epsilon(\delta)/2$. Consider now the other term in the logarithm, $\left(\frac{5}{2\rho_g^2}\right)^{\epsilon|\beta|}$. Since $\frac{5}{2\rho_g^2} < 1$, this term will approach zero for fixed ϵ and $\beta \to -\infty$. Clearly then, there exists a $\beta(\delta, \epsilon(\delta)) = \beta(\delta)$ such that for all $\beta < \beta(\delta)$

$$\left(\frac{5}{2\rho_g^2}\right)^{\frac{\epsilon(\delta)}{2}|\beta|} < \frac{\delta}{2}$$

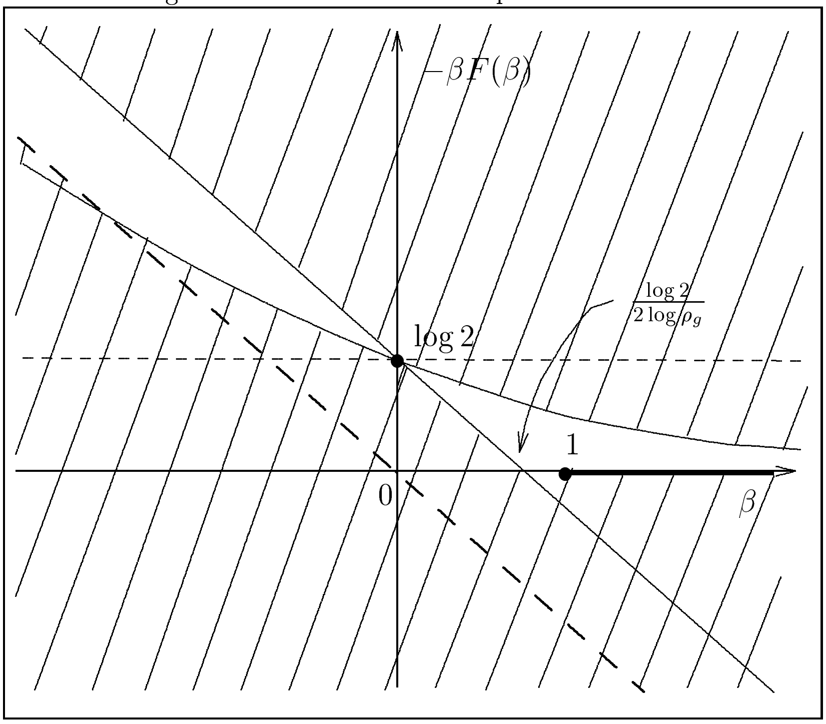
Returning to equation (4.121), these observations show that for all $\beta < \beta(\delta)$

$$-\beta F(\beta) - 2|\beta|\log\rho_g < \log\left(1 + \frac{\delta}{2} + \frac{\delta}{2}\right) = \log(1+\delta) < \delta \tag{4.122}$$

This completes the proof of theorem (4.15). □

All of the information obtained regarding the pressure function thus far, see figures (4.3),(4.4) and (4.5), is displayed in figure (4.6). The bold dashed line indicates the asymptote $y = -2\beta \log 2$.

Figure 4.6: All bounds on the pressure function



4.2.5 The phase transition at $\beta = 1$

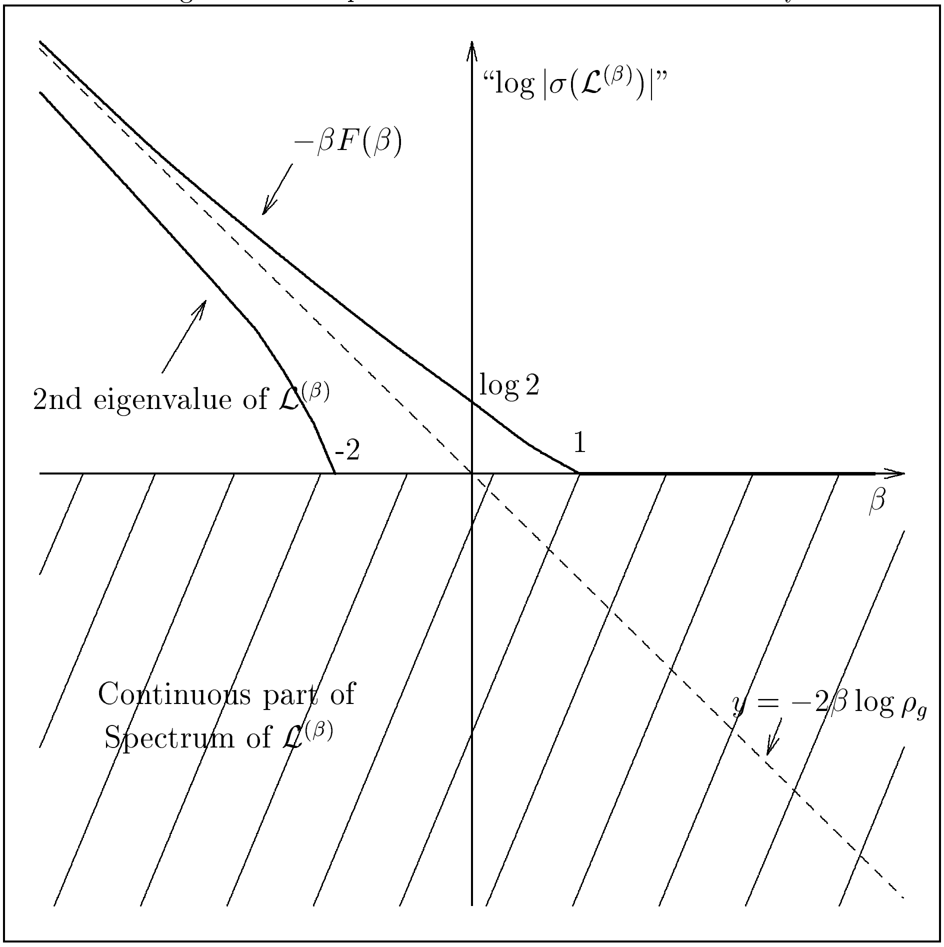
It has been demonstrated that a phase transition exists at some value of β in the interval $(0, 1]$. It has in fact been shown for a certain class of maps of the interval in [22] that the critical value of β is actually 1. The proof uses the induced transfer operator and the fact that this has a simple leading eigenvalue. Perturbation theory of simple eigenvalues, see [16], may then be applied to show that the pressure function is analytic for $0 \leq \beta < 1$. This implies that the non-analytic point must occur at $\beta = 1$. The asymptotic behaviour of the pressure function is also found for this class of maps and is reproduced here for the special case of the Farey map:

$$-\beta F(\beta) \sim \frac{1-\beta}{-\log(1-\beta)} \text{ as } \beta \to 1^- \tag{4.123}$$

4.3 Concluding Remarks

The constraints found on the pressure function are enough to give a very solid picture of its shape. In particular, the scaling behaviour at the phase transition and the asymptotic behaviour for large negative β have been demonstrated analytically. Moreover, the findings are seen to agree with previous numerical calculations of the pressure function, ([6]). Numerical observations have identified that for the Farey map there are no other eigenvalues with a magnitude greater than 1 for all $-2 \leq \beta < 1$. For $-7 < \beta < -2$, it has been observed that a single secondary eigenvalue appears. This structure is qualitatively represented in figure (4.7). (Note that the figure displays the logarithm of the magnitude of each element of the spectrum of the transfer operator, the largest such value corresponding to the pressure function).

Figure 4.7: The pressure function obtained numerically



It was also shown analytically in [22] that there exists a spectral gap in the spectrum of the transfer operator for all $0 \leq \beta < 1$. The correlation length of the system may be obtained from the distance of the leading eigenvalue (i.e., the exponential of the pressure function) to the second largest eigenvalue. So while there is knowledge of a spectral gap, it is the secondary spectra of the transfer operator which will provide exact details of correlations in the system. This structure has proved to be very resistant to analytic investigation. Note that in the positive temperature range, i.e. $\beta > 0$, it is observed that the leading eigenvalue of the transfer operator is the only eigenvalue outside of the unit disc of continuous spectrum (i.e. the essential radius).

The spin system created through the symbolic dynamics of the Farey map is seen to freeze into one state when $\beta \geq 1$ or $T = 1/\beta \leq 1$. This state is just the one with all spins being 'up' which corresponds to an infinite string of 'ones'. In terms of the map, this says that with probability 1, a typical orbit of the Farey map will stay near the indifferent fixed point forever. In terms of intermittent behaviour, the 'signal' has become completely smooth and regular. So in this toy model of intermittency, it is observed that there is a sudden transition to laminar behaviour, reflected in the singular nature of the invariant density of the original Farey map.

Finally, it is remarked that the counting and ordering exercises involved are interesting in their own right. In particular, the reader's attention is drawn to conjecture (A.3) in the first appendix. This certainly stands by itself as a worthy result away from the main content of the thesis and it is hoped that a proof may be found in the near future.

Appendix A

Continued Fractions

The theory presented here is standard work and is taken primarily from the book of Rockett and Szüsz, [24].

A.1 General definitions

In general, a continued fraction C is the following entity:

$$C = i_0 + \cfrac{1}{i_1 + \cfrac{1}{i_2 + \cfrac{1}{i_3 + \cfrac{1}{\cdots + \cfrac{1}{i_n}}}}} = [i_0; i_1, i_2, \ldots, i_n] \tag{A.1}$$

where the i_i are positive integers, i_0 may also be 0, and $n \in N \bigcup \{\infty\}$. A continued fraction with $i_0 = 0$ will be abbreviated as $C = [i_1, i_2, \ldots, i_n]$ $(= [0; i_1, i_2, \ldots, i_n])$.

Irrational numbers have an infinite continued fraction expansion while the expansion for any rational terminates at some finite n. The 'convergents' of a continued fraction, C_k, naturally arise as the rational number given by the truncated continued fraction

$$C_k = \frac{A_k}{B_k} = [i_0; i_1, i_2, \ldots, i_k] \tag{A.2}$$

where $k \leq n$, the number of terms in the expansion, and the sequences $\{A_k\}$ and $\{B_k\}$ are generated by the recursion relations

$$\left[\begin{array}{l} A_{k+1} = i_{k+1} A_k + A_{k-1} \\ \\ B_{k+1} = i_{k+1} B_k + B_{k-1} \end{array} \right. \quad \text{for } k = 0, 1, 2, \ldots, n-1 \tag{A.3}$$

with the seed values $A_{-1} = 1, A_0 = i_0$ and $B_{-1} = 0, B_0 = 1$. It can be easily shown using these relations that

$$A_m B_{m-1} - A_{m-1} B_m = (-1)^{m+1} \tag{A.4}$$

The k$^{\text{th}}$ complete quotient μ_k of a continued fraction $C = [i_0; i_1, i_2, \ldots]$ is defined by:

$$\mu_k = [i_k; i_{k+1}, i_{k+2}, \ldots] \tag{A.5}$$

A simple proof by induction shows that

$$C = \mu_0 = \frac{A_k \mu_{k+1} + A_{k-1}}{B_k \mu_{k+1} + B_{k-1}} \quad \text{for } k \geq 0 \tag{A.6}$$

Finally, a small result used in the text is that the terms of the sequence $\{\alpha_k\} = \{A_k + B_k\}$ are always positive and increasing. To show this, note that $\alpha_{-1} = 1$ and $\alpha_0 = 1 + i_0$ and that $\{\alpha_k\}$ obeys the same recursion relation as $\{A_k\}$ and $\{B_k\}$:

$$A_{k+1} + B_{k+1} = i_{k+1}(A_k + B_k) + (A_{k-1} + B_{k-1}) \tag{A.7}$$

Thus, for $k \geq 0$, $\alpha_{k+1} - \alpha_k = (i_{k+1} - 1)\alpha_k + \alpha_{k-1}$ for $k = 0, 1, \ldots, n-1$. Since $i_{k+1} \geq 1$ for all $k \geq 0$ the first term on the righthand side is ≥ 0. By inspection, α_k is always > 0 so the second term is always > 0. Thus, the terms must increase with the possible exception of the initial terms α_{-1} and α_0 when $i_0 = 0$.

A.2 Periodic Continued Fractions

The fixed points of the induced Farey map are periodic continued fractions. These are simply those continued fractions whose entries are periodic. In particular, the sequence of entries representing a periodic continued fraction, $\{i_i\}$, is made up of a finite initial sequence $\{i_0, i_1, \ldots, i_{n-1}\}$ followed by a repeating sequence $\{i_n, i_{n+1}, \ldots, i_{n+m-1}\}$ where $i_{n+j+km} = i_{n+j}$ for all $0 \leq j \leq m-1$ and $k = 1, 2, \ldots$. If $n = 0$, i.e. there is no preliminary sequence, the continued fraction is said to be purely periodic. Most of the main work involves purely periodic continued fractions and only these will be considered here.

The major result regarding periodic continued fractions is a bijective map between them and quadratic surds. A quadratic surd is a solution to a quadratic equation such that the discriminant $(b^2 - 4ac)$ is not an perfect square. This result is quoted as a theorem in the main body of the text and a proof can be found in Rockett.

A useful result discovered by the author and subsequently found to be a lemma on p.54 of [24](!) is the following:

Lemma A.1 *Let* $C = [\overline{i_0; i_1, \ldots, i_{m-1}}]$. *Then*

$$\prod_{k=0}^{m-1} \mu_k = \mu_0 \cdot \mu_1 \cdot \cdots \cdot \mu_{m-1} = B_{m-1}C + B_{m-2}$$

This product is referred to as an 'orbit product'.

Proof. Note firstly that $C = \mu_0 = \mu_m$. Therefore

$$\prod_{k=0}^{m-1} \mu_k = \mu_0 \cdot \mu_1 \cdot \cdots \cdot \mu_{m-1} = \mu_1 \cdot \mu_2 \cdot \cdots \cdot \mu_{m-1} \cdot \mu_m \tag{A.8}$$

In general, if it is assumed that $\mu_k = B_{k-1}\mu_k + B_{k-2}$ then

$$\begin{aligned} \mu_k \cdot \mu_{k+1} &= (B_{k-1}\mu_k + B_{k-2}) \cdot \mu_{k+1} = \left(B_{k-1}(i_k + \frac{1}{\mu_{k+1}}) + B_{k-2}\right) \cdot \mu_{k+1} \\ &= (i_k B_{k-1} + B_{k-2})\mu_{k+1} + B_{k-1} = B_k \mu_{k+1} + B_{k-1} \end{aligned} \tag{A.9}$$

Since $\mu_1 = B_0\mu_1 + B_{-1}$ it follows by induction that

$$\prod_{k=0}^{m-1} \mu_k = B_{m-1}\mu_m + B_{m-2} = B_{m-1}C + B_{m-2} \tag{A.10}$$

□

Using equation (A.6), a purely periodic continued fraction with period m can be written in terms of itself as

$$C = \frac{A_{m-1}C + A_{m-2}}{B_{m-1}C + B_{m-2}} \tag{A.11}$$

Rearranging this gives a quadratic equation for C with the positive solution necessarily being C.

$$C = \frac{A_{m-1} - B_{m-2} + \sqrt{(A_{m-1} - B_{m-2})^2 + 4A_{m-2}B_{m-1}}}{2B_{m-1}} \tag{A.12}$$

This observation allows for the following corollary to lemma (A.1).

Corollary A.2 *The orbit product of a purely periodic continued fraction*

$$C = [\overline{i_0; i_1, \ldots, i_{m-1}}]$$

is given by

$$\prod_{k=0}^{m-1} \mu_k = \frac{(A_{m-1} + B_{m-2}) + \sqrt{(A_{m-1} + B_{m-2})^2 + 4(-1)^{m-1}}}{2}$$

Proof. Relationship (A.4) shows that

$$(A_{m-1} - B_{m-2})^2 + 4A_{m-2}B_{m-1} = (A_{m-1} + B_{m-2})^2 + 4(-1)^{m-1}$$

The corollary is then shown by inserting this into the solution for C, equation (A.12), and then all this into the result of lemma (A.1):

$$\begin{aligned}
\prod_{k=0}^{m-1} \mu_k &= B_{m-1}\left(\frac{A_{m-1} - B_{m-2} + \sqrt{(A_{m-1} + B_{m-2})^2 + 4(-1)^{m-1}}}{2B_{m-1}}\right) + B_{m-2} \\
&= \frac{A_{m-1} - B_{m-2} + 2B_{m-2} + \sqrt{(A_{m-1} + B_{m-2})^2 + 4(-1)^{m-1}}}{2} \\
&= \frac{(A_{m-1} + B_{m-2}) + \sqrt{(A_{m-1} + B_{m-2})^2 + 4(-1)^{m-1}}}{2} \qquad \text{(A.13)}
\end{aligned}$$

□

Note that the orbit product depends only on one quantity $(A_{m-1} + B_{m-2})$. Finally, a very helpful relationship would be the following inequality which, as yet, has not been proven and only verified numerically for small values of m and n.

Conjecture A.3 *Let* $\{i_0; i_1, \ldots, i_{m-1}\}$ *be a positive integer sequence with* $\sum_{k=0}^{m-1} i_k = n$. *Then*

$$\prod_{k=0}^{m-1} [\overline{i_k, i_{k+1}, \ldots, i_{m-1}, i_0, \ldots, i_{k-1}}] \leq \prod_{k=0}^{m-1} [\overline{i_k;}]$$

with the equality only holding when $i_1 = i_2 = \cdots = i_{m-1}$.

This would certainly simplify the proof of theorem (4.15) as well as being an elegant result in its own right.

Appendix B

Miscellaneous

B.1 The spectral radius of $\mathcal{L}_1^{(\beta)}$

The operator $\mathcal{L}_1^{(\beta)}$ was defined in chapter (2) by the relation $\mathcal{L}_1^{(\beta)}\psi = \mathcal{L}^{(\beta)}(\chi_{J^c}\psi)$. From the definitions of the Farey map and the transfer operator (equation (1.1) and definition (1.3) respectively), $\mathcal{L}_1^{(\beta)}$ is found to be

$$\mathcal{L}_1^{(\beta)}\psi(x) = (1+x)^{-2\beta}\psi(\frac{x}{1+x}) \tag{B.1}$$

There is the following result regarding the spectral radius of this operator.

Lemma B.1 *The spectral radius of the operator $\mathcal{L}_1^{(\beta)}$, $r(\mathcal{L}_1^{(\beta)})$, is equal to 1.*

Proof. Consider the following formula for the spectral radius of an operator $\mathcal{O}$ taken from [22]:

$$r(\mathcal{O}) \equiv lim_{n\to\infty}\|\mathcal{O}^n\|^{\frac{1}{n}} \tag{B.2}$$

where the norm of $\mathcal{O}$ is defined as $\|\mathcal{O}\| = \sup_{\|\psi\|=1} \|\mathcal{O}\psi\|$. It follows from the definition of $\mathcal{L}_1^{(\beta)}$ that its n^{th} iterate is given by

$$\mathcal{L}_1^{(\beta)}\psi(x) = (1+nx)^{-2\beta}\psi(\frac{x}{1+nx}) \tag{B.3}$$

Therefore, using expression (B.2)

$$\|(\mathcal{L}_1^{(\beta)})^n\|^{\frac{1}{n}} = \sup_{\|\psi\|=1} \|(\mathcal{L}_1^{(\beta)})^n\psi\| = \sup_{\|\psi\|=1} \sup_x |1+nx|^{-2\beta} \left|\psi(\frac{x}{1+nx})\right| \tag{B.4}$$

Now, setting $\psi = 1$ shows that $\|(\mathcal{L}_1^{(\beta)})^n\|^{\frac{1}{n}} \geq \sup_x |1+nx|^{-2\beta}$. On the other hand, it follows from equation (B.4) that

$$\|(\mathcal{L}_1^{(\beta)})^n\|^{\frac{1}{n}} \leq \sup_x |1+nx|^{-2\beta} \sup_{\|\psi\|=1,x} \left|\psi(\frac{x}{1+nx})\right| \leq \sup_x |1+nx|^{-2\beta} \tag{B.5}$$

Since $|1+nx|^{\frac{-2\beta}{n}} \to 1$ as $n \to \infty$, the proof is finished. $\square$

B.2 The trace of the operator $\mathcal{R}_k^{(z,\beta)}$

The work in section 3.6 hinges upon the calculation or at least analysis of the single eigenvalue of the operator $\mathcal{R}_k^{(z,\beta)}$:

$$\lambda_k = \frac{(-1)^k z}{k!}\phi_k^{(k)}(1) = \frac{(-1)^k z}{k!}\frac{d^k}{d\xi^k}\, \xi^{-2\beta}\Phi(z, 2\beta+k, \frac{1}{\xi}+1)\Big|_{\xi=1} \tag{B.6}$$

where $k > 0$. The following lemma provides an expression for λ_k from which the singularity structure may be easily obtained.

Lemma B.2 *The sole eigenvalue of $\mathcal{R}_k^{(z,\beta)}$ is given by*

$$\begin{aligned}\lambda_k &= \frac{z}{k!}\frac{\Gamma(2\beta+k)}{\Gamma(2\beta)}\Phi(z, 2\beta+k, 2) \\ &\quad + \frac{z}{k!}\sum_{l=1}^{k}\sum_{j=1}^{l}(-1)^j a_{(l,j)}\binom{k}{l}\frac{\Gamma(2\beta+k-l)}{\Gamma(2\beta)}\frac{\Gamma(2\beta+k+j)}{\Gamma(2\beta+k)}\Phi(z, 2\beta+k+j, 2)\end{aligned}$$

where each $a_{(l,j)}$ is some positive integer, $\Gamma(z)$ is the gamma function and $\Phi(z,s,\nu)$ is the Lerch transcendent function. For $k > 0$, this eigenvalue is holomorphic in z for $|z| < 1$ and $\beta \in C$ and is also holomorphic in β for $\beta \in C$ and $|z| \leq 1$.

Proof. Using equations 3.89 and 3.90, the expression for λ_k may be written as follows:

$$\begin{aligned}\lambda_k &= \frac{(-1)^k z}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{(k-l)}2\beta(2\beta+1)(2\beta+2)\ldots(2\beta+k-l-1)\\ &\quad\times\frac{d^l}{d\xi^l}\Phi(z,2\beta+k,\frac{1}{\xi}+1)\Big|_{\xi=1}\\ &= \frac{z}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^l\frac{\Gamma(2\beta+k-l)}{\Gamma(2\beta)}\frac{d^l}{d\xi^l}\Phi(z,2\beta+k,\frac{1}{\xi}+1)\Big|_{\xi=1} \qquad \text{(B.7)}\end{aligned}$$

where Γ is the Gamma function. Γ satisfies the factorial-like relationship $\Gamma(s+1) = s\Gamma(s)$, see [7], and hence has been used to express λ_k in a more compact form. This section is mainly concerned with the term

$$T(l,z,2\beta+k) = \frac{d^l}{d\xi^l}\Phi(z,2\beta+k,\frac{1}{\xi}+1)\Big|_{\xi=1} \qquad \text{(B.8)}$$

the righthand part of the summand in B.7.

Let $u = \frac{1}{\xi}$. The operator $\frac{d^l}{d\xi^l}$ then becomes:

$$\frac{d^l}{d\xi^l} = \left(\frac{d}{d\xi}\right)^l = \left(\frac{du}{d\xi}\frac{d}{du}\right)^l = \left(-u^2\frac{d}{du}\right)^l = (-1)^l\left(u^2\frac{d}{du}\right)^l \qquad \text{(B.9)}$$

So, with this change of variable, equation B.8 transforms to

$$(-1)^l T(l,z,2\beta+k) = \left(u^2\frac{d}{du}\right)^l\Phi(z,2\beta+k,u+1)\Big|_{u=1} \qquad \text{(B.10)}$$

Now assume that for $l = L \geq 1$ the righthand side of the above (without the evaluation at $u = 1$) is of the form

$$\left(u^2\frac{d}{du}\right)^L\Phi(z,2\beta+k,u+1) = \sum_{j=1}^{L}a_{(L,j)}u^{L+j}\frac{d^j}{du^j}\Phi(z,2\beta+k,u+1) \qquad \text{(B.11)}$$

where the $a_{(L,j)}$ are positive integers and have an extended definition to $j = 0$ and $j = L+1$ where $a_{(L,L+1)} = a_{L,0} = 0$. Note that this is clearly true for $L = 1$. Then it follows that

$$\begin{aligned}
(-1)^{L+1}T(L+1,z,2\beta+k) &= \left(u^2\frac{d}{du}\right)\sum_{j=1}^{L} a_{(L,j)}u^{L+j}\frac{d^j}{du^j}\Phi(z,2\beta+k,u+1) \\
&= u^2\sum_{j=1}^{L} a_{(L,j)}\left[(L+j)u^{L+j-1}\frac{d^j}{du^j}\Phi(z,2\beta+k,u+1)\right. \\
&\quad \left. + u^{L+j}\frac{d^{j+1}}{du^{j+1}}\Phi(z,2\beta+k,u+1)\right] \\
&= \sum_{j=1}^{L} a_{(L,j)}(L+j)u^{L+1+j}\frac{d^j}{du^j}\Phi(z,2\beta+k,u+1) \\
&\quad + \sum_{j=1}^{L} a_{(L,j)}u^{L+2+j}\frac{d^{j+1}}{du^{j+1}}\Phi(z,2\beta+k,u+1) \\
&= \sum_{j=1}^{L+1} a_{(L+1,j)}u^{L+j+1}\frac{d^j}{du^j}\Phi(z,2\beta+k,u+1) \qquad \text{(B.12)}
\end{aligned}$$

where the $a_{(L+1,j)} \equiv (L+j)a_{(L,j)} + a_{(L,j-1)}$ and must be positive integers since the $a_{(L,j)}$ are never negative and the sum of consecutive values are never zero. Therefore, by induction, it is clear that $T(l,z,2\beta+k)$ is of the form

$$(-1)^l T(l,z,2\beta+k) = \delta_{l,0}\Phi(z,2\beta+k,2) + \sum_{j=1}^{l} a_{(l,j)}\frac{d^j}{du^j}\Phi(z,2\beta+k,u+1)\bigg|_{u=1} \qquad \text{(B.13)}$$

where the case $l = 0$ has been included with the help of the Kronecker delta, $\delta_{m,n}$, and the understanding that the second sum disappears when $l = 0$. So, without exactly determining the coefficients $a_{(l,j)}$ of the differentiations of the Lerch transcendents, it can be seen that they are positive integers and of course finite for l finite. Note that $a_{(l,1)} = l!$ and $a_{(l,l)} = 1$ for $l = 1, 2, \ldots$.

Equation 3.91 shows that $\frac{d}{du}\Phi(z,s,u+1) = -s\Phi(z,s+1,u+1)$ from which it follows that

$$\begin{aligned}
\frac{d^j}{du^j}\Phi(z,s,u+1) &= (-1)^j s(s+1)\cdots(s+j-1)\Phi(z,s+j,u+1) \\
&= (-1)^j \frac{\Gamma(s+j)}{\Gamma(s)}\Phi(z,s+j,u+1) \qquad \text{(B.14)}
\end{aligned}$$

Inserting this expression with $u = 1$ and $s = 2\beta + k$ into the equation B.13 gives the result

$$\begin{aligned}
(-1)^l T(l,z,2\beta+k) &= \delta_{l,0}\Phi(z,2\beta+k,2) \qquad \text{(B.15)} \\
&\quad + \sum_{j=1}^{l} a_{l,j}(-1)^j \frac{\Gamma(2\beta+k+j)}{\Gamma(2\beta+k)}\Phi(z,2\beta+k+j,2)
\end{aligned}$$

where Finally, this result for $T(l,z,2\beta+k)$ may be substituted into the expression for λ_k, equation B.7:

$$\begin{aligned}
\lambda_k = {} & \frac{z}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^l\frac{\Gamma(2\beta+k-l)}{\Gamma(2\beta)} \qquad \text{(B.16)} \\
& \times(-1)^l\left[\delta_{l,0}\Phi(z,2\beta+k,2) + \sum_{j=1}^{l}(-1)^j a_{(l,j)}\frac{\Gamma(2\beta+k+j)}{\Gamma(2\beta+k)}\Phi(z,2\beta+k+j,2)\right] \\
= {} & \frac{z}{k!}\frac{\Gamma(2\beta+k)}{\Gamma(2\beta)}\Phi(z,2\beta+k,2) \\
& + \frac{z}{k!}\sum_{l=1}^{k}\sum_{j=1}^{l}(-1)^j a_{(l,j)}\binom{k}{l}\frac{\Gamma(2\beta+k-l)}{\Gamma(2\beta)}\frac{\Gamma(2\beta+k+j)}{\Gamma(2\beta+k)}\Phi(z,2\beta+k+j,2)
\end{aligned}$$

The comment in the main body of the thesis that λ_k is a holomorphic function of β when $z = 1$ and $k > 0$ is supported by the form of equation B.17. To see this, note that $\Gamma(s)$ has no zeroes on the s-plane and has simple poles at $s = 0, -1, -2, \ldots$, see [7]. Thus the quotient $\frac{\Gamma(s+n)}{\Gamma(s)}$, where n is a positive integer, has simple zeroes at $s = 0, -1, -2, \ldots, -n+1$. It has no poles as these can only come from the simple poles of the numerator $\Gamma(s+n)$, at $s = -n, -n-1, -n-2, \ldots$, but these are cancelled by the poles of the denominator at those same points.

So the two quotients of Gamma functions in the second term of the last line of equation B.17 provide no poles since $k-l \geq 0$ and $j > 0$. They do however have zeroes at $2\beta = 0, -1, -2, \ldots, -k+l+1$ and at $2\beta + k = 0, -1, -2, \ldots, -j+1$, respectively. All of this was clear, of course, from the fact that these quotients were introduced to represent finite polynomials with simple factors which are very well behaved entities. Together, the two quotients provide zeroes at the following values of β:

$$2\beta = 0, -1, -2, \ldots, -k+l+1; -k, -k-1, -k-2, \ldots, -k-j+1 \qquad \text{(B.17)}$$

For $z = 1$, the Lerch transcendent $\Phi(z, 2\beta + k + j, 2)$ reduces to the Riemann zeta function less 1, i.e. $\zeta_R(2\beta + k + j) - 1$. This can be analytically continued to the whole of the β-plane with a simple pole at $2\beta + k + j = 1 \Leftrightarrow \beta = \frac{1-k-j}{2}$ with residue $\frac{1}{2}$; again see [7]. This pole is cancelled by the most negative zero of the quotients, the last value in B.17. Thus, the second term is a holomorphic function of β for $z = 1$.

Since $k \geq 1$, the quotient of gamma functions for the first term has at least one zero. The zeroes in general are at $2\beta = 0, -1, -2, \ldots, 1-k$. The Lerch transcendent at $z = 1$ reduces to the function $\zeta_R(2\beta + k) - 1$ which has one simple pole at $2\beta = 1 - k$ which is balanced by the most negative zero of the preceding quotient. Thus, both terms are holomorphic in β for all $\beta \in C$ and $z = 1$. In conclusion, for $k > 0$, the following is true:

- The function $(z, \beta) \to \lambda_k$ is holomorphic in z for $|z| < 1$ and $\beta \in C$.
- The function $(z, \beta) \to \lambda_k$ is holomorphic in β for $\beta \in C$ and $|z| \leq 1$.

as is required for the completion of the proof of the lemma. $\square$

B.3 Generalised Fibonacci Numbers

This section provides a derivation of an exact expression for generalised Fibonacci numbers $\gamma_n^{(k)}$. This sequence of numbers is generated by setting $\gamma_{-1}^{(k)} = 0$ and $\gamma_0^{(k)} = 1$ and applying the recursion formula

$$\gamma_{n+1}^{(k)} = k\gamma_n^{(k)} + \gamma_{n-1}^{(k)} \tag{B.18}$$

Note that $k \geq 1$ and for $k = 1$ this produces the normal Fibonacci series.

Lemma B.3 *The generalised Fibonacci number, $\gamma_n^{(k)}$, has an exact expression of the form*

$$\gamma_n^{(k)} = \frac{1}{1+\rho_k^2}\left(\rho_k^{n+2} + \left(\frac{-1}{\rho_k}\right)^n\right)$$

where $\rho_k = \frac{k+\sqrt{k^2+4}}{2} = [\overline{k};]$ is the positive solution to the quadratic equation $x^2-kx-1 = 0$. Moreover, this relationship holds for all integers n.

Proof. Firstly, note that $\rho_k = k + \frac{1}{\rho_k}$. Assume the statement is true for $n = m$ and $n = m-1$. Then the term $\gamma_{m+1}^{(k)}$ is given by

$$\begin{aligned}
\gamma_{m+1}^{(k)} &= k\gamma_m^{(k)} + \gamma_{m-1}^{(k)} = \frac{1}{1+\rho_k^2}\left(k\rho_k^{m+2} + k\left(\frac{-1}{\rho_k}\right)^m + \rho_k^{m+1} + \left(\frac{-1}{\rho_k}\right)^{m-1}\right) \\
&= \frac{1}{1+\rho_k^2}\left(\rho_k^{m+1}(k\rho_k+1) + (1-\frac{k}{\rho_k})\left(\frac{-1}{\rho_k}\right)^{m-1}\right) \\
&= \frac{1}{1+\rho_k^2}\left(\rho_k^{m+3} + \left(\frac{-1}{\rho_k}\right)^{m+1}\right) \qquad \text{(B.19)}
\end{aligned}$$

since $(k\rho_k+1) = \rho_k^2$ and $(1-\frac{k}{\rho_k}) = \frac{1}{\rho_k^2} = \left(\frac{-1}{\rho_k}\right)^2$. A similar procedure shows that this the relationship holds for $n = m-2$ as well. Also, the formula holds true for $n = 0$

and $n = -1$ since

$$\gamma_0^{(k)} = \frac{1}{1+\rho_k^2}(\rho_k^2 + 1) = 1$$

and

$$\gamma_{-1}^{(k)} = \frac{1}{1+\rho_k^2}\left(\rho_k - \left(\frac{1}{\rho_k}\right)^{-1}\right) = 0$$

Therefore, by induction, the proof is finished. $\square$

Note that $\rho_1 = \rho_g = \frac{1+\sqrt{5}}{2}$ is the golden ratio. Also, $\rho_2 = \rho_s = 1 + \sqrt{2}$ is sometimes called the 'silver ratio'. Both of these numbers and their corresponding Fibonacci sequences are used in the final section of the thesis.

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