Maps of Intervals with Indifferent Fixed Points: Thermodynamic Formalism and Phase Transitions

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(ABSTRACT)

We develop the thermodynamic formalism for a large class of maps of the interval with indifferent fixed points. For such systems the formalism yields one-dimensional systems with many-body infinite range interactions for which the thermodynamics is well defined while the Gibbs states are not. (Piecewise linear systems of this kind yield the soluble, in a sense, Fisher models.)

We prove that such systems exhibit phase transitions, the order of which depends on the behavior at the indifferent fixed points. We obtain the critical exponent describing the singularity of the pressure and analyse the decay of correlations of the equilibrium states at all temperatures.

Our technique relies on establishing and exploiting a relationship between the transfer operators of the original map and its suitable (expanding) induced version. The technique allows one to also obtain a version of the Bowen-Ruelle formula for the Hausdorff dimension of repellers for maps with indifferent fixed points, and to generalize Fisher results to some non-soluble models.

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Chapter 1

Introduction

1.1 General Remarks

The Thermodynamic Formalism [33, 36] proved to be a powerful tool in the ergodic theory of hyperbolic and, in particular, expanding maps [34]. A central role is played here by the transfer (or Ruelle-Perron-Frobenius) operator. That the map is expanding allows one to express thermodynamic and statistical characteristics of the system (free energy, equilibrium states, etc.) in terms of the transfer operator resulting in regularity properties of both. In particular, one obtains a Statistical Mechanics system with a fast decaying interaction and, correspondingly, a transfer operator with compactness properties which allows for quite a complete analysis of such systems. Consequentially, one has fast convergence to the thermodynamic limit and smoothness of the thermodynamic functions (no phase transitions).

These regularity properties disappear when one passes to non-hyperbolic maps, as has been demonstrated convincingly in recent works, mostly by theoretical physicists. Numerical analysis and calculations in some soluble models exhibit both singularities and slow convergence to the thermodynamic limit [8, 9, 14]. Some insights have been gained into the origin of the singularities, in particular by relating the phase transitions to that of Fisher Models [23, 37, 40, 41].

On the other hand, in a number of mathematical works, the method of *inducing* and its variants have been used to investigate absolutely continuous invariant measures for non-hyperbolic maps of the interval [1, 5, 20, 28].

Apart from a remark of Walters [39] on a relation between pressures of a soluble system and its induced version, I am aware of no work relating thermodynamics and transfer operators of a system and its induced version.

The aim of this dissertation is to establish such a relation and to show that it yields a complete version of the Thermodynamic Formalism for almost expanding maps with an indifferent fixed point with good insight into the nature of singularities in such systems; this relation can be considered a version of the Renormalization Group concept.

Maps with indifferent fixed points arise in a number of different problems. They exhibit the phenomenon of *intermittency* [30], as the dynamical system is at the transition point from a periodic state to a chaotic one. The time evolution of an intermittent system is characterized by long "laminar" phases, interrupted by "chaotic outbursts". The Farey map, one of the examples treated here, arises in phenomena as mode-locking of coupled nonlinear oscillators [21], and has been investigated as a model for intermittency in [12]. Using the thermodynamic formalism on a linearized version of intermittent maps, the statistical mechanics of the system has been investigated by [40, 41, 37]. This linearization gives rise to Fisher models, i.e. reduction of the interaction to single-cluster interactions. However, in these approximations the discarded parts of the interactions are not small in any obvious sense, as they have infinite "energy norm".

Our method extends these results to a larger class of maps, also including the smooth fixed points of the intermittency renormalization (see Chapter 3).

Chapter 1 reviews the thermodynamic formalism for continuous transformations of compact metric spaces, especially the connection between the topological pressure and the transfer operator $\mathcal{L}$ and the existing results for expansive and expanding mappings. We conclude this chapter with the introduction of the function class $\mathcal{C}_r$ which we wish to investigate.

In Chapter 2 we present the necessary modifications of the above formalism for piecewise continuous mappings in order to deal with this function class. Further, the concept of inducing is defined and a modified transfer operator $\mathcal{M}_z$ for the induced system is introduced. Relations between the modified transfer operator and the transfer operator of the original system are presented here as well.

Chapter 3 develops needed results on asymptotics of iterations near the indifferent fixed point. These are of interest in their own right as they give insight into the behavior of the intermittency renormalization transformation near its fixed point.

Chapter 4 applies the formalism of Chapter 2 to the function class $\mathcal{C}_r$ with specified interaction $-\beta \log |f'|$. The existence of a phase transition at $\beta = 1$ is proven using the relation between $\mathcal{M}_{\beta z}$ and $\mathcal{L}_\beta$.

Chapter 5 gives the asymptotic expansion of the topological pressure at the phase transition and the computation of the critical index.

Appendix A serves as a reference for notations and general definitions.

1.2 Review of the Thermodynamic Formalism for Continuous Mappings

Before we introduce the class of functions which we will be interested in, we will review the thermodynamic formalism as it exists for continuous mappings [5, 33, 35, 38].

Let f be a continuous transformation of a compact metrizable space X with metric d.

The set $M(X)$ of all probability measures on the σ-algebra of Borel subsets of X is a convex set which is compact in the weak*-topology and the subset $M(f)$ of all f-invariant probability measures is a closed subset of $M(X)$.

We call f *(positively) expansive* if and only if there exists a $\delta > 0$ such that

$$x \neq y \qquad \Rightarrow \qquad \exists n \in \mathbf{N}: \quad d(f^n x, f^n y) > \delta. \tag{1.1}$$

This is equivalent to the existence of an open cover $\mathcal{A}$ of X such that $\bigcap_{n=0}^{\infty} \mathrm{cl}(f^{-n} A_{i_n})$ contains at most one point whenever $A_{i_n} \in \mathcal{A}$. $\mathcal{A}$ is then called a *(one-sided) generator.*

f is *expanding* if and only if there exists a $\delta > 0$ and a $\lambda_0 > 1$ such that

$$d(x, y) < \delta \qquad \Rightarrow d(fx, fy) \geq \lambda_0 d(x, y). \tag{1.2}$$

An expanding map is expansive.

For a finite open cover $\mathcal{A}$ of X write $\operatorname{diam}(\mathcal{A}) = \sup_{A\in\mathcal{A}} \sup_{x,y\in A} d(x,y)$ and

$$\mathcal{A}_n = \left\{ A_{i_1} \cap f^{-1}A_{i_2} \cap \ldots f^{-(n-1)}A_{i_n} : \quad A_{i_j} \in \mathcal{A},\ j = 1, \ldots, n \right\}.$$

1.2.1 The Entropy

Given $m \in M(f)$, for a finite partition $\mathcal{A}$ of X, the entropy of $\mathcal{A}$ is defined as

$$H_m(\mathcal{A}) = -\sum_{A\in\mathcal{A}} m(A) \log m(A)$$

and the entropy of f with respect to $\mathcal{A}$ is defined as

$$H_m(f, \mathcal{A}) = \lim_{n\to\infty} H(\mathcal{A}_n).$$

Finally, the *entropy* h_m of the transformation f is given as

$$h_m = \sup \{ H_m(f, \mathcal{A}) : \quad \mathcal{A} \text{ finite partition of } X \}. \tag{1.3}$$

If f is expansive then h_m is upper semi-continuous.

1.2.2 The Pressure Function

Given a continuous "interaction" $\varphi \in C(X, \mathbf{R})$, write

$$S_n\varphi(x) = \sum_{i=0}^{n-1} \varphi(f^i x).$$

Then define the *partition function*

$$P_n(\varphi, \mathcal{A}) = \inf \left\{ \sum_{A\in\alpha} \sup_{x\in A} \exp S_n\varphi(x) : \alpha \subset \mathcal{A}_n \text{ finite subcover of } X \right\}$$

and

$$P(\varphi, \mathcal{A}) = \lim_{n\to\infty} \frac{1}{n} \log P_n(\varphi, \mathcal{A})$$

Finally, the *(topological) pressure* $P(\varphi)$ is defined as

$$P(\varphi) = \sup \{P(\varphi, \mathcal{A}) : \quad \mathcal{A} \text{ open cover of } X\}. \tag{1.4}$$

$P(0)$ is equal to $h(f)$, the *(topological) entropy.* $P(\varphi)$ is finite if and only if $h(f)$ is finite. Then, $P : C(X, \widetilde{R}) \to \mathbf{R}$ is convex and continuous.

If f is expansive then $h(f)$ is finite. If, in addition, $\mathcal{A}$ is a generator for f then the pressure is given by

$$P(\varphi) = \lim_{n\to\infty} \frac{1}{n} \log P_n(\varphi, \mathcal{A}). \tag{1.5}$$

1.2.3 The Variational Principle and Equilibrium States

Given $\varphi \in C(X, \mathbf{R})$, then

$$P(\varphi) = \sup_{m\in M(f)} \{h_m(f) + m(\varphi)\}. \tag{1.6}$$

$\mu \in M(f)$ is called an *equilibrium state* for φ if and only if the above supremum is attained for $m = \mu$:

$$P(\varphi) = h_\mu(f) + \mu(\varphi). \tag{1.7}$$

If f is expansive then each $\varphi \in C(X, \mathbf{R})$ has equilibrium states. In this case one also has a converse variational principle. For $\mu \in M(f)$,

$$h_\mu(f) = \sup_{\varphi\in C(X,\mathbf{R})} \{P(\varphi) - m(\varphi)\}.$$

If f is expanding and φ Hölder-continuous then there is a unique equilibrium state.

1.2.4 The Transfer Operator

Given φ in $C(X, \mathbf{R})$, the transfer operator $\mathcal{L}_\varphi$ acting on $C(X)$ is defined as

$$\mathcal{L}_\varphi \Phi(x) = \sum_{fy=x} \exp \varphi(y) \Phi(y). \tag{1.8}$$

As a motivation for the study of the transfer operator, we remark that iteration of 1 gives

$$\mathcal{L}_\varphi^n 1(x) = \sum_{f^n y=x} \exp S_n \varphi(y). \tag{1.9}$$

This can again be seen as a *partition function*, and the dependence on x can be interpreted as *boundary conditions.* The close resemblence of this partition function (1.9) to the above definition (1.4) of the pressure might motivate that definition. Due to this correspondence, we might expect $P(\varphi) = \lim_{n\to\infty} \frac{1}{n} \log \mathcal{L}_\varphi^n 1$.

The spectral properties of $\mathcal{L}_\varphi$ govern the behavior of $\mathcal{L}_\varphi^n$. In the case of expanding f and Hölder-continuous $\varphi \in C^\alpha$, we have the following result [34], where $\mathcal{L}_\varphi$ acts on $C^\alpha(X)$.

Theorem 1.1 $\exp P(\varphi)$ *is equal to the spectral radius of* $\mathcal{L}_\varphi$*, and the essential spectral radius is strictly smaller.* $\exp P(\varphi)$ *is a simple eigenvalue of* $\mathcal{L}_\varphi$ *(resp.* $\mathcal{L}_\varphi^*$*), and it has a strictly positive eigenfunction* Φ *(resp. a nonnegative measure* μ*). All other eigenvalues have strictly smaller modulus.*

Thus, we have a spectral gap for $\mathcal{L}$, which implies exponential convergence of $\frac{1}{n} \log \mathcal{L}_\varphi^n 1$ to the pressure $P(\varphi)$. Moreover, $\mu(\Phi)$ is the unique equilibrium state for φ.

We will show that this is no longer the case if one admits an indifferent fixed point of f, i.e. violates the expanding property of f.

1.2.5 Analyticity of the Pressure

Introducing the real parameter β (temperature), we clearly have continuity of the map $\beta \mapsto P(\beta\varphi)$. It is natural to investigate analyticity properties of this map, as the non-analyticities can be interpreted as phase transitions.

If the transfer operator $\mathcal{L}_{\beta\varphi}$ has an isolated leading eigenvalue that can be identified with the pressure, then, by standard perturbation theory [22], the analyticity of the interaction $\beta\varphi$ gives rise to analyticity of $P(\beta\varphi)$.

Thus, in the case of expanding f and Hölder-continuous φ, Theorem 1.1 implies real-analyticity of $P(\beta\varphi)$ in β, so that there are no phase transitions.

In the statistical mechanics interpretation, Hölder-continuity of φ corresponds to exponential decay of interactions.

However, in our setting this is not the case, as we have long-range interactions, the origin of which is the influence of the indifferent fixed point on the dynamics.

1.3 Continuous Maps of the Interval

Choose X to be the interval $I = [0, 1]$ and the interaction $\varphi = -\log|f'|$. Then, the transfer operator $\mathcal{L}_\beta$ (omitting φ) is

$$\mathcal{L}_\beta \Phi(x) = \sum_{fy=x} \frac{\Phi(y)}{|f'(y)|^\beta}.$$

This is a generalization of the Perron-Frobenius operator (for $\beta = 1$) which is used to describe densities of invariant measures of f. (Note that $\mathcal{L}_1^* \mu_L = \mu_L$ for μ_L Lebesgue measure on I.)

The expanding property of f is equivalent to $|f'| \geq \lambda_0 > 1$. By Theorem 1.1, an expanding $f \in C^{1+\alpha}$ leads to a spectral gap of $\mathcal{L}_\beta$ on $C^\alpha(X)$ and, thus, analyticity of the pressure $P(\beta)$ for all β.

In this dissertation, we want to investigate what happens if one weakens the expanding property of the map f and, in particular, admits an indifferent fixed point for f where the slope of f approaches 1. We will show that this gives rise to a phase transition of $P(\beta)$, the order of which depends crucially on the behavior of f near the indifferent fixed point.

1.4 The Class of Functions $\mathcal{C}_r$

In extension of the above formalism, we wish to consider certain piecewise monotone transformations of the interval $[0, 1]$ with an indifferent fixed point at the boundary.

The technical difficulty which arises through requiring only *piecewise* continuity will be dealt with in the next chapter. Of more interest is the existence of the indifferent fixed point, as it considerably alters the behavior of the dynamical system under iterations as opposed to the expanding case.

We specify the considered class of functions as follows. The functions will have two branches, one of which contains the indifferent fixed point and is responsible for the "laminar" phase. The asymptotic behavior in the neighborhood of the fixed point influences this laminar behavior and will turn out to be crucial for the dynamics. The other branch is the "chaotic" branch which facilitates reinjection into the "laminar" phase.

A typical $f \in \mathcal{C}_r$ is shown in Figure 1.1.

More specifically, for $r > 0$ we define

Definition 1.1 *A function f of the interval belongs to class $\mathcal{C}_r$ if*

1. *f is a map of the interval $I = [0,1]$ with fixed point 0, i. e.*

$$f : [0,1] \to [0,1], \quad f(0) = 0,$$

2. *there exists an $a \in]0,1[$ such that*

$$f[0,a[= [0,1[, \quad f[a,1] = [0,1],$$

3. *$f|_{[0,a[}$ extends to a C^1-diffeomorphism f_0 on $K = [0,a]$, and $f_1 = f|_{[a,1]}$ is a C^1-diffeomorphism on $J = [a,1]$, then the inverses are denoted as*

$$F_i = f_i^{-1}, \quad i = 1,\ 2,$$

4. *f is* almost expanding,

$$|f'| > 1 \text{ on }]0,a[\text{ and }]a,1[,$$

5. *for technical reasons (to guarantee that the induced system on J is expanding,)*

$$|f'| \geq \lambda_0 > 1 \text{ on } f^{-1}[a,1],$$

6. *and that the asymptotic behavior of f near the fixed point 0 is given by*

$$f(x) = x + cx^{1+r}(1 + r(x)) \tag{1.10}$$

with exponent $1 + r > 1$, some constant $c > 0$, and

$$r'(x) = O(x^{\alpha-1}), \quad x \to 0$$

for some $\alpha > 0$. (Without loss of generality, we will assume that $\alpha < r$.)

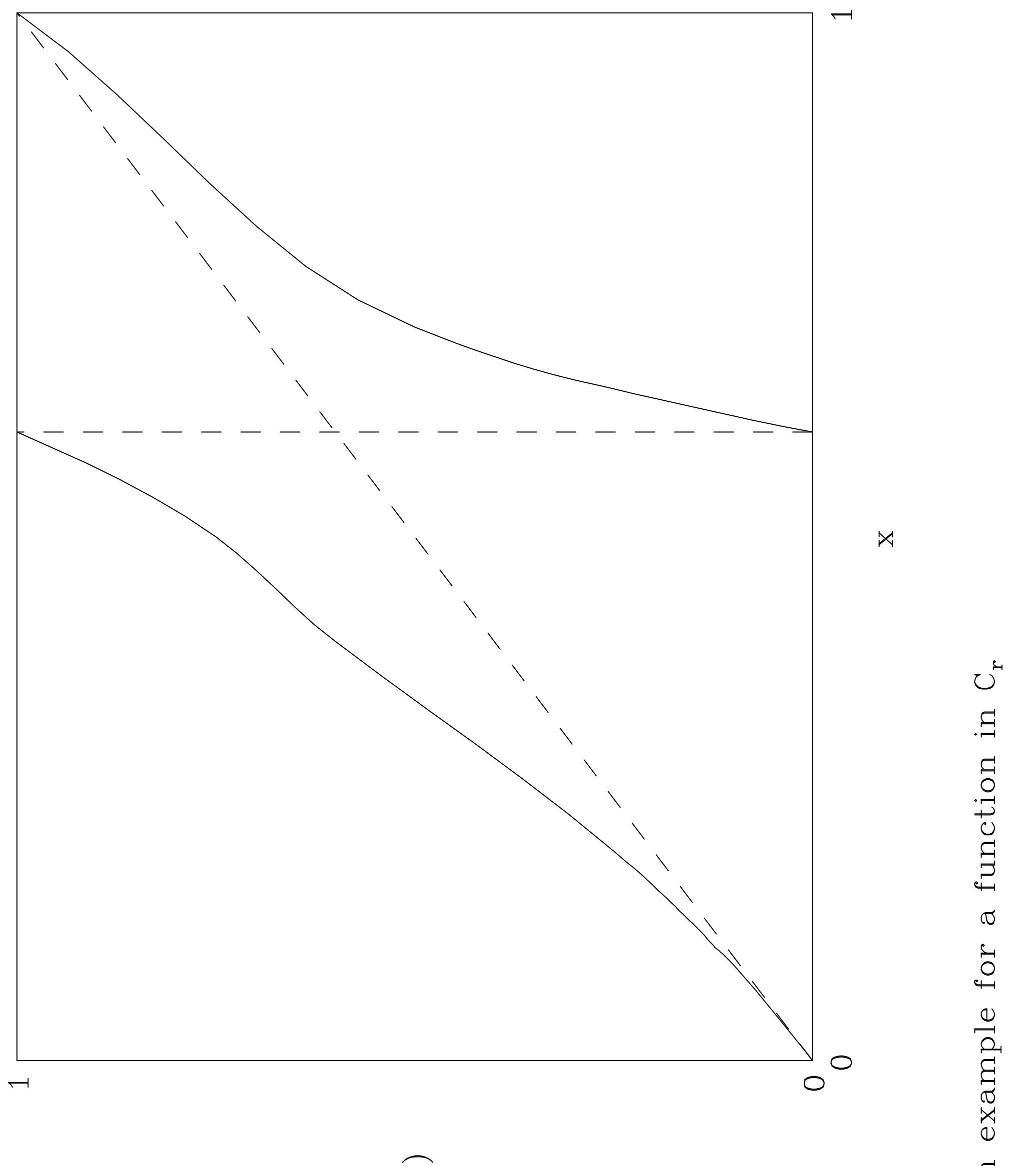


Figure 1.1:

The last property implies for the inverse function

$$F_0(x) = x - cx^{1+r}(1 + R(x)) \tag{1.11}$$

with $R'(x) = O(x^{\alpha-1})$, $x \to 0$.

(Note that $f'(x) = O(x^{\alpha-1})$ for $x \to 0$ implies $f(x) = O(x^{\alpha})$ for $x \to 0$ whereas the converse is only true under additional assumptions, e. g. if f' is monotone in a neighborhood of 0.)

Later, we will also consider F_0 and F_1 with suitable Hölder-continuity of their derivatives. Taking into account the asymptotic behavior of F_0, we demand

$$x^{-r}(F_0{}'(x) - 1) \in C^{\epsilon}(I) \qquad \text{and} \qquad F_1{}' \in C^{\epsilon}(I)$$

for some $\epsilon \leq \alpha$. Equivalent to the first condition, we can write $xR'(x) \in C^{\epsilon}(I)$ where the multiplication with x compensates for the $O(x^{\alpha-1})$ behavior of $R'(x)$.

Further, we write

$$a_n = F_0{}^n(1), \quad n \in \mathbf{N_0}; \qquad b_n = F_1 F_0{}^{n-1}(1), \quad n \in \mathbf{N}. \tag{1.12}$$

1.4.1 Example: Piecewise Linear Map

Lacking the smoothness required above, the below defined map $\tilde{f}$ does not belong to $\mathcal{C}_r$ itself. However, it can be seen as an exactly solvable toy model [40, 41, 37]. As stated in the introduction, this linearization gives rise to Fisher models with cluster interaction. The essential features of this model can be transfered to the case of smooth mappings. Due to its simplicity, it is instructive to include this example in our exposition.

Given a sequence (a_n) of real numbers such that

$$a_0 = 1, a_1, a_2, \ldots \searrow 0, \tag{1.13}$$

we define the piecewise linear map $\tilde{f}$ such that $\tilde{f}(a_n) = a_{n-1}$ for $n \in \mathbf{N}$:

$$\tilde{f}(x) = \begin{cases} a_n + (x - a_{n+1})\dfrac{a_{n-1} - a_n}{a_n - a_{n+1}}, & a_{n+1} \le x < a_n, \quad n \in \mathbf{N} \\ 1 - \dfrac{x - a_1}{1 - a_1}, & a_1 \le x \le 1 \end{cases} \tag{1.14}$$

Writing the magnitude of the slopes of $\tilde{f}$ on $]a_n, a_{n+1}]$ as

$$d_n = \begin{cases} \dfrac{a_{n-1} - a_n}{a_n - a_{n+1}}, & n \in \mathbf{N} \\ \dfrac{1}{1 - a_1}, & n = 0 \end{cases}, \tag{1.15}$$

this simplifies to

$$\tilde{f}(x) = \begin{cases} a_n + d_n(x - a_{n+1}), & a_{n+1} \le x < a_n, \quad n \in \mathbf{N} \\ 1 - d_0(x - a_1), & a_1 \le x \le 1 \end{cases} \tag{1.16}$$

and the inverses are given as

$$\begin{aligned} \tilde{F}_0(x) &= a_{n+1} + (x - a_n)/d_n, \quad a_n < x \le a_{n-1}, \quad n \in \mathbf{N} \\ \tilde{F}_1(x) &= 1 - x/d_0, \qquad\qquad 0 \le x \le 1. \end{aligned}$$

The asymptotic behavior of $\tilde{f}$ close to the origin is given by the asymptotic behavior of the sequence (a_n). In particular, if the sequence (1.13) is given by (1.12) for a function $f \in \mathcal{C}_r$ we can interpret $\tilde{f}$ as a linearized version of f in $\mathcal{C}_r$. By construction, we have $a_n = {F_0}^n(1)$ and (1.12) implies

$$b_n = 1 - (1 - a_1)a_{n-1}. \tag{1.17}$$

As we will show later, for a function in $\mathcal{C}_r$ we have an asymptotic behavior of (a_n) as

$$a_n \sim (rcn)^{-\frac{1}{r}}, \quad n \to \infty. \tag{1.18}$$

1.4.2 Example: The Farey Map

The Farey map [12], whose significance we mentioned in the introduction, is defined as

$$f(x) = \begin{cases} x/(1-x), & 0 \le x < 1/2 \\ (1-x)/x, & 1/2 \le x \le 1 \end{cases} \tag{1.19}$$

with the inverses

$$F_0(x) = x/(1+x), \qquad F_1(x) = 1/(1+x) \tag{1.20}$$

The Farey map is in class $\mathcal{C}_r$ for $r = 1$.

Here,

$$a_n = \frac{1}{n+1}, \quad n \in \mathbf{N_0}; \qquad b_n = \frac{n}{n+1}, \quad n \in \mathbf{N} \tag{1.21}$$

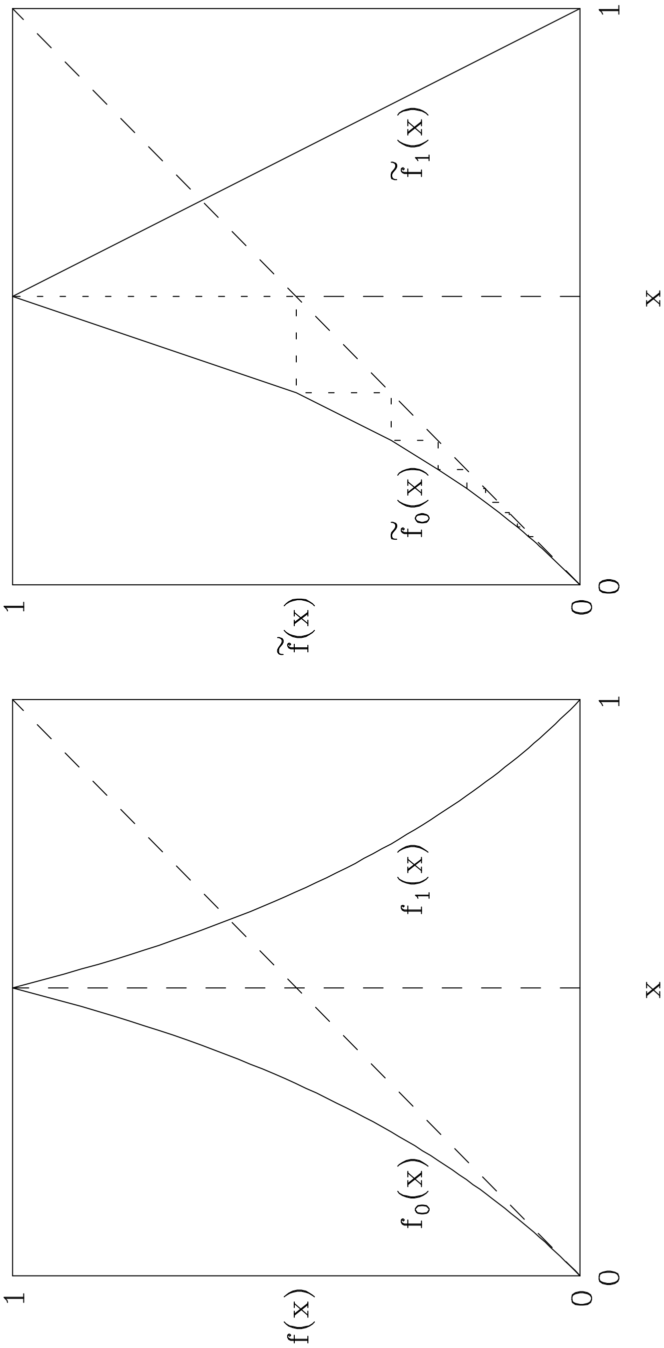
so that its piecewise linearized version is given by

$$\tilde{f}(x) = \begin{cases} \frac{n+2}{n}\, x - \frac{1}{n(n+1)}, & \frac{1}{n+2} \le x < \frac{1}{n+1}, \qquad n \in \mathbf{N} \\ 1 - 2x, & \frac{1}{2} \le x \le 1. \end{cases} \tag{1.22}$$

The inverses are given as

$$\begin{aligned} \widetilde{F}_0(x) &= \frac{n}{n+2}\, y + \frac{1}{(n+1)(n+2)}, \qquad \frac{1}{n+1} \le x < \frac{1}{n}, \quad n = \mathbf{N} \\ \widetilde{F}_1(x) &= 1 - \tfrac{1}{2}x, \qquad 0 \le x \le 1. \end{aligned}$$

The Farey map and its linearized version are shown in Figure 1.2.



The Farey map f with the piecewise linearized version $\tilde{f}$

Figure 1.2:

Chapter 2

The Thermodynamic Formalism for Piecewise Continuous Mappings

Following [3, 18], we introduce the class of piecewise continuous mappings of the interval and adapt a suitable version of the thermodynamic formalism for these mappings by embedding the interval I into a larger space X in which the transformation becomes continuous. A suitable function space for the transfer operator $\mathcal{L}$ is the space of functions with bounded variation $BV(X)$ resp. $BV(I)$, as one has identification of $\log \mathrm{r}(\mathcal{L})$ with the pressure [3]. Moreover, the extension of I to X does not change the spectral properties of $\mathcal{L}$ significantly. In the space BV, we also have a formula for the essential spectral radius $\mathrm{r}_{\mathrm{ess}}(\mathcal{L})$ [3].

In order to prepare the investigation of the existence of a spectral gap, we then introduce a *modified transfer operator* $\mathcal{M}_z$ for the induced map on a subinterval $J \subset I$. We show that there is a natural extension ${\mathcal{M}_z}^+$ of this operator to $BV(J)$ and that there are operator relations connecting $\mathcal{M}_z$ with $\mathcal{L}$. Using this new formalism, it is possible to get information on spectral properties of $\mathcal{L}$ through the investigation of $\mathcal{M}_z$.

2.1 Piecewise Continuous Mappings

Let I be a closed interval and suppose that f is a piecewise monotone transformation of I, i.e. there is an at most countable partition $\mathcal{Z} = \{Z_i : \ i \in \mathcal{I}\}$ of I into intervals such that, for each $Z \in \mathcal{Z}$, $f|_Z$ is strictly monotone and continuous. Write $F_i = {f|_{Z_i}}^{-1}$, $i \in I$. Moreover, suppose that $\mathcal{Z}$ is a generating partition for f. A typical piecewise monotone map is shown in Figure 2.1.

Out of convenience, we would like to view f and F_i as being continuously extended to each $\mathrm{cl}(Z_i)$ respectively $\mathrm{cl}(fZ_i)$.

Technically, this involves the investigation of the dynamical system on a somewhat larger space [18]. One doubles all boundary points of the partition inside the interval and their preimages as follows:

Write $\partial\mathcal{Z} = \bigcup_{Z_i \in \mathcal{Z}} \partial Z_i$. Substitute each $x \in \bigcup_{k=0}^{\infty} f^{-k}(\partial\mathcal{Z}) \setminus \partial I$ by two points x^- and x^+ and denote this new space by X. Setting $u < x^- < x^+ < v$ in X if $u < x < v$ in I extends the order to X, and the order topology on X is compact.

The space X, enlarged by at most countably many points, has a "Cantor set"-like structure. The partition $\mathcal{Z}$ consists of closed intervals.

Now, f extends in an obvious way to a continuous transformation on X: $f(x^-) = \lim_{y \nearrow x} f(y)$, $f(x^+) = \lim_{y \searrow x} f(y)$. ($s$ extends in a similar way.)

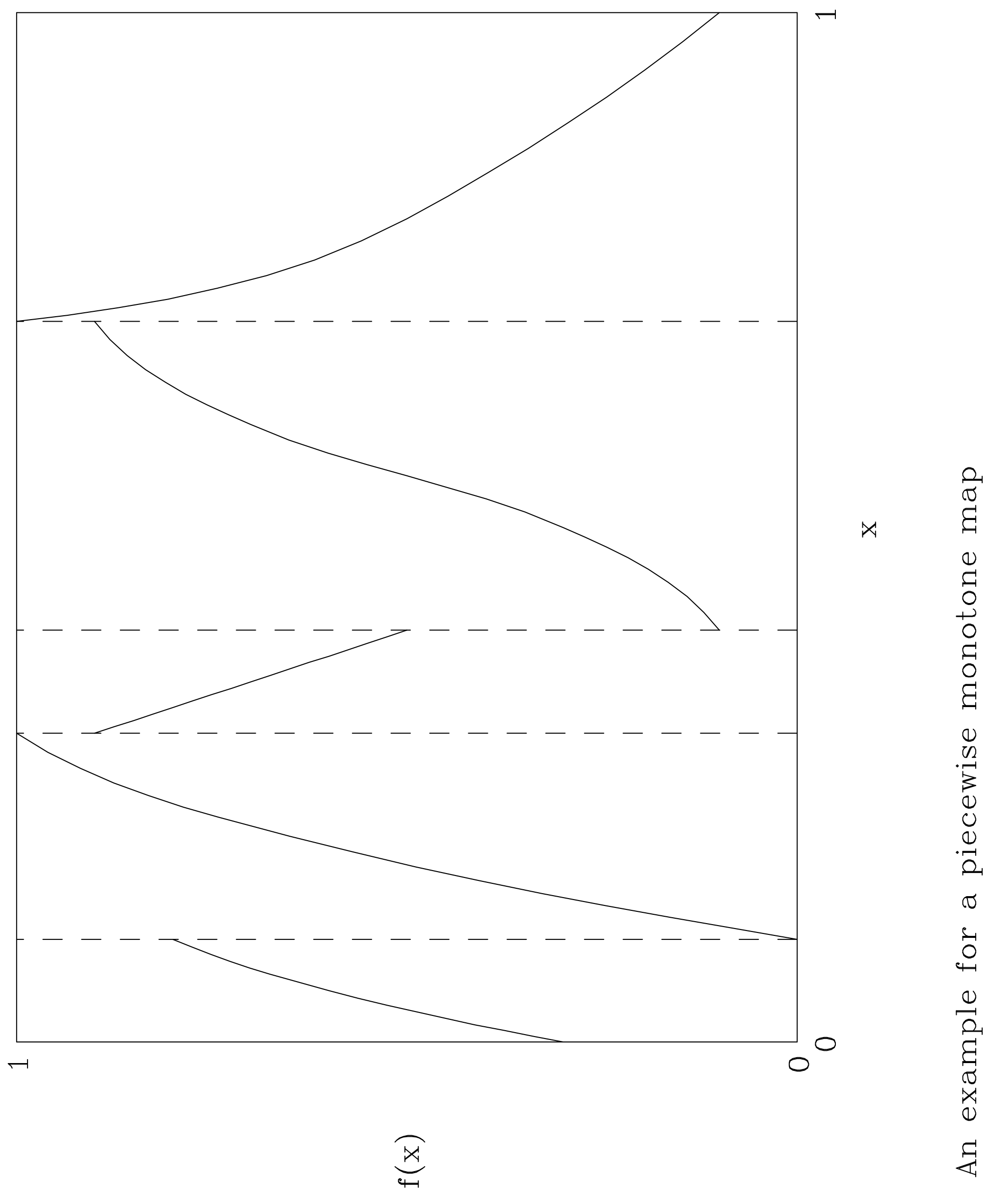


Figure 2.1:

A nonatomic measure on I can be lifted to X and the resulting dynamical system still has the same spectral properties as the original one. Also, any measure on X induces a measure on I.

2.2 The Essential Spectrum and a Decomposition Theorem

Before we introduce the transfer operator for this dynamical system, we need to recall some spectral properties of operators. Let A be a bounded linear operator on a Banach space B.

We define

Definition 2.1 *The* essential spectrum $\sigma_{\text{ess}}(A)$ *is the set of all* $\lambda \in \sigma(A)$ *such that one of the following holds:*

i) the range of $(A - \lambda)$ *is not closed.*

ii) $\bigcup_{n=0}^{\infty} \ker((A-\lambda)^n)$ *is of infinite dimension.*

iii) λ *is a limit point of* $\sigma(A)$.

The essential spectral radius *is defined as*

$$\mathrm{r}_{\text{ess}}(A) = \sup \{|\lambda| : \quad \lambda \in \sigma_{\text{ess}}(A)\}.$$

Nussbaum [27] has shown that $\mathrm{r}_{\text{ess}}(A)$ can be computed by an approximation with compact operators as follows:

$$\mathrm{r_{ess}}(A) = \lim_{n\to\infty} \left(\inf\left\{||A^n - K|| : \quad K \text{ compact}\right\}\right)^{1/n}.$$

This means that outside the essential spectral radius the spectrum of A is like the spectrum of a compact operator, as stated in the following theorem.

Theorem 2.1 (Lemma VIII.8.2 of [11]) *We have the following spectral decomposition of A:*

For each $\Theta > \mathrm{r_{ess}}(A)$, the operator A can be decomposed as

$$A = \sum_{i=1}^{N(\Theta)} \lambda_i \mathcal{P}_i \Lambda_i + \mathcal{P} A, \tag{2.1}$$

where $\mathcal{P}_i$, for $i = 1, \ldots, N(\Theta)$, and $\mathcal{P}$ are mutually orthogonal projections commuting with A such that $\mathcal{P} + \sum_{i=1}^{N(\Theta)} \mathcal{P}_i = \mathrm{Id}$. For each $i = 1, \ldots, N(\Theta)$, we have $|\lambda_i| > \Theta$, $\mathrm{rank}(\mathcal{P}_i) < \infty$, and $\Lambda_i = \mathcal{P}_i + N_i$, where N_i is nilpotent and $\mathcal{P}_i N_i = N_i \mathcal{P}_i = N_i$. Finally, $||\mathcal{P} A^m||_{BV} \leq \mathrm{const} \cdot \Theta^m$.

(It may happen that $\theta = \mathrm{r}(A)$, the spectral radius of A. In this case, $N(\Theta) = 0$.)

2.3 The Transfer Operator $\mathcal{L}$

Definition 2.2 *Given a function $s \in BV(I)$ such that*

$$\sum_{Z_i \in \mathcal{Z}} ||s||_{Z_i} < \infty, \tag{2.2}$$

the transfer operator $\mathcal{L}$ *is defined as*

$$\mathcal{L} : BV(I) \to BV(I), \quad \mathcal{L}\psi(x) = \sum_{fy=x} s(y)\ \psi(y) = \sum_{Z_i \in \mathcal{Z}} s \circ F_i(x)\ \psi \circ F_i(x). \tag{2.3}$$

Condition (2.2) in connection with $s \in BV(I)$ ensures that $||\mathcal{L}||_{BV(I)}$ is finite, since we have

$$
\begin{aligned}
||\mathcal{L}\psi||_I &\leq \left(\sum_{Z_i \in \mathcal{Z}} ||s||_{Z_i}\right) ||\psi||_I && (2.4) \\
\mathrm{var}_I(\mathcal{L}\psi) &\leq \mathrm{var}_I(s)||\psi||_I + ||s||_I \mathrm{var}_I(\psi). && (2.5)
\end{aligned}
$$

Again, it would be convenient to have F_i and $s \circ F_i$ continuously extended to $\mathrm{cl} f(Z_i)$, so that we will not have to bother with the endpoints of the partition. Technically, we therefore have to consider $\mathcal{L}$ on $BV(X)$ instead of $BV(I)$. (We will write $\mathcal{L}$ in both cases.)

However, the following theorem shows that this does not change the spectral properties of $\mathcal{L}$ outside the essential spectral radius $\mathrm{r}_{\mathrm{ess}}(\mathcal{L})$.

We write

$$
\psi \approx 0 \quad \text{if } \{x : \psi(x) \neq 0\} \text{ is at most countable.} \tag{2.6}
$$

Theorem 2.2 (Baladi, Keller [3]) *Let Y be I or X and let $\mathcal{L}$ act on $BV(Y)$.*

(1) If there exist $k \geq 1$ and $\lambda > \mathrm{r}(\mathcal{L})$ such that $(\mathcal{L} - \lambda)^k\ \psi = 0$ and $\psi \approx 0$, then $\psi(x) = 0$ for all $x \in X$.

(2) If there exist $k \geq 1$ and $\lambda \neq 0$ such that $(\mathcal{L} - \lambda)^k\ \psi \approx 0$, then there is $\psi_1 \in BV(X)$ such that $\psi_1 \approx \psi$ and $(\mathcal{L} - \lambda)^k\ \psi_1 = 0$.

Theorem 2.2 in particular says that there are corresponding eigenvalues and eigenfunctions of $\mathcal{L}$ in $BV(X)$ in $BV(I)$.

Corollary 2.1 *(a) Suppose that, for $\psi_I \in BV(I)$ and $\lambda \in \mathbf{C}$, $\mathcal{L}\psi_I = \lambda\psi_I$. Then there exists a function $\psi_X \in BV(X)$ such that $\mathcal{L}\psi_X = \lambda\psi_X$.*

(b) Suppose that, for $\psi_X \in BV(X)$ and $\lambda \in \mathbf{C}$, $\mathcal{L}\psi_X = \lambda\psi_X$. Then there exists a function $\psi_I \in BV(I)$ such that $\mathcal{L}\psi_I = \lambda\psi_I$.

Moreover, in both cases we have $\psi_I' \approx \psi_X$ with ψ_I' being the natural embedding of ψ_I in $BV(X)$.

PROOF: Suppose, for $\psi_I \in BV(I)$, $\mathcal{L}\psi_I = \lambda\psi_I$. Then the trivial embedding ψ_I' of ψ_I in $BV(X)$ fulfills $\mathcal{L}\psi_I' \approx \lambda\psi_I'$, and by Theorem 2.2 we have the existence of $\psi_X \approx \psi_I'$ such that $\mathcal{L}\psi_X = \lambda\psi_X$.

Conversely, if $\mathcal{L}\psi_X = \lambda\psi_X$ for $\psi_X \in BV(X)$, then any restriction ψ_X' of ψ_X in $BV(I)$ fulfills $\mathcal{L}\psi_X' \approx \lambda\psi_X'$, and by Theorem 2.2 we have the existence of $\psi_I \approx \psi_X'$ such that $\mathcal{L}\psi_I = \lambda\psi_I$. Clearly, for the embedding ψ_I' of ψ_I in $BV(X)$, we have also $\psi_X \approx \psi_I'$. $\square$

For convenience, we introduce

$$s_n(x) = s(f^{n-1}x) \cdot \ldots \cdot s(fx)\, s(x). \tag{2.7}$$

Then we can write powers of $\mathcal{L}$ as

$$\mathcal{L}^n\psi(x) = \sum_{f^n(y)=x} s_n(y)\, \psi(y). \tag{2.8}$$

The following theorem gives a formula for the essential spectral radius of $\mathcal{L}$.

Theorem 2.3 (Ibid.) *Let Y be I or X and let $\mathcal{L}$ act on $BV(Y)$. The essential spectral radius $\mathrm{r}_{\mathrm{ess}}(\mathcal{L})$ is given by*

$$\mathrm{r}_{\mathrm{ess}}(\mathcal{L}) = \inf_n ||s_n||_Y^{1/n}. \tag{2.9}$$

Corollary 2.2 $\mathrm{r}_{\mathrm{ess}}(\mathcal{L})$ *is equal for* $\mathcal{L}$ *acting on* $BV(I)$ *and* $BV(X)$*.*

Proof: Clearly, $||s_n||_I = ||s_n||_X$ by construction of the extension to X. $\square$

The next theorem establishes a connection between properties of $\mathcal{L}$ and the pressure of the dynamical system. In particular, it shows that the space of functions with bounded variation is indeed suitably chosen.

Theorem 2.4 (Ibid.) *Suppose* s_{Z_i} *is continuous for each* $Z_i \in \mathcal{Z}$*. Then*

$$P(f, \log|s|) \geq \mathrm{r}(\mathcal{L}). \tag{2.10}$$

Suppose that additionally $s \geq 0$*. Then*

$$P(f, \log|s|) = \mathrm{r}(\mathcal{L}), \tag{2.11}$$

and $\lambda = \mathrm{r}(\mathcal{L})$ *is an eigenvalue of* $\mathcal{L}$*, provided* $\mathrm{r}(\mathcal{L}) > \mathrm{r}_{\mathrm{ess}}(\mathcal{L})$*.*

We add a theorem with a more detailed result on the largest eigenvalue.

Theorem 2.5 *Suppose* $s > 0$ *and* $\mathrm{r}(\mathcal{L}) > \mathrm{r}_{\mathrm{ess}}(\mathcal{L})$*. Then* $\lambda = \mathrm{r}(\mathcal{L})$ *is an eigenvalue of multiplicity* 1 *with a positive eigenfunction* Ψ_λ *bounded away from* 0*.*

Proof: Choose $0 \neq \Psi_\lambda \in BV(I)$ with $\mathcal{L}\Psi_\lambda = \lambda\Psi_\lambda$. Then

$$\mathcal{L}(\Re\Psi_\lambda) = \Re(\mathcal{L}\Psi_\lambda) = \Re(\lambda\Psi_\lambda) = \lambda\Re(\Psi_\lambda)$$

and we can assume Ψ_λ to be real.

Suppose $\Psi_\lambda(x) \geq 0$ for some $x \in I$. $\mathcal{Z}$ generates, whence $\Psi_\lambda \geq 0$ on a dense subset $\Omega(x) \subset I$, given by all inverse images of x.

If in addition $\Psi_\lambda(y) \leq 0$ for some $y \in I$, then $\Psi_\lambda \leq 0$ on the dense subset $\Omega(y) \subset I$. For $\Psi_\lambda \in BV(I)$ left and right limits exists. Taking these limits over $\Omega(x)$ and $\Omega(y)$ must give the same result, thus all the left and right limits are 0. Hence $\Psi_\lambda \approx 0$, and by the above theorem we see that $\Psi_\lambda = 0$ on I.

Thus, $\Psi_\lambda > 0$. Assume that Ψ_λ is not bounded away from zero, i.e. there is $(x_n) \in I$ such that $\Psi_\lambda(x_n) \to 0$. Then there are y and a subsequence $x_{n_k} \to y$ with $\Psi_\lambda(x_{n_k}) \to 0$.

Thus, $\Omega(y)$ consists of points where the left or right limit of Ψ_λ is equal to 0. Now, for each $x \in I$ there is a sequence $(y_n) \subset \Omega(y)$ that converges to y and there are sequences $(x_k^{(y_n)}) \subset I$ converging to y_n with $\Psi_\lambda(x_k^{(y_n)})$ converging to 0 for each n.

Then the diagonal sequence $(x_n^{(y_n)})$ converges to x, and $\Psi_\lambda(x_n^{(y_n)})$ converges to 0. Choosing (y_n) suitably, we get that all left and right limits are equal to zero. Hence, $\Psi_\lambda \approx 0$ in contradiction to $\Psi_\lambda > 0$.

So far we have proven that $\Psi_\lambda \geq \alpha > 0$ for any eigenfunction Ψ_λ of $\mathcal{L}$ with eigenvalue $\lambda = \mathrm{r}(\mathcal{L})$. Now, using a standard argument [29], suppose that there is a $\Psi_\lambda \neq {\Psi_\lambda}' \in BV(I)$ with $\mathcal{L}{\Psi_\lambda}' = \lambda {\Psi_\lambda}'$. Again, we can pick ${\Psi_\lambda}'$ real, and we choose $t \in \mathbf{R}$ such that $\Psi_\lambda - t{\Psi_\lambda}' \geq 0$ with $\Psi_\lambda(x) = t{\Psi_\lambda}'(x)$ for some $x \in I$. Repeating the above argument, we see that $\Psi_\lambda = t{\Psi_\lambda}'$, i.e. the eigenvalue λ is simple. $\square$

At this point, one would also like to show the existence of a *spectral gap* for $\mathcal{L}$, i.e. that the following holds for $\mathrm{r}_{\mathrm{ess}}(\mathcal{L}) < \mathrm{r}(\mathcal{L})$:

$$\sup\{|\lambda| : \mathrm{r}(\mathcal{L}) \neq \lambda \in \sigma(\mathcal{L})\} < \mathrm{r}(\mathcal{L}). \tag{2.12}$$

For this to hold, it is enough to show that there are no other eigenvalues of $\mathcal{L}$ with magnitude equal to $\mathrm{r}(\mathcal{L})$. However, in general it is only possible to show that the set of eigenvalues with magnitude equal to $\mathrm{r}(\mathcal{L})$ is cyclic (see e.g. [11]).

Thus, we are left with two questions. First, one would like to know under what conditions one has a leading eigenvalue or, equivalently, when $\mathrm{r}_{\mathrm{ess}}(\mathcal{L}) < \mathrm{r}(\mathcal{L})$ holds. Second, even if this is shown one still has to investigate separately whether $\mathcal{L}$ has a spectral gap.

(In a different setting [33], Ruelle has shown that this is always the case for an expanding map f with Hölder-continuous interaction $\log s$.)

In order to accomplish this task, it will be convenient to investigate certain transfer operators in induced subsystems as outlined in the next section.

2.4 The Induced Map and the Modified Transfer Operator $\mathcal{M}_z$

Let $J \subset I$ be a closed interval. We define

$$\begin{aligned} J_n &= \Big\{x \in J : \ f^i(x) \notin J,\ i = 1, \ldots, n-1,\ f^n(x) \in J\Big\}, \qquad n \in \mathbf{N} \quad (2.13) \\ K_n &= \Big\{x \in J : \ f^i(x) \notin J,\ i = 1, \ldots, n-1\Big\} \ = \ \bigcup_{k=n}^{\infty} J_k, \qquad n \in \mathbf{N} \quad (2.14) \end{aligned}$$

$\mathcal{Z}$ is generating for f, hence

$$J_\infty = J \setminus \bigcup_{k=1}^{\infty} J_k \qquad (2.15)$$

is at most countable. Now, define

$$n(x) = n \qquad \text{if} \qquad x \in J_n, \quad n \in \mathbf{N} \cup \{\infty\}. \qquad (2.16)$$

After this preparation, as in [31], we define

Definition 2.3 *The* induced *or* first-return map *is the map* $g: \; J \setminus J_\infty \to J \setminus J_\infty$ *given by*

$$g(x) = f^{n(x)}(x). \tag{2.17}$$

Defining $g(x)$ arbitrarily on J_∞, we extend g to all of J. Due to the fact that f is piecewise continuous, each J_n can be written as an at most countable union of intervals. The partition of J into these intervals is generating for g. (In the special case of functions in $\mathcal{C}_r$ and inducing on $J = [a, 1]$, each J_n is itself an interval. Then we will write $G_n = g|_{J_n}{}^{-1}$.)

Thus, both f and g are piecewise continuous mappings of the interval with generating partitions.

Now, we can proceed to introducing a new transfer operator for the induced dynamical system. We define the modified transfer operator as follows:

Definition 2.4 *For* $z \in \mathbf{C}$ *and* $s \in BV(I)$, *the* modified *transfer operator* $\mathcal{M}_z$ *is defined as*

$$\begin{aligned} \mathcal{M}_z: \; BV(J) \to BV(J), \quad \mathcal{M}_z\phi(x) &= \sum_{g(y)=x} z^{n(y)} s_{n(y)}(y)\, \phi(y) \\ &= \sum_{n=1}^{\infty} z^n \sum_{g(y)=x,\, y \in J_n} s_n(y)\, \phi(y). \end{aligned} \tag{2.18}$$

Thus, $\mathcal{M}_z$ is an operator-valued power series in z,

$$\mathcal{M}_z = \sum_{n=1}^{\infty} z^n \mathcal{M}_n \tag{2.19}$$

with

$$\mathcal{M}_n\phi(x) = \sum_{g(y)=x,\ y\in J_k} s_n(y)\ \phi(y) \tag{2.20}$$

which has a radius of convergence equal to $r = \lim_{n\to\infty} ||\mathcal{M}_n||_{BV(J)}^{-1/n}$.

(For functions in $\mathcal{C}_r$ and inducing on $J = [a, 1]$, we can write the operator as $\mathcal{M}_n\phi(x) = s_n \circ G_n(x)\ \phi \circ G_n(x)$.)

Analoguous to above, we extend J to an enlarged space Y and extend $\mathcal{L}$ by continuous extension along the inverse branches of g.

Theorem 2.3 can be applied to $\mathcal{M}_z$, as we have a piecewise monotone transformation of the interval with a generating partition. (Naturally, the conditions on $\mathcal{M}_z$ still have to be verified.)

Denoting $\underline{n}_k(y) = (n_1, \dots, n_k)$, where $g^{i-1}y \in J_{n_i}$, $i = 1, \dots, k$, we write

$$s_{\underline{n}_k(y)}(y) = s_{n_k}(g^{k-1}y) \cdot \ldots \cdot s_{n_2}(gy) s_{n_1}(y).$$

Using $|\underline{n}_k| = (n_1, \dots, n_k)$, the formula for the essential spectral radius is

$$\mathrm{r}_{\mathrm{ess}}(\mathcal{M}_z) = \inf_k \left|\left| z^{|\underline{n}_k(y)|} s_{\underline{n}_k(y)}(y) \right|\right|_J^{1/k} \tag{2.21}$$

2.5 The Extension $\mathcal{M}_z^{\ +}$ of $\mathcal{M}_z$

In order to relate the operators $\mathcal{M}_z$ and $\mathcal{L}$ to each other, we must analyze $\mathcal{M}_z$ further. We decompose

$$\mathcal{L}\psi = \mathcal{L}(\chi_{J^c}\psi) + \mathcal{L}(\chi_J\psi) = \mathcal{L}_0\psi + \mathcal{L}_1\psi. \tag{2.22}$$

Then one can check that

$$\mathcal{M}_n\phi = \mathcal{L}^n(\chi_{J_n}\phi) = \chi_J {\mathcal{L}_0}^{n-1}\mathcal{L}_1\phi \tag{2.23}$$

holds. Here we see that multiplication with χ_J alone is "responsible" for the restriction to $BV(J)$. Thus, it is natural to define an extension ${\mathcal{M}_n}^+$ of $\mathcal{M}_n$ by omitting the χ_J. This leads to

$$ {\mathcal{M}_n}^+ \phi = \mathcal{L}^n(\chi_{K_n}\phi) = {\mathcal{L}_0}^{n-1}\mathcal{L}_1\phi \tag{2.24} $$

and we define

$$ {\mathcal{M}_z}^+ = \sum_{n=1}^{\infty} z^n {\mathcal{M}_n}^+. \tag{2.25} $$

(We will use ${\mathcal{M}_z}^+$ both for the operator mapping $BV(J)$ into $BV(I)$ and for the operator acting on $BV(I)$.)

Relations (2.23) and (2.24) imply

Lemma 2.1 *The radii of convergence for both $\mathcal{M}_z$ and ${\mathcal{M}_z}^+$ are bounded below by $1/\mathrm{r}(\mathcal{L}_0)$.*

Unfortunately, the extension ${\mathcal{M}_z}^+$ is no longer given by a piecewise monotone transformation, so that Theorem 2.3 does not apply in this case. (However, a theorem of Ruelle [35], generalizing Theorem 2.3, can be applied and gives an upper bound on $\mathrm{r}_{\mathrm{ess}}({\mathcal{M}_z}^+)$.)

Still, we can get information on the eigenvalues of ${\mathcal{M}_z}^+$ through the following Lemma.

Lemma 2.2 *Suppose that ${\mathcal{M}_z}^+$ is bounded. Then*

$$ \sigma_p(\mathcal{M}_z) \setminus \{0\} = \sigma_p({\mathcal{M}_z}^+) \setminus \{0\} $$

and the geometric multiplicity of the eigenspaces to an eigenvalue $\lambda \neq 0$ are identical. Moreover, the corresponding eigenfunctions ϕ of $\mathcal{M}_z$ and ψ of ${\mathcal{M}_z}^+$ are related by

$$\lambda\psi = {\mathcal{M}_z}^+\phi, \qquad \phi = \psi_J.$$

PROOF: Suppose $\phi \in BV(I)$ is eigenfunction of ${\mathcal{M}_z}^+$ with eigenvalue λ. Then, $\mathcal{M}_z\phi_J = ({\mathcal{M}_z}^+\phi_J)_J = ({\mathcal{M}_z}^+\phi)_J = \lambda\phi_J$, Moreover, $\phi_J \neq 0$, for, if $\phi_J = 0$ then $\phi = \lambda^{-1}{\mathcal{M}_z}^+\phi = {\mathcal{M}_z}^+\phi_J = 0$. Thus, ϕ_J is eigenfunction of $\mathcal{M}_z$ with the same eigenvalue.

Conversely, suppose that $\psi \in BV(J)$ is an eigenfunction of $\mathcal{M}_z$ with eigenvalue λ. Then, extending to $\phi = \lambda^{-1}{\mathcal{M}_z}^+\psi$, ϕ is nonzero and eigenfunction of ${\mathcal{M}_z}^+$ with the same eigenvalue, for we have $\phi_J = \lambda^{-1}({\mathcal{M}_z}^+\psi)_J = \psi$ and thus, ${\mathcal{M}_z}^+\phi = {\mathcal{M}_z}^+\phi_J = {\mathcal{M}_z}^+\psi = \lambda\phi$. □

Thus, we can restrict ourselves to the investigation of eigenfunctions of $\mathcal{M}_z$.

2.6 Operator Relations between $\mathcal{M}_z$ and $\mathcal{L}$

Now, the following operator relations emerge:

Theorem 2.6

$$(1 - z\mathcal{L}_0)(1 - {\mathcal{M}_z}^+)\psi = (1 - z\mathcal{L})\psi, \qquad \psi \in BV(I) \tag{2.26}$$

$$(1 - z\mathcal{L}){\mathcal{M}_z}^+\phi = z\mathcal{L}_1(1 - \mathcal{M}_z)\phi, \qquad \phi \in BV(J). \tag{2.27}$$

PROOF: On $BV(I)$ we have

$$(1 - z\mathcal{L}_0)(1 - {\mathcal{M}_z}^+) \;\; = \;\; (1 - z\mathcal{L}_0)(1 - \sum_{n=1}^{\infty} z^n {\mathcal{L}_0}^{n-1}\mathcal{L}_1)$$

$$
\begin{aligned}
&= 1 - z\mathcal{L}_0 + \sum_{n=1}^{\infty} z^{n+1}{\mathcal{L}_0}^n\mathcal{L}_1 - \sum_{n=1}^{\infty} z^n{\mathcal{L}_0}^{n-1}\mathcal{L}_1 \\
&= 1 - z\mathcal{L}_0 - z\mathcal{L}_1 = 1 - z\mathcal{L}
\end{aligned}
$$

and on $BV(J)$

$$
\begin{aligned}
z\mathcal{L}_1(1-\mathcal{M}_z) &= z\mathcal{L}(\chi_J(1-\chi_J{\mathcal{M}_z}^+)) \\
&= z\mathcal{L}_1(1-{\mathcal{M}_z}^+) \\
&= z\mathcal{L}_1 - z\mathcal{L}_1\sum_{n=1}^{\infty} z^n{\mathcal{L}_0}^{n-1}\mathcal{L}_1) \\
&= \sum_{n=1}^{\infty} z^n{\mathcal{L}_0}^{n-1}\mathcal{L}_1 - \sum_{n=1}^{\infty} z^{n+1}{\mathcal{L}_0}^n\mathcal{L}_1 - z\mathcal{L}_1\sum_{n=1}^{\infty} z^n{\mathcal{L}_0}^{n-1}\mathcal{L}_1) \\
&= (1 - z\mathcal{L}_0 - z\mathcal{L}_1)\sum_{n=1}^{\infty} z^n{\mathcal{L}_0}^{n-1}\mathcal{L}_1) \\
&= (1 - z\mathcal{L}){\mathcal{M}_z}^+. \;\Box
\end{aligned}
$$

Remark 2.1 *The proof of the operator relations is purely algebraic and thus independent of the specific choice of function spaces.*

2.7 Eigenvalues of $\mathcal{L}$

The operator relations of Theorem 2.6 provide a method to show the existence of eigenvalues and eigenfunctions of $\mathcal{L}$. It is exactly this new relation which will enable us to investigate the behavior of the transfer operator in Chapter 4.

Theorem 2.7 *Suppose $0 \neq |z| < 1/\mathrm{r}(\mathcal{L}_0)$. Then $z^{-1} \in \sigma_p(\mathcal{L})$ if and only if $1 \in \sigma_p(\mathcal{M}_z)$. Moreover, the geometric multiplicity of the eigenvalues 1 and z^{-1} are the same and the corresponding eigenfunctions ψ of $\mathcal{L}$ and ϕ of $\mathcal{M}_z$ are related by*

$$\psi = {\mathcal{M}_z}^+\phi.$$

PROOF: Suppose $\mathcal{M}_z\phi = \phi$. Then, extending $\psi = {\mathcal{M}_z}^+\phi$ and applying (2.27), we have

$$(1 - z\mathcal{L})\psi = (1 - z\mathcal{L}){\mathcal{M}_z}^+\phi = z\mathcal{L}_1(1 - \mathcal{M}_z)\phi = z\mathcal{L}_1 0 = 0.$$

Thus, ψ is eigenvector of $\mathcal{L}$ with eigenvalue z^{-1}.

Conversely, suppose $z\mathcal{L}\psi = \psi$. Then, applying (2.26) we have

$$(1 - z\mathcal{L}_0)(1 - {\mathcal{M}_z}^+)\psi = (1 - z\mathcal{L})\psi = 0$$

and thus, $(1 - {\mathcal{M}_z}^+)\psi = 0$. Restricting $\phi = \psi_J$, clearly ϕ is an eigenfunction of $\mathcal{M}_z$ with eigenvalue 1.

As ${\mathcal{M}_z}^+$ uniquely extends ϕ to $\psi = {\mathcal{M}_z}^+\phi$, the geometric multiplicity of the corresponding eigenvalues is the same. $\square$

2.8 Generalizations of the Formalism

Note that the above developed formalism still holds in a more general setting. Formally, for an arbitrary transformation f of a set I, the notion of the induced map g along with the definition of the sets J_n and K_n holds for any subset J. Choosing a suitable topology on I and a "nice" subset J, $\mathcal{L}$ and $\mathcal{M}_z$ make sense on suitable function spaces, so that one can expect to relate their properties via the operator relations of Theorem 2.6. Thus, this method enjoys a wide range of possible applications.

Chapter 3

Asymptotics of Iterations

We investigate the iteration of a function F_0 with indifferent fixed points. First, we present the relation of this iteration to the intermittency renormalization. Next, the relation to Abel's equation and the theory of iterative functional equations is explored. The asymptotic behavior of a solution of Abel's equation is used to get sharp bounds on the asymptotics of the iteration of F_0. Moreover, these estimates enable us to show convergence of the intermittency renormalization.

3.1 Intermittency Renormalization and Abel's Equation

We want to be able to control the iteration of functions $f \in \mathcal{C}_r$. This is closely related to the investigation of the *intermittency renormalization transformation* (see e. g.

[19].) This is the transformation

$$T_\gamma g(x) = \gamma g^2(x/\gamma) \tag{3.1}$$

with intermittency boundary condition

$$g(0) = 0, \quad g'(0) = 1. \tag{3.2}$$

The fixed point $g = T_\gamma g$ of this transformation is explicitly known,

$$g(x) = \{x^{-r} + a\}^{-\frac{1}{r}} \tag{3.3}$$

with $\gamma = 2^{\frac{1}{r}}$. The expansion at $x = 0$ yields

$$g(x) = x - \frac{a}{r}x^{r+1} + O(x^{2r+1}), \quad x \to 0 \tag{3.4}$$

which coincides with the expansion (1.11) of $F_0(x)$ for $a = cr$ if F_0 is given by a function $f \in \mathcal{C}_r$.

We will see that iterates of F_0 converge against iterates of this fixed point g. More specifically, we need an uniform asymptotic estimate of the difference between iterates of F_0 and iterates of the fixed point.

In order to get this estimate, we utilize a connection of the renormalization transformation to Abel's functional equation,

$$G(F_0(x)) - G(x) = 1. \tag{3.5}$$

(For a review of the theory of iterative functional equations, see e. g. [25].) Solving this equation for G for given F_0 makes it possible to explicitly determine higher iterates of this function by

$$F_0{}^n(x) = G^{-1}(n + G(x)). \tag{3.6}$$

The advantage of this formula is that it explicitely contains the number of iterations n.

The connection to Abel's equation emerges out of a smooth conjugation of F_0 to the fixed point g:

$$h(F_0(x)) = g(h(x)) \tag{3.7}$$

with h being a C^1-diffeomorphism such that $h(0) = 0$ and $h(1) = 1$. Rewriting this equation we get

$$(h(F_0(x)))^{-r} = (h(x))^{-r} + a.$$

Choosing $G(x) = \frac{1}{a}(h(x))^{-r}$, this is equivalent to Abel's equation (3.5).

Thus, we see that if F_0 is equal to the fixed point $g = T_\gamma g$ then Abel's equation is explicitely solvable with $G(x) = \frac{1}{a}x^{-r}$ and iterates of g are given by $G^{-1}(n + G(x))$, i.e.

$$g^n(x) = \{x^{-r} + na\}^{-\frac{1}{r}}. \tag{3.8}$$

3.2 Abel's Equation and Asymptotics of Iterations

For convenience, we now define the function F_0 explicitely instead of relating F_0 to $f \in \mathcal{C}_r$:

Definition 3.1

$$F_0 : \quad [0, 1] \to [0, a]$$

is a C^1-diffeomorphism with

$$0 < F_0' < 1 \ \textit{on} \ ]0, 1[$$

and asymptotic behavior

$$F_0(x) = \left\{x^{-r} + rc(1 + \widetilde{R}(x))\right\}^{-\frac{1}{r}} \tag{3.9}$$

with $r, c > 0$ *and*

$$\widetilde{R}'(x) = O(x^{\alpha-1}), \quad x \to 0$$

for some $\alpha > 0$. *(Without loss of generality, choose* $\alpha < r$.*)*

Remark 3.1 *For* $f \in \mathcal{C}_r$ *the asymptotic behavior (3.9) is equivalent to (1.11) with* $\widetilde{R}'(x) - R'(x) = O(x^{r-1})$ *for* $x \to 0$.

Along the ideas outlined in the previous section, we now derive the asymptotic behavior of ${F_0}^n(x)$. The main result of this chapter is

Theorem 3.1 *The following asymptotic expressions hold uniformly in* n *for* $x \to 0$.

$$\begin{aligned} {F_0}^n(x) &= g^n(x)\left\{1 + O(\{g^n(x)\}^\alpha)\right\} && (3.10) \\ ({F_0}^n)'(x) &= (g^n)'(x)\left\{1 + O(x^\alpha)\right\}. && (3.11) \end{aligned}$$

This will be proven by determining the existence of a solution G to Abel's equation for F_0. The explicit formula for this solution is given and used for a derivation of an asymptotic expression for G. Using this result, we get the desired asymptotic estimation of iterates of F_0 directly in terms of iterates of the fixed point of the intermittency renormalization (3.8).

Theorem 3.1 is also the basis for a precise statement about the convergence under intermittency renormalization:

Corollary 3.1 *For $\gamma = 2^{\frac{1}{r}}$ and g given by (3.3), we have*

$$T_\gamma{}^n F_0 \to g \quad \textit{and} \quad (T_\gamma{}^n F_0)' \to g' \tag{3.12}$$

uniformly on $[0, 1]$.

Proof: We have

$$\begin{aligned} T_\gamma{}^n F_0(x) &= \gamma^n F_0{}^{2^n}(\gamma^{-n} x) \\ &= \gamma^n g^{2^n}(\gamma^{-n} x) \left\{ 1 + O(\{g^n(\gamma^{-n} x)\}^\alpha) \right\} \\ &= g(x) \left\{ 1 + O(\{\gamma^{-n} g^n(x)\}^\alpha) \right\} \end{aligned}$$

uniformly in n and x. Thus

$$T_\gamma{}^n F_0(x) - g(x) = O(1)\gamma^{-n\alpha}, \quad n \to \infty$$

with $\gamma > 1$. Similarly,

$$\begin{aligned} (T_\gamma{}^n F_0)'(x) &= (F_0{}^{2^n})'(\gamma^{-n} x) \\ &= (g^{2^n})'(\gamma^{-n} x) \left\{ 1 + O(\{\gamma^{-n} x\}^\alpha) \right\} \\ &= g'(x) \left\{ 1 + O(\{\gamma^{-n} x\}^\alpha) \right\} \end{aligned}$$

uniformly in n and x. Thus, also

$$(T_\gamma{}^n F_0)'(x) - g'(x) = O(1)\gamma^{-n\alpha}, \quad n \to \infty. \ \Box$$

In order to prepare the proof of Theorem 3.1, we first need some inequalities:

Lemma 3.1 *$F_0{}^n$ is bounded by*

$$\{x^{-r} + nrc(1 + a_2)\}^{-\frac{1}{r}} \le F_0{}^n(x) \le \{x^{-r} + nrc(1 + a_1)\}^{-\frac{1}{r}} \tag{3.13}$$

with

$$a_1 = \inf_x \widetilde{R}(x) > -1 \quad \textit{and} \quad a_2 = \sup_x \widetilde{R}(x). \tag{3.14}$$

Moreover, for $({F_0}^n)'$ *we have*

$$\frac{1 - Cx^{\alpha}}{\{1 + nrc(1+a_2)x^r\}^{\frac{1}{r}+1}} \leq ({F_0}^n)'(x) \leq \frac{1 + Cx^{\alpha}}{\{1 + nrc(1+a_1)x^r\}^{\frac{1}{r}+1}}, \tag{3.15}$$

and

$$\left|\sum_{i=0}^{n-1} (\widetilde{R} \circ {F_0}^i)'(x)\right| \leq \frac{C}{c} x^{-(1+r-\alpha)}, \tag{3.16}$$

for some constant $C > 0$.

Proof: We have

$${F_0}^n(x) = \left\{x^{-r} + rc\left(n + \sum_{i=0}^{n-1} (\widetilde{R} \circ {F_0}^i)(x)\right)\right\}^{-\frac{1}{r}}. \tag{3.17}$$

Thus, with a_1 and a_2 given by (3.14), we have the bounds (3.13).

Clearly $1 + a_1 > 0$ due to the asymptotic behavior of $F_0(x)$ in connection with ${F_0}'(x) \leq 1$.

Differentiating (3.17), we get

$$({F_0}^n)'(x) = \frac{1 - cx^{r+1} \sum_{i=0}^{n-1} (\widetilde{R} \circ {F_0}^i)'(x)}{\left\{1 + rcx^r \left(n + \sum_{i=0}^{n-1} \widetilde{R}({F_0}^i(x))\right)\right\}^{\frac{1}{r}+1}}. \tag{3.18}$$

In order to get bounds on this expression, we have to estimate the sum

$$\widetilde{R}_n(x) = cx^{r+1} \sum_{i=0}^{n-1} (\widetilde{R} \circ {F_0}^i)'(x). \tag{3.19}$$

We have from (3.18)

$$\begin{aligned} \left(1 - |\widetilde{R}_n(x)|\right) \{1 + nrc(1+a_2)x^r\}^{-\frac{1+r}{r}} &\leq ({F_0}^n)'(x) \\ \leq \quad \left(1 + |\widetilde{R}_n(x)|\right) \{1 + nrc(1+a_1)x^r\}^{-\frac{1+r}{r}}. \end{aligned} \tag{3.20}$$

The asymptotic behavior of $\widetilde{R}'$ implies that

$$a' = \sup_x \left| x^{1-\alpha} \widetilde{R}'(x) \right| < \infty \tag{3.21}$$

and we can estimate

$$\begin{aligned}
|\widetilde{R}_n(x)| &\leq ca'x^{r+1} \sum_{i=0}^{n-1} (F_0{}^i(x))^{\alpha-1} (F_0{}^i)'(x) \\
&\leq ca'x^{r+1} \sum_{i=0}^{n-1} \left\{ x^{-r} + irc(1+a_2) \right\}^{\frac{1-\alpha}{r}} (F_0{}^i)'(x) \\
&\leq \left(\frac{1+a_2}{1+a_1} \right)^{\frac{1-\alpha}{r}} ca'x^{r+\alpha} \sum_{i=0}^{n-1} \{1 + irc(1+a_1)x^r\}^{\frac{1-\alpha}{r}} (F_0{}^i)'(x). \qquad (3.22)
\end{aligned}$$

In the last estimation

$$1 \leq \sup_{n,x} \frac{1 + nrc(1+a_2)x^r}{1 + nrc(1+a_1)x^r} \leq \frac{1+a_2}{1+a_1}$$

was used. The bound

$$\begin{aligned}
\sum_{i=0}^{n-1} \{1 + irc(1+a_1)x^r\}^{\frac{1-\alpha}{r}} &\leq 1 + \int_0^n \{1 + trc(1+a_1)x^r\}^{\frac{1-\alpha}{r}} dt \\
&\leq 1 + \frac{1}{c(1+a_1)(r+1-\alpha)x^r} \{1 + nrc(1+a_1)x^r\}^{\frac{r+1-\alpha}{r}},
\end{aligned}$$

and $0 \leq (F_0{}^n)' \leq 1$, implies along with (3.22) the initial estimate

$$|\widetilde{R}_n(x)| \leq A_0 x^\alpha \left(1 + \{1 + nrc(1+a_1)x^r\}^{\frac{r+1-\alpha}{r}} \right) \tag{3.23}$$

with some positive A_0.

Next, we proceed iteratively. Inserting (3.20) into (3.22), we get

$$|\widetilde{R}_n(x)| \leq \left(\frac{1+a_2}{1+a_1} \right)^{\frac{1-\alpha}{r}} ca'x^{r+\alpha} \sum_{i=0}^{n-1} \left(1 + |\widetilde{R}_i(x)| \right) \{1 + irc(1+a_1)x^r\}^{-\frac{r+\alpha}{r}}. \tag{3.24}$$

Inserting

$$\begin{aligned}
\sum_{i=0}^{n-1} \{1 + irc(1+a_1)x^r\}^{-\frac{r+\alpha}{r}} &\leq 1 + \int_0^\infty \{1 + trc(1+a_1)x^r\}^{-\frac{r+\alpha}{r}} dt \\
&= 1 + \frac{1}{c(1+a_1)\alpha x^r}
\end{aligned}$$

into (3.24), we get

$$|\widetilde{R}_n(x)| \leq Bx^\alpha \left(1 + x^r \sum_{i=0}^{n-1} |\widetilde{R}_i(x)| \left\{1 + irc(1+a_1)x^r\right\}^{-\frac{r+\alpha}{r}}\right) \tag{3.25}$$

With some positive constant B. Assuming that for some δ and positive A we have the bound

$$|\widetilde{R}_n(x)| \leq Ax^\alpha \left(1 + \left\{1 + nrc(1+a_1)x^r\right\}^{\frac{r+\delta}{r}}\right), \tag{3.26}$$

we insert this into the (3.25) in order to improve the bound.

$$\begin{aligned}
|\widetilde{R}_n(x)| &\leq \\
&\leq \; Bx^\alpha \left(1 + Ax^{r+\alpha} \left\{\sum_{i=0}^{n-1} \left\{1 + irc(1+a_1)x^r\right\}^{-\frac{r+\alpha}{r}} + \sum_{i=0}^{n-1} \left\{1 + irc(1+a_1)x^r\right\}^{\frac{\delta-\alpha}{r}}\right\}\right) \\
&\leq \; Bx^\alpha \left(1 + Ax^{r+\alpha} \left\{\left(1 + \frac{1}{c(1+a_1)\alpha x^r}\right) + \right.\right. \\
&\quad \left.\left. + \left(1 + \int_0^n \left\{1 + trc(1+a_1)x^r\right\}^{\frac{\delta-\alpha}{r}} dt\right)\right\}\right).
\end{aligned} \tag{3.27}$$

First, suppose that $\delta \geq \alpha - r$. Then we can estimate

$$|\widetilde{R}_n(x)| \leq A'x^\alpha \left(1 + \left\{1 + nrc(1+a_1)x^r\right\}^{\frac{r+\delta-\alpha}{r}}\right), \tag{3.28}$$

(if $\delta = \alpha - r$, simply choose a slightly larger δ to avoid the logarithm in the integration). Thus, the estimate (3.26) has been improved by $\delta' = \delta - \alpha$. We can repeat this step finitely many times until $\delta' < \alpha - r$.

Now suppose that $\delta < \alpha - r$. Then the right expression is bounded by a constant and we get a bound independent of n,

$$|\widetilde{R}_n(x)| \leq Cx^\alpha, \tag{3.29}$$

as the final estimate which directly implies the bounds (3.16) and (3.15). $\square$

Whereas the proof of Lemma 3.1 was still quite straightforward, we will have to use results related to Abel's equation (3.5) in order to get even sharper results.

Lemma 3.2 *Abel's equation (3.5) has a real solution G on $(0,1]$ which can be written as*

$$G(x) = \frac{1}{rc}x^{-r} + \int_\gamma^x \frac{\widetilde{R}(t)}{t - F_0(t)}dt + \lim_{n\to\infty}\left(\sum_{i=0}^{n-1}\widetilde{R}({F_0}^i(x)) - \int_{{F_0}^n(x)}^x \frac{\widetilde{R}(t)}{t - F_0(t)}dt\right) \quad (3.30)$$

with γ constant, the last term being of order $O(x^\alpha)$ for $x \to 0$. Moreover, an expression for its derivative is given by

$$G'(x) = -\frac{1}{c}x^{-(r+1)} + \sum_{i=0}^{\infty}(\widetilde{R}\circ {F_0}^i)'(x). \quad (3.31)$$

Demanding that

$$\lim_{x\to 0} x^{r+1}G'(x) \quad (3.32)$$

exists, this solution is unique (up to an additive constant).

Proof: Write

$$G(x) = \frac{1}{rc}x^{-r} + \int_C^x \frac{\widetilde{R}(t)}{t - F_0(t)}dt + G_1(x). \quad (3.33)$$

Inserting this into Abel's equation (3.5) leads to

$$G_1(F_0(x)) - G_1(x) = h(x) \quad (3.34)$$

with

$$h(x) = \int_{F_0(x)}^x \frac{\widetilde{R}(t)}{t - F_0(t)}dt - \widetilde{R}(x). \quad (3.35)$$

Showing that $h(x)$ is "small enough" for small x will enable us to get an explicit expression of $G_1(x)$.

We recall that $R(x)$ is given by

$$F_0(x) = \left\{x^{-r} + rc(1 + \widetilde{R}(x))\right\}^{-\frac{1}{r}} = x - cx^{1+r}(1 + R(x)). \quad (3.36)$$

Applying the mean value theorem, for some $y \in [F_0(x), x]$ one can write

$$\begin{aligned} h(x) &= \widetilde{R}(y)\frac{x - F_0(x)}{y - F_0(y)} - \widetilde{R}(x) = \widetilde{R}(y)\frac{x^{r+1}R(x)}{y^{r+1}R(y)} - \widetilde{R}(x) \\ &= O(1)\{R(y) - R(x)\} + O(1)\{\widetilde{R}(y) - \widetilde{R}(x)\} + O(1)x^r R(x), \quad x \to 0. \end{aligned} \tag{3.37}$$

Moreover, $|\widetilde{R}(x) - \widetilde{R}(y)| < (x - F_0(x)) \sup_{y \in [F_0(x), x]} |\widetilde{R}'(y)|$. Using $\widetilde{R}'(x) = O(x^{\alpha-1})$ for $x \to 0$, one can also show that

$$\widetilde{R}'(y) = O(x^{\alpha-1}) \quad \text{for} \quad x \to 0 \quad \text{and} \quad y \in [F_0(x), x]. \tag{3.38}$$

The same consideration applies for $|R(x) - R(y)|$. Inserting this into (3.37), we have

$$h(x) = O(1)x^{r+1}x^{\alpha-1} + O(1)x^r x^\alpha = O(x^{r+\alpha}), \quad x \to 0. \tag{3.39}$$

Using (3.39) in connection with the upper bound (3.15) on iterates of F_0, we can show the existence of a solution $G_1(x)$ of (3.34) and that its asymptotic behavior is

$$G_1(x) = O(x^\alpha) \quad x \to 0. \tag{3.40}$$

Inserting

$$G_1(x) = -\sum_{n=0}^{\infty} h({F_0}^n(x)) = \lim_{N\to\infty}\left(\int_{{F_0}^N(x)}^{x} \frac{\widetilde{R}(t)}{t - F_0(t)}dt - \sum_{n=0}^{N-1} \widetilde{R}({F_0}^n(x))\right) \tag{3.41}$$

into (3.34), one checks that this is a solution, provided the sum converges. Indeed, using (3.13) for fixed x, each term is of order

$$h({F_0}^n(x)) = O(n^{-\frac{r+\alpha}{r}}),$$

thus making the sum convergent. The sum can further be estimated by

$$\begin{aligned} G_1(x) &= O(1)\sum_{n=0}^{\infty}\left(\{x^{-r} + nc(1 + a_1)\}^{-\frac{1}{r}}\right)^{r+\alpha} \\ &= O(1)\left(Nx^{r+\alpha} + \sum_{n=N+1}^{\infty} n^{-\frac{r+\alpha}{r}}\right) = O(1)\left(Nx^{r+\alpha} + N^{-\frac{\alpha}{r}}\right). \end{aligned}$$

Choosing $N = x^{-r}$, we get

$$G_1(x) = O(1)x^\alpha$$

which implies the asserted asymptotic behavior of the solution (3.30).

Next, we show that $G'(x)$ is given by direct differentiation of (3.30). Differentiating without taking the limit yields

$$-\frac{1}{c}x^{-(r+1)} + \left(\sum_{i=0}^{n-1}(\widetilde{R}{F_0}^i)'(x) + \frac{\widetilde{R}({F_0}^n(x))}{{F_0}^n(x) - {F_0}^{n+1}(x)}({F_0}^n)'(x)\right).$$

The sum has already been estimated in Lemma 3.1. Fixing x and using the bounds for ${F_0}^n$ and $({F_0}^n)'$ from Lemma 3.1, we get that the last term converges to zero as

$$(n^{-\frac{1}{r}})^\alpha n^{-\frac{1}{r}-1}(n^{-\frac{1}{r}})^{-r-1} = n^{-\frac{\alpha}{r}}.$$

Thus, we get (3.31) in the limit.

A direct computation shows that this expression fulfills the differentiated Abel equation,

$$G'(F_0(x)){F_0}'(x) - G'(x) = 0. \tag{3.42}$$

Now, assume (3.32), i.e. that $\lim_{x\to 0} x^{r+1}G'(x) = A$ exists. Then

$$\begin{aligned}
A &= \lim_{n\to\infty}({F_0}^n(x))^{r+1}G'({F_0}^n(x)) = \lim_{n\to\infty}({F_0}^n(x))^{r+1}\frac{G'(x)}{({F_0}^n)'(x)} \\
&= G'(x)\lim_{n\to\infty}\left\{x^{-r} + rc\left(n + \sum_{i=0}^{n-1}(\widetilde{R}\circ {F_0}^i)(x)\right)\right\}^{-(\frac{1}{r}+1)} \times \\
&\quad \times \frac{\left\{1 + rcx^r\left(n + \sum_{i=0}^{n-1}\widetilde{R}({F_0}^i(x))\right)\right\}^{\frac{1}{r}+1}}{1 - cx^{r+1}\sum_{i=0}^{n-1}(\widetilde{R}\circ {F_0}^i)'(x)} \\
&= \frac{x^{r+1}G'(x)}{1 - cx^{r+1}\sum_{i=0}^{\infty}(\widetilde{R}\circ {F_0}^i)'(x)}.
\end{aligned} \tag{3.43}$$

Thus, $G'(x)$ is determined by (3.43) up to a multiplicative constant, which is uniquely fixed by demanding that $G(x)$ solves (3.5). $\square$

It immediately follows that

Corollary 3.2

$$G(x) = \frac{1}{rc} x^{-r}(1 + O(x^\alpha)), \quad x \to 0 \tag{3.44}$$

and

$$G'(x) = -\frac{1}{c} x^{-(r+1)}(1 + O(x^\alpha)), \quad x \to 0. \tag{3.45}$$

PROOF: For an asymptotic estimate for the integral, write

$$\int_\gamma^x \frac{\widetilde{R}(t)}{t - F_0(t)} dt = \frac{x - \gamma}{c(1 + R(y))} \frac{\widetilde{R}(y)}{y^r}, \quad y \in [x, x_0]$$

with fixed positive γ. $\widetilde{R}(y)y^{-r}$ is at most of order $O(x^{-r+\alpha})$ for $x \to 0$, whereas the other terms on the right hand side are bounded away from zero and infinity. The second equation follows directly from (3.16). □

We also need information about the asymptotic behavior of the inverse G^{-1}.

Lemma 3.3 *Let $G(x)$ be a solution of Abel's equation (3.5). Then G^{-1} exists, and we have the asymptotic expressions*

$$G^{-1}(y) = (rcy)^{-\frac{1}{r}} \left(1 + O(y^{-\frac{\alpha}{r}})\right), \quad y \to \infty \tag{3.46}$$

and

$$(G^{-1})'(y) = -c(rcy)^{-\frac{1}{r}-1} \left(1 + O(y^{-\frac{\alpha}{r}})\right), \quad y \to \infty. \tag{3.47}$$

PROOF: The existence and asymptotic behavior of $G(x)$ are given by Lemma 3.2. Also, by virtue of the asymptotic behavior, $G'(x) < 0$. For, if it were zero for some x_0 then it would be zero for all ${F_0}^n(x_0)$. This is a clear contradiction to (3.45) which implies that $G'({F_0}^n(x_0))$ is negative for n large enough.

Thus, the inverse G^{-1} exists. Clearly, $G^{-1}(y) = O(1)y^{-\frac{1}{r}}, \;\; y \to \infty$, but we need more. Write

$$G(x) = \frac{1}{rc} x^{-r} \left\{1 + \phi(x)\right\} \quad \text{with} \quad \phi(x) = O(x^{\alpha}), \;\; x \to 0 \tag{3.48}$$

and

$$G^{-1}(y) = (rcy)^{-\frac{1}{r}} + \Psi(y). \tag{3.49}$$

Then,

$$x = (G^{-1} \circ G)(x) = x\left(1 + \phi(x)\right)^{-\frac{1}{r}} + \Psi\left(\frac{1}{rc} x^{-r}\left\{1 + \phi(x)\right\}\right). \tag{3.50}$$

Thus, in order to get an estimation for $\Psi(x)$, we have to get an estimate on the expression $\Psi(y+h) - \Psi(y)$. Using Lemma (3.1), write for $0 < x \le y < 1$

$$\begin{aligned} \left|G^{-1}(G(x)+n) - G^{-1}(G(y)+n)\right| &= \left|{F_0}^n(x) - {F_0}^n(y)\right| \\ &\le |x-y| \sup_{\xi \in]x,y[} ({F_0}^n)'(\xi) \\ &\le \frac{(1+C)y}{\left\{1 + nrc(1+a_2)x^r\right\}^{1+\frac{1}{r}}}. \end{aligned} \tag{3.51}$$

Denote $G(x) = s$ and $G(y) = t$, (this implies $s \ge t$). Then, using $G^{-1}(y) = O(1)y^{-\frac{1}{r}}$ we continue (3.51)

$$G^{-1}(s+n) - G^{-1}(t+n) = O(1)\frac{t^{-\frac{1}{r}}}{\left\{1 + ns^{-1}\right\}^{1+\frac{1}{r}}} = O(1)\frac{s^{1+\frac{1}{r}}}{t^{\frac{1}{r}}\left\{s+n\right\}^{1+\frac{1}{r}}}.$$

This leads to

$$G^{-1}(y+h) - G^{-1}(y) = O(1)\frac{(y+h-n)^{1+\frac{1}{r}}}{(y-n)^{\frac{1}{r}}(y+h)^{1+\frac{1}{r}}} = O(1)\frac{h}{(y+h)^{1+\frac{1}{r}}}. \tag{3.52}$$

The last step follows from choosing a suitable n. Clearly, the estimation (3.52) remains true uniformly for general $h > -y$, as long as $y - h$ remains suitably bounded away from zero. Thus, for $-h/y > \delta > 0$ with some $\delta > 0$, instead of (3.52) we can write

$$G^{-1}(y+h) - G^{-1}(y) = O(1)\frac{h}{y^{1+\frac{1}{r}}}. \tag{3.53}$$

Now, (3.53) implies the desired estimation. Write

$$\begin{aligned}
\Psi(y+h)-\Psi(y) &= \left\{G^{-1}(y+h)-G^{-1}(y)\right\}-\left\{\frac{1}{rc}(y+h)^{-\frac{1}{r}}-\frac{1}{rc}y^{-\frac{1}{r}}\right\} \\
&= O(1)\frac{h}{y^{1+\frac{1}{r}}}+O(1)\frac{h}{y^{1+\frac{1}{r}}} \qquad (3.54)
\end{aligned}$$

and insert (3.54) into (3.50):

$$x=(G^{-1}\circ G)(x)=x\,(1+\phi(x))^{-\frac{1}{r}}+\Psi\left(\frac{1}{rc}x^{-r}\right)+O(1)x\phi(x).$$

Thus, we get

$$\Psi\left(\frac{1}{rc}x^{-r}\right)=O(1)x\phi(x).$$

Writing $y=\frac{1}{rc}x^{-r}$ and using $\phi(x)=O(x^{\alpha})$, this finally leads to

$$\Psi(y)=O(1)y^{-\frac{1}{r}-\frac{\alpha}{r}}.$$

This implies (3.46).

The asymptotic formula (3.47) for the derivative can be derived directly by use of the asymptotic formulas (3.46) and (3.45) for G' and G^{-1}:

$$\begin{aligned}
(G^{-1})'(y) &= \left(G'\circ G^{-1}(y)\right)^{-1} \\
&= -c\left(G^{-1}(y)\right)^{r+1}\left(1+O(1)G^{-1}(y)\right)^{-1} \\
&= -c(rcy)^{-(1+\frac{1}{r})}\left(1+O(1)y^{-\frac{\alpha}{r}}\right)^{r+1}\left\{1+O(1)y^{-\frac{\alpha}{r}}\right\} \\
&= -c(rcy)^{-(1+\frac{1}{r})}\left\{1+O(1)y^{-\frac{\alpha}{r}}\right\}, \quad y\to\infty. \ \square
\end{aligned}$$

Having arrived at this point, we can deduce a short Lemma about the conjugation between $F_0(x)$ and $g(x)$.

Lemma 3.4 *There exists a unique C^1-diffeomorphism h of the interval $[0,1]$ conjugating F_0 and g, i.e. $h(F_0(x))=g(h(x))$. Asymptotically,*

$$h'(x)=1+O(x^{\alpha}),\quad x\to 0$$

and

$$(h^{-1})'(x) = 1 + O(x^\alpha), \quad x \to 0.$$

PROOF: As shown in Section 3.1, each function h satisfying $h(F_0(x)) = g(h(x))$ corresponds one-to-one to a positive solution of Abel's equation (3.5), and we have

$$h(x) = (rcG(x))^{-\frac{1}{r}}.$$

Demanding that $\lim_{x\to 0} h'(x)$ exists and is nonzero, we get the uniqueness condition (3.32) of Lemma 3.2. This determines $G(x)$, and the additive constant is uniquely fixed by $1 = h(1) = G(1)$.

Due to the properties of $G(x)$, $h^{-1}(x)$ exists. Moreover, using the asymptotic behavior of $G(x)$ we see that $h(x)$ and $h^{-1}(x)$ are in $C^1[0,1]$ and get their asymptotic behavior:

$$\begin{aligned}
h'(x) &= (rc)^{-\frac{1}{r}}(-\frac{1}{r})(G(x))^{-(1+\frac{1}{r})}G'(x) \\
&= (rc)^{-\frac{1}{r}}(-\frac{1}{r})\left(\frac{1}{rc}x^{-r}(1+O(x^\alpha))\right)^{-(1+\frac{1}{r})}\left(-\frac{1}{c}x^{-(r+1)}(1+O(x^\alpha))\right) \\
&= 1 + O(x^\alpha), \quad x \to 0
\end{aligned}$$

and

$$\begin{aligned}
(h^{-1})'(x) &= (G^{-1})'(\frac{1}{rc}x^{-r})\left(-\frac{1}{c}x^{-(r+1)}\right) \\
&= -c(x^{-r})^{-\frac{1}{r}-1}\left(1+O(x^\alpha)\right)\left(-\frac{1}{c}x^{-(r+1)}\right) \\
&= 1 + O(x^\alpha), \quad x \to 0. \ \square
\end{aligned}$$

Using Lemmas 3.2 and 3.3, we now conclude this chapter with

Proof of Theorem 3.1: Using the asymptotic behavior of $G(x)$, we can write

$$
\begin{aligned}
&{F_0}^n(x) = G^{-1}(n+G(x)) = \\
&= \left\{rcn + x^{-r}\left(1+O(1)x^{\alpha}\right)\right\}^{-\frac{1}{r}} \left(1+O(1)\left\{rcn + x^{-r}\left(1+O(1)x^{\alpha}\right)\right\}^{-\frac{\alpha}{r}}\right) \\
&= \left\{rcn + x^{-r}\right\}^{-\frac{1}{r}} \left\{1+O(1)\frac{x^{-r+\alpha}}{rcn+x^{-r}}\right\}^{-\frac{1}{r}} \left(1+O(1)\left\{rcn+x^{-r}\right\}^{-\frac{\alpha}{r}}\right) \\
&= \left\{rcn + x^{-r}\right\}^{-\frac{1}{r}} \left(1+O(1)\left(\frac{x^{-r+\alpha}}{rcn+x^{-r}} + \left\{rcn+x^{-r}\right\}^{-\frac{\alpha}{r}}\right)\right) \\
&= \left\{rcn + x^{-r}\right\}^{-\frac{1}{r}} \left(1+O(1)\left\{rcn+x^{-r}\right\}^{-\frac{\alpha}{r}} \left(\{1+rcnx^{r}\}^{-\frac{(r-\alpha)}{r}} + 1\right)\right) \\
&= g^n(x)\left(1+O(1)\left(g^n(x)\right)^{\alpha}\right), \quad x \to 0
\end{aligned}
$$

and

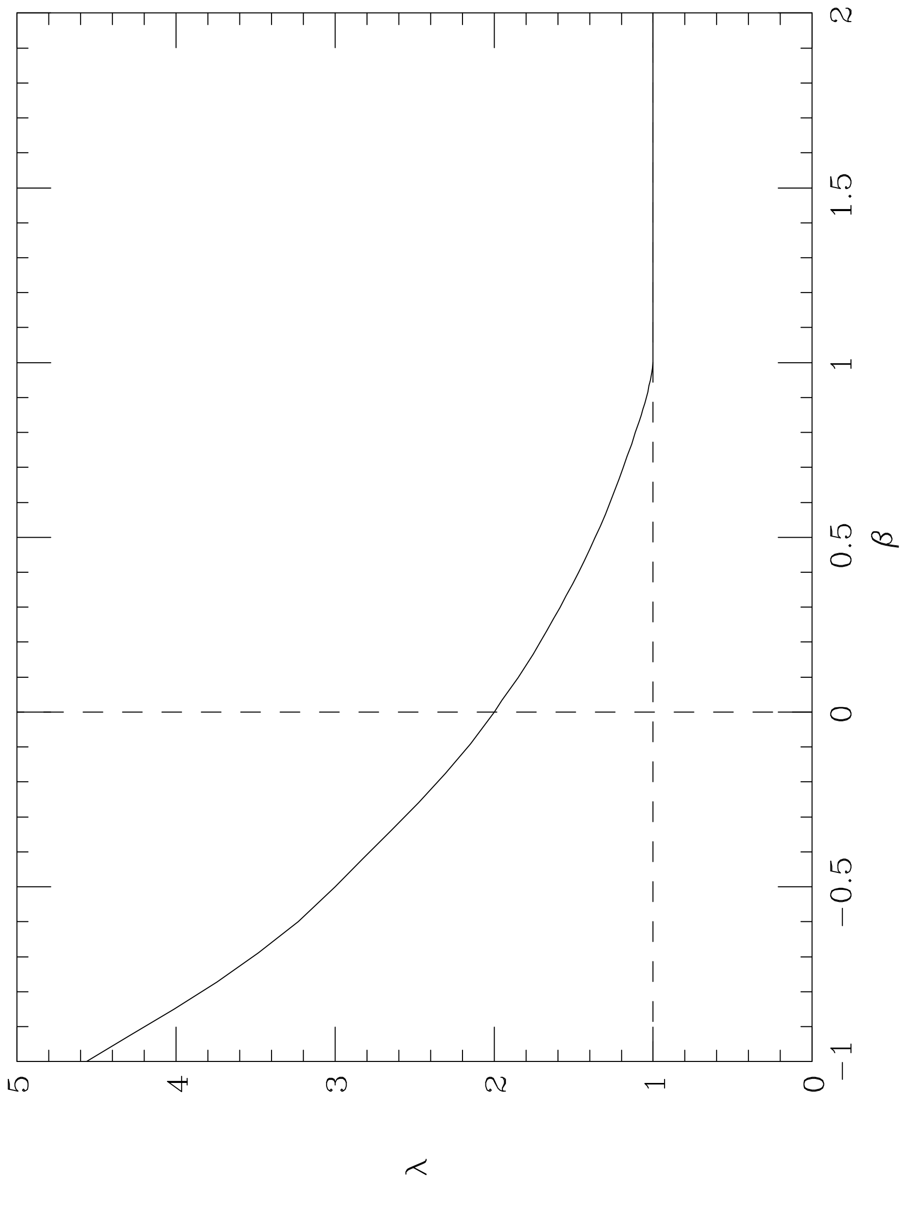
$$
\begin{aligned}
({F_0}^n)'(x) &= \frac{G'(x)}{G'({F_0}^n(x))} \\
&= \left(\frac{{F_0}^n(x)}{x}\right)^{r+1} \frac{1+O(x^{\alpha})}{1+O(\{{F_0}^n(x)\}^{\alpha})} \\
&= \left(\frac{g^n(x)}{x}\right)^{r+1} \left(1+O(\{g^n(x)\}^{\alpha})\right) \frac{1+O(x^{\alpha})}{1+O(\{g^n(x)\}^{\alpha})} \\
&= (g^n)'(x)\left\{1+O(\{g^n(x)\}^{\alpha}+x^{\alpha})\right\} \\
&= (g^n)'(x)\left\{1+O(x^{\alpha})\right\}.
\end{aligned}
$$

These asymptotic estimates are uniform in n. □

Chapter 4

Inducing and Phase Transitions

In this chapter we apply the thermodynamic formalism to the functions in class $\mathcal{C}_r$ with interaction $-\beta \log |f'|$. The existence of the indifferent fixed point is shown to imply a phase transition. This is done by using an expanding induced subsystem on which the modified transfer operator $\mathcal{M}_{\beta z}$ has "nice" properties. The spectral properties of $\mathcal{M}_{\beta z}$ give information on the spectrum of $\mathcal{L}_\beta$, in particular on the existence of a leading eigenvalue. It is shown that this eigenvalue depends analytically on β for $\beta < 1$ and approaches the essential spectral radius 1 at $\beta = 1$. For $\beta \geq 1$, $\mathrm{r}(\mathcal{L}_\beta) = \mathrm{r}_{\mathrm{ess}}(\mathcal{L}_\beta) = 1$, so that we have a phase transition at $\beta_c = 1$. A typical graph of $\mathrm{r}(\mathcal{L}_\beta) = \exp P(\beta)$ is depicted in Figure 4.1, where our numerical results for the Farey map are presented (repeating the calculations of [14].) The two examples of the piecewise linear map and the Farey map are worked out further.



The β−dependence of the largest eigenvalue λ of the transfer operator.

Figure 4.1:

4.1 Definition of $\mathcal{L}_\beta$ and $\mathcal{M}_{\beta z}$

Specifying the theory of Chapter 2, we are interested in the case where the function s in the definition (2.2) of the transfer operator $\mathcal{L}$ is given by the derivative of the transformation f. More specifically, if f is piecewise C^1 we introduce the real parameter β and set $s = |f'|^{-\beta}$.

Thus, for functions in class $\mathcal{C}_r$, we are interested in investigating the transfer operator $\mathcal{L}_\beta$ given as

$$\mathcal{L}_\beta \psi(x) = \sum_{f(y)=x} \frac{\psi(y)}{|f'(y)|^\beta}. \tag{4.1}$$

Using the inverse functions, this leads to

$$\mathcal{L}_\beta \psi = |F_0{}'|^\beta \ \psi \circ F_0 + |F_1{}'|^\beta \ \psi \circ F_1. \tag{4.2}$$

(Here, we redefine the transfer operator at the point $x = 1$, which only has one inverse image under f, by using the continuously extended inverse functions. However, due to the remarks in Chapter 2 it is clear that this does not change the spectral properties of $\mathcal{L}_\beta$.)

In an analogous way, we write the modified transfer operator $\mathcal{M}_{\beta z}$ on a subinterval $J \subset I$: s_n is equal to $|g'(x)|^{-\beta}|_{J_n} = |(f^n)'(x)|^{-\beta}$, and thus

$$\mathcal{M}_{\beta z} \phi(x) = \sum_{g(y)=x} z^{n(y)} \frac{\phi(y)}{|g'(y)|^\beta}. \tag{4.3}$$

If we choose inducing on $J = [a, 1]$ the induced map g is expanding, as we have $|f'| \geq \lambda_0 > 1$ on $f^{-1}J$ by definition of $\mathcal{C}_r$. The operators $\mathcal{L}_{0\beta}$ and $\mathcal{L}_{1\beta}$ are given as

$$\mathcal{L}_{0\beta}\psi = |F_0{}'|^\beta \ \psi \circ F_0, \quad \mathcal{L}_{1\beta}\psi = |F_1{}'|^\beta \ \psi \circ F_1 \tag{4.4}$$

and the sets J_n and K_n are intervals, given as

$$J_n = [b_n, b_{n+1}[, \quad K_n = [b_n, 0[. \tag{4.5}$$

The inverses of $g = f^n$ on J_n are $F_1 {F_0}^{n-1}$ and extend continuously to J. Thus, $\mathcal{M}_{\beta z}$ takes the simple form

$$\mathcal{M}_{\beta z}\phi = \sum_{n=1}^{\infty} z^n |{G_n}'|^\beta \ \phi \circ G_n \tag{4.6}$$

and its extension $\mathcal{M}_{\beta z}^+$ is gained by extending $G_n = F_1 {F_0}^{n-1}$ to all of I.

Summarizing, we define

Definition 4.1 *Given $f \in \mathcal{C}_r$ for some $r > 0$ and $\beta \in \mathbf{R}$, the transfer operator associated with f is*

$$\mathcal{L}_\beta : \ \ BV(I) \to BV(I), \quad \mathcal{L}_\beta \psi = |{F_0}'|^\beta \ \psi \circ F_0 + |{F_1}'|^\beta \ \psi \circ F_1. \tag{4.7}$$

For $z \in \mathbf{C}$, the modified transfer operator associated with the induced function g on J is

$$\mathcal{M}_{\beta z} : \ \ BV(J) \to BV(J), \quad \mathcal{M}_{\beta z}\phi = \sum_{n=1}^{\infty} z^n |{G_n}'|^\beta \ \phi \circ G_n. \tag{4.8}$$

with

$$G_n : \ \ I \to J, \quad G_n = F_1 {F_0}^{n-1}, \qquad n \in \mathbf{N}. \tag{4.9}$$

The extension of $\mathcal{M}_{\beta z}$ to I is

$$\mathcal{M}_{\beta z} : \ \ BV(J) \to BV(I), \quad \mathcal{M}_{\beta z}\phi = \sum_{n=1}^{\infty} z^n |{G_n}'|^\beta \ \phi \circ G_n. \tag{4.10}$$

For convenience, we use the same notation for

$$\mathcal{M}_{\beta z}^+ : \ \ BV(I) \to BV(I), \quad \mathcal{M}_{\beta z}^+ \psi = \mathcal{M}_{\beta z}^+(\chi_J \cdot \psi). \tag{4.11}$$

Later, we will need the following estimations:

Lemma 4.1

$$\begin{aligned}\mathrm{var}_J(|{G_n}'|^\beta) &\le \frac{\left\||{G_n}'|^\beta\right\|_J}{\||f'|^{-\beta}\|_I}\mathrm{var}_I(|f'|^{-\beta}) && (4.12)\\ \mathrm{var}_I(|{G_n}'|^\beta) &\le n\frac{\left\||{G_n}'|^\beta\right\|_I}{\||f'|^{-\beta}\|_I}\mathrm{var}_I(|f'|^{-\beta}). && (4.13)\end{aligned}$$

PROOF: We have

$$\begin{aligned}\mathrm{var}_{J_n}(|(f^n)'|^{-\beta}) &= \mathrm{var}_J(|(F_1{F_0}^{n-1})'|^\beta)\\ &\le \mathrm{var}_J(|{F_1}'{F_0}^{n-1}|^\beta)\left\||({F_0}^{n-1})'|^\beta\right\|_J +\\ &\quad +\sum_{i=1}^{n-1}\left\||(F_1{F_0}^{n-1-i})'\circ {F_0}^i\cdot({F_0}^{i-1})'|^\beta\right\|_J\mathrm{var}_J(|{F_0}'{F_0}^{i-1}|^\beta)\\ &\le \left\||(F_1{F_0}^{n-1})'|^\beta\right\|_J\left\{\frac{\mathrm{var}_J(|{F_1}'{F_0}^{n-1}|^\beta)}{\inf_J(|{F_1}'{F_0}^{n-1}|^\beta)}+\sum_{i=1}^{n-1}\frac{\mathrm{var}_J(|{F_0}'{F_0}^{i-1}|^\beta)}{\inf_J(|{F_0}'{F_0}^{i-1}|^\beta)}\right\}\\ &\le \frac{\left\||{G_n}'|^\beta\right\|_J}{\||f'|^{-\beta}\|_I}\left\{\mathrm{var}_J(|{F_1}'{F_0}^{n-1}|^\beta)+\sum_{i=1}^{n-1}\mathrm{var}_J(|{F_0}'{F_0}^{i-1}|^\beta)\right\}\end{aligned}$$

and the expression in parentheses can be estimated by

$$\begin{aligned}&\mathrm{var}_J(|{F_1}'{F_0}^{n-1}|^\beta)+\sum_{i=1}^{n-1}\mathrm{var}_J(|{F_0}'{F_0}^{i-1}|^\beta)\\ &= \mathrm{var}_{{F_0}^{n-1}(J)}(|{F_1}'|^\beta)+\sum_{i=1}^{n-1}\mathrm{var}_{{F_0}^{i-1}(J)}(|{F_0}'|^\beta)\\ &\le \mathrm{var}_I(|{F_1}'|^\beta)+\mathrm{var}_I(|{F_0}'|^\beta)\\ &= \mathrm{var}_J(|f'|^{-\beta})+\mathrm{var}_{J^c}(|f'|^{-\beta})\\ &= \mathrm{var}_I(|f'|^{-\beta}).\end{aligned}$$

Repeating this estimation with J instead of J_n, we get

$$\mathrm{var}_J(|(f^n)'|^{-\beta}) \le \frac{\left\||{G_n}'|^\beta\right\|_I}{\||f'|^{-\beta}\|_I}\left\{\mathrm{var}_I(|{F_1}'{F_0}^{n-1}|^\beta)+\sum_{i=1}^{n-1}\mathrm{var}_I(|{F_0}'{F_0}^{i-1}|^\beta)\right\}$$

$$
\begin{aligned}
&= \frac{\left|\left||G_n{}'|^\beta\right|\right|_I}{|||f'|^{-\beta}||_I} \left\{ \mathrm{var}_{F_0{}^{n-1}(I)}(|F_1{}'|^\beta) + \sum_{i=1}^{n-1} \mathrm{var}_{F_0{}^{i-1}(I)}(|F_0{}'|^\beta) \right\} \\
&\leq \frac{\left|\left||G_n{}'|^\beta\right|\right|_I}{|||f'|^{-\beta}||_I} \left\{ \mathrm{var}_I(|F_1{}'|^\beta) + n\, \mathrm{var}_I(|F_0{}'|^\beta) \right\} \\
&\leq n \frac{\left|\left||G_n{}'|^\beta\right|\right|_I}{|||f'|^{-\beta}||_I} \mathrm{var}_I(|f'|^{-\beta}).
\end{aligned}
$$

The essential difference is that the sets $\mathrm{int}(F_0{}^{n-1}(J))$ are mutually disjoint, whereas $F_0{}^n(I) \subset F_0{}^{n-1}(I)$. □

We also will need the asymptotic behavior of $G_n{}'$.

Lemma 4.2 *There exist constants c_1, c_2 such that*

$$c_1 n^{-(1+\frac{1}{r})} \leq |G_n{}'(x)| \leq c_2 n^{-(1+\frac{1}{r})}, \qquad x \in J \tag{4.14}$$

$$c_1 n^{-\frac{1}{r}} \leq |F_1(0) - G_n(x)| \leq c_2 n^{-\frac{1}{r}}, \qquad x \in J \tag{4.15}$$

and

$$c_1 n^{-(1+\frac{1}{r})} \leq |G_n{}'(x)| \leq 1, \qquad x \in I \tag{4.16}$$

$$c_1 n^{-\frac{1}{r}} \leq |F_1(0) - G_n(x)| \leq 0, \qquad x \in I. \tag{4.17}$$

Remark 4.1 *Note that $G_n{}'(0) = F_1{}'(0)$ and $G_n(0) = F_1(0)$ are independent of n.*

Proof: A direct application of Theorem 3.1. □

4.1.1 Example: Piecewise Linear Map

Given the map $\tilde{f}$ defined by (1.16) through sequence (a_n) from (1.13), the transfer operator is

$$\widetilde{\mathcal{L}}_\beta \psi(x) = {d_n}^{-\beta}\psi(a_{n+1} + (x - a_n)/d_n) + {d_0}^{-\beta}\psi(1 - x/d_0), \quad a_n \le x \le a_{n-1}, \quad n \in \mathbf{N}. \tag{4.18}$$

Of particular interest is the subspace of piecewise constant functions,

$$\psi|_{]a_n, a_{n-1}]} = \rho_n. \tag{4.19}$$

This leads to the space of sequences (ρ_n) of bounded variation

$$\left\{(\rho_n): \quad \rho_n \in \mathbf{C}, \ \sum_{n=1}^{\infty} |\rho_n - \rho_{n+1}| < \infty\right\}. \tag{4.20}$$

Here, $\widetilde{\mathcal{L}}_\beta$ has the simple form

$$\widetilde{\mathcal{L}}_\beta \rho_n = {d_n}^{-\beta}\rho_{n+1} + {d_0}^{-\beta}\rho_1, \quad n \in \mathbf{N}. \tag{4.21}$$

The advantage of $\tilde{f}$ becomes clear when we induce on $J = [a_1, 1]$. Here, the functions $\widetilde{G}_n$ turn out to be linear, and we get

$$\widetilde{G}_n(x) = b_{n+1} - q_n(x - a_1), \quad n \in \mathbf{N} \tag{4.22}$$

with

$$q_n = \frac{b_{n+1} - b_n}{a_0 - a_1}, \qquad n \in \mathbf{N} \tag{4.23}$$

so that the modified transfer operator on J can be written as

$$\widetilde{\mathcal{M}}_{\beta z}\phi(x) = \sum_{n=1}^{\infty} z^n q_n^\beta \phi(b_{n+1} - q_n(x - a_1)). \tag{4.24}$$

Inducing on the above defined subspace leads to an even simpler form. Here, $\widetilde{\mathcal{M}}_{\beta z}$ reduces to a multiplication by a constant,

$$\widetilde{\mathcal{M}}_{\beta z}\rho_1 = \lambda_{\beta z}\rho_1, \qquad \text{with} \quad \lambda_{\beta z} = \sum_{n=1}^{\infty} z^n q_n^{\beta}. \tag{4.25}$$

For the sake of completeness, we state the formulas for the extension $\widetilde{\mathcal{M}}_{\beta z}^{+}$. Using $G_n(x) = \widetilde{F}_1 \widetilde{F}_0^{n-1}(x)$ we get

$$G_n(x) = b_{n+m} - q_{n,m}(x - a_m), \quad a_m < x \leq a_{m-1}, \quad m \in \mathbf{N} \tag{4.26}$$

with

$$q_{n,m} = \frac{b_{n+m} - b_{n+m-1}}{a_{m-1} - a_m} \qquad n \in \mathbf{N}. \tag{4.27}$$

Thus, we can write

$$\widetilde{\mathcal{M}}_{\beta z}^{+}\phi(x) = \sum_{n=1}^{\infty} z^n q_{n,m}^{\beta}\phi(b_{n+m} - q_{n,m}(x - a_m)), \quad a_m < x \leq a_{m-1}, \quad m \in \mathbf{N}. \tag{4.28}$$

On the subspace of piecewise constant functions, we get

$$(\widetilde{\mathcal{M}}_{\beta z}^{+}\rho)_m = \lambda_{\beta z}^{(m)}\rho_1, \qquad \text{with} \quad \lambda_{\beta z}^{(m)} = \sum_{n=1}^{\infty} z^n q_{n,m}^{\beta}. \tag{4.29}$$

Clearly, the action of $\widetilde{\mathcal{M}}_{\beta z}^{+}$ on ρ is only dependent on ρ_1.

The fact that the subspace of piecewise constant functions is an invariant subspace for $\widetilde{\mathcal{L}}_\beta$ and $\widetilde{\mathcal{M}}_{\beta z}^{+}$ reflects the reduction to cluster interaction. Since the single clusters do not interact with each other, we get piecewise constant densities on intervals corresponding to clusters.

4.1.2 Example: The Farey Map

For the map f given by (1.19), the transfer operator is

$$\mathcal{L}_\beta\psi(x) = (1+x)^{-\beta}\left\{\psi\left(\frac{x}{1+x}\right) + \psi\left(\frac{1}{1+x}\right)\right\}. \tag{4.30}$$

In the computation of $\mathcal{M}_{\beta z}$, this example shows its full algebraic beauty. Inducing yields

$$G_n(x) = 1 - \frac{x}{1+nx}, \quad , {G_n}'(x) = -\frac{1}{(1+nx)^2} \tag{4.31}$$

so that

$$\mathcal{M}_{\beta z}\phi(x) = \sum_{n=1}^{\infty} z^n (1+nx)^{-2\beta} \phi\left(1 - \frac{x}{1+nx}\right). \tag{4.32}$$

The extension $\mathcal{M}^+_{\beta z}$ can be written identically.

Inducing for the Farey map and its linearized version are shown in Figures 4.2 and 4.3. Note in Figure 4.3 that the branches of g are piecewise linear inbetween the dashed lines.

4.2 Analysis of $\mathcal{M}_{\beta z}$ and $\mathcal{M}^+_{\beta z}$

4.2.1 Boundedness of $\mathcal{M}_{\beta z}$ and $\mathcal{M}^+_{\beta z}$

Lemma 4.3 *$\mathcal{M}_{\beta z}$ and $\mathcal{M}^+_{\beta z}$ are power series in z with radius of convergence* 1.

At the radius of convergence $|z| = 1$, $\mathcal{M}_{\beta z}$ is bounded for $\beta > \frac{r}{1+r}$.

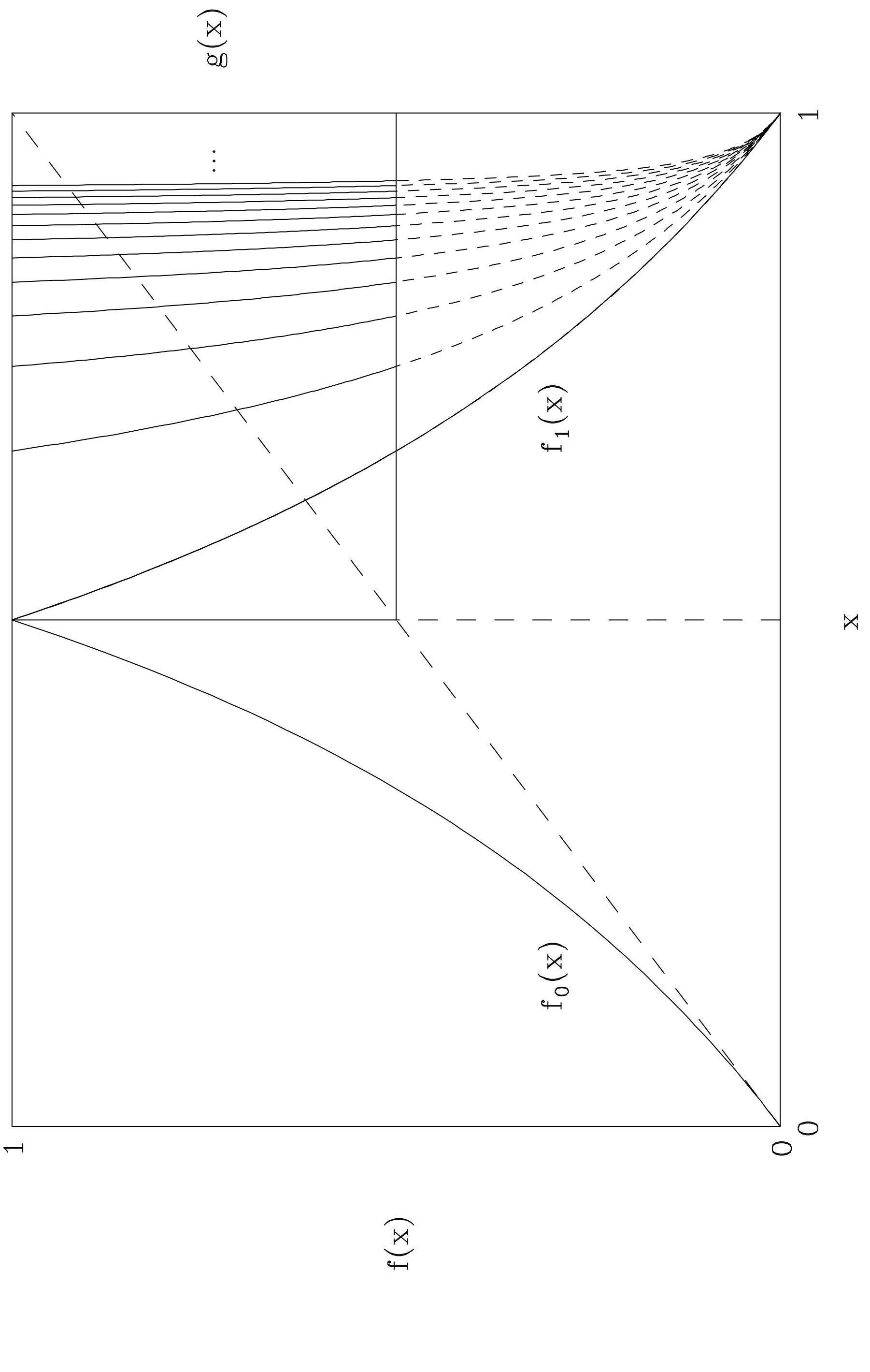
For $z = 1$, $\mathcal{M}_{\beta 1}$ is unbounded if $\beta \leq \frac{r}{1+r}$. Moreover, $\mathcal{M}^+_{\beta 1}$ is unbounded for all β.

Proof: We write

$$\mathcal{M}_{\beta z} = \sum_{n=1}^{\infty} z^n {\mathcal{M}_{\beta z}}^{(n)} \quad \text{with} \quad \mathcal{M}_n^{(\beta)}\phi = \left|{G_n}'\right|^\beta \phi \circ G_n. \tag{4.33}$$

and compute $\left\|\mathcal{M}_n^{(\beta)}\right\|_{BV(J)}$. We have

$$\left\|\mathcal{M}_n^{(\beta)}\right\|_{BV(J)} =$$



The Farey map f with the induced map g in the upper right corner.

Figure 4.2:

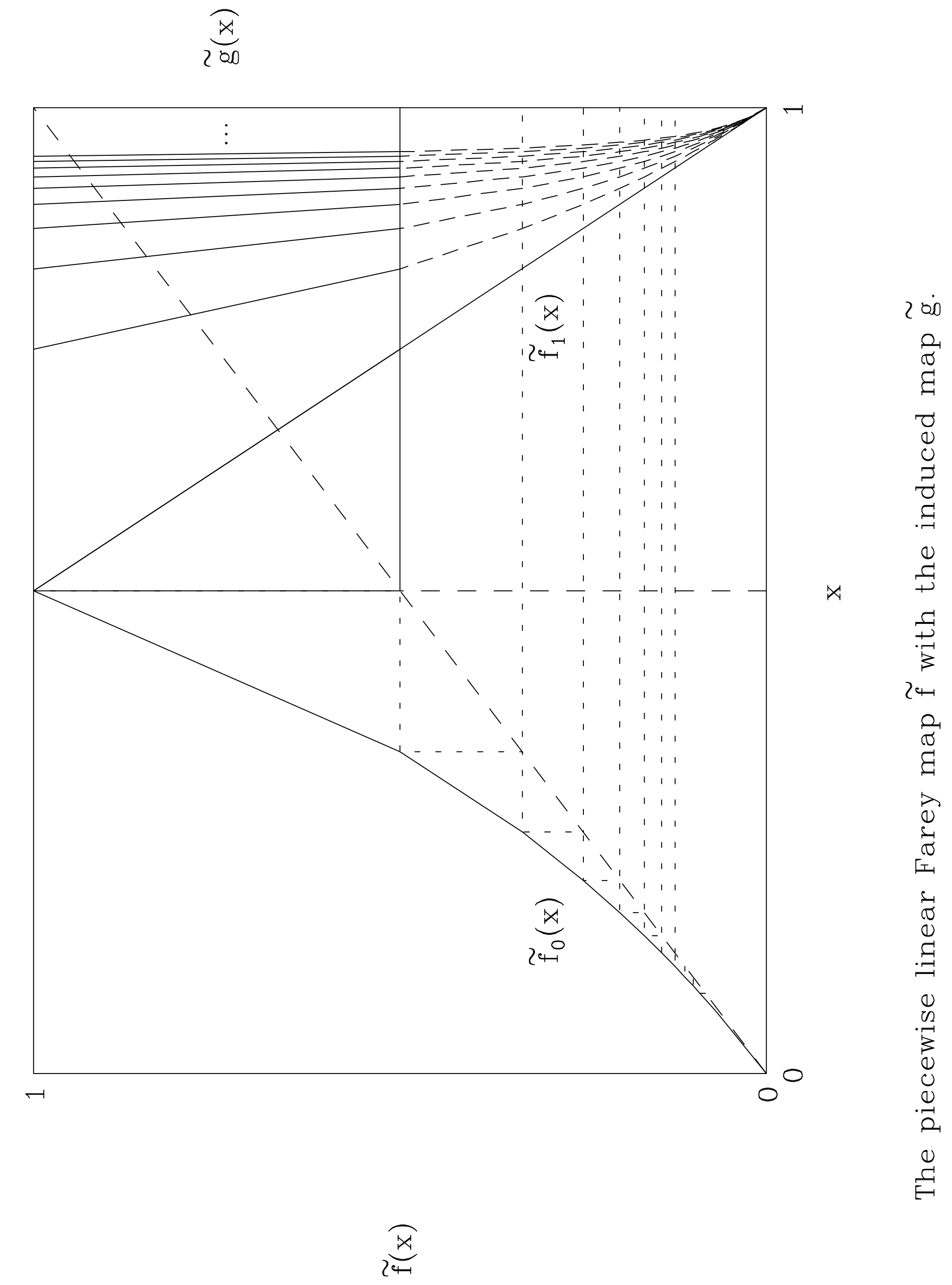

g̃(x)
1
...
f̃₁(x)
f̃₀(x)
x
0
0
1
f̃(x)
The piecewise linear Farey map f̃ with the induced map g̃.

Figure 4.3:

$$
\begin{aligned}
&= \sup_{||\phi||_{BV(J)}=1} \max\left\{\left|\left|\mathcal{M}_n^{(\beta)}\phi\right|\right|_J,\ \mathrm{var}_J(\mathcal{M}_n^{(\beta)}\phi)_J\right\} \\
&\le \sup_{||\phi||_{BV(J)}=1} \max\left\{\left|\left||{G_n}'|^\beta\right|\right|_J ||\phi||_J,\ \mathrm{var}_J(|{G_n}'|^\beta)||\phi||_J + \left|\left||{G_n}'|^\beta\right|\right|_J \mathrm{var}_J(\phi)\right\} \\
&\le 2\left|\left||{G_n}'|^\beta\right|\right|_{BV(J)}.
\end{aligned}
$$

Moreover,

$$\left|\left|\mathcal{M}_n^{(\beta)}\right|\right|_{BV(J)} \ge \left|\left|\mathcal{M}_n^{(\beta)}\right|\right|_{BV(J)} = \left|\left||{G_n}'|^\beta\right|\right|_{BV(J)}.$$

Thus, the radius of convergence for $\mathcal{M}_{\beta z}$ is given by

$$\lim_{n\to\infty} \left|\left||{G_n}'|^\beta\right|\right|_{BV(J)}^{-1/n}. \tag{4.34}$$

Using the asymptotic behavior of $|{G_n}'|^\beta = O(1)n^{-\beta(1+\frac{1}{r})}$ on J, the radius of convergence for $\mathcal{M}_{\beta z}$ is equal to 1.

The same argument carries through for $\mathcal{M}_{\beta z}^+$ due to the fact that $|{G_n}'|^\beta = O(1)$ on I for $\beta \ge 0$ and $|{G_n}'|^\beta = O(1)n^{-\beta(1+\frac{1}{r})}$ on I for $\beta < 0$.

For $|z| = 1$ we have

$$||\mathcal{M}_{\beta z}||_{BV(J)} \le \sum_{n=1}^{\infty} 2\left|\left||{G_n}'|^\beta\right|\right|_{BV(J)} \le O(1)\sum_{n=1}^{\infty} n^{-\beta(1+\frac{1}{r})}$$

which is convergent for $\beta > \frac{r}{1+r}$. Also,

$$\mathcal{M}_{\beta 1}1(x) = \sum_{n=1}^{\infty} |{G_n}'(x)|^\beta \ge c_1 \sum_{n=1}^{\infty} n^{-\beta(1+\frac{1}{r})} \tag{4.35}$$

is divergent for $\beta \le \frac{r}{1+r}$. However,

$$\mathcal{M}_{\beta 1}^+1(0) = \sum_{n=1}^{\infty} |{F_1}'(0)|^\beta \tag{4.36}$$

is divergent for all β. □

We recall that by Lemma 2.2 we can relate the nonzero eigenvalues and the corresponding eigenfunctions of $\mathcal{M}_{\beta z}$ and $\mathcal{M}_{\beta z}^+$, whenever $\mathcal{M}_{\beta z}^+$ is bounded. Therefore, in order to investigate eigenfunctions of $\mathcal{M}_{\beta z}^{+}$, we only need to consider eigenfunctions of $\mathcal{M}_{\beta z}$.

4.2.2 The Essential Spectral Radius of $\mathcal{M}_{\beta z}$ and the Existence of a Leading Eigenvalue

One way to show the existence of a leading eigenvalue is the comparison of $\mathrm{r}(\mathcal{M}_{\beta z})$ and $\mathrm{r}_{\mathrm{ess}}(\mathcal{M}_{\beta z})$, as stated in Theorem 2.4. Therefore, we provide explicit formulas for each. We use Theorem 2.3 for $\mathcal{M}_{\beta z}$.

Theorem 4.1 *Suppose that*

(a) $|z| < 1$ *or*

(b) $|z| = 1$ *and* $\beta > \frac{r}{1+r}$.

Then, writing $G_{\underline{n}_k} = G_{n_1} \circ G_{n_2} \circ \ldots \circ G_{n_k}$, *we have*

$$\mathrm{r}_{\mathrm{ess}}(\mathcal{M}_{\beta z}) = \inf_k \left(\sup_{\underline{n}_k} \left\{ |z|^{|\underline{n}_k|} \left|\left| |G_{\underline{n}_k}{}'|^{\beta} \right|\right|_J \right\} \right)^{1/k}. \tag{4.37}$$

Proof: In order to apply Theorem 2.3, we have to check whether $s = z^{n(x)}(f^n(x))'(x)$ is in $BV(J)$ and whether condition (2.2) holds for s.

By Lemma 4.1,

$$\begin{aligned} \mathrm{var}_J(s) &\leq \sum_{n=1}^{\infty} |z|^n \mathrm{var}(|(f^n)'|_{J_n} \\ &\leq \frac{\mathrm{var}_I(|f'|^{-\beta})}{|||f'|^{-\beta}||_I} \sum_{n=1}^{\infty} |z|^n \left|\left| |G_n{}'|^{\beta} \right|\right|_J \end{aligned}$$

which is finite by Lemma 4.2. The same holds for condition (2.2), as

$$\sum_{n=1}^{\infty} ||s||_{J_n} = \sum_{n=1}^{\infty} |z|^n \left|\left| |{G_n}'|^\beta \right|\right|_J .$$

Now, the formula (4.37) follows directly from (2.21). $\square$

A direct consequence of this theorem is

Corollary 4.1 $\mathrm{r_{ess}}(\mathcal{M}_{\beta z}) \leq |z| \lambda_0^{-\beta}$.

Now, we give a formula for $\mathrm{r}(\mathcal{M}_{\beta z})$.

Lemma 4.4 *Suppose that*

(a) $0 \leq z < 1$ *or*

(b) $z = 1$ *and* $\beta > \frac{r}{1+r}$.

Then

$$\mathrm{r}(\mathcal{M}_{\beta z}) = \inf_k \left|\left| \sum_{\underline{n}_k} z^{|\underline{n}_k|} |{G_{\underline{n}_k}}'|^\beta \right|\right|_J^{1/k} . \tag{4.38}$$

Proof: We have

$$\mathrm{r}(\mathcal{M}_{\beta z}) = \inf_k \left|\left| {\mathcal{M}_{\beta z}}^k \right|\right|_{BV(J)}^{1/k} \geq \inf_k \left|\left| {\mathcal{M}_{\beta z}}^k 1 \right|\right|_{BV(J)}^{1/k} \geq \inf_k \left|\left| {\mathcal{M}_{\beta z}}^k 1 \right|\right|_J^{1/k},$$

which is equal to the r.h.s. of formula (4.38). Moreover,

$$\left|\left| {\mathcal{M}_{\beta z}}^k \psi \right|\right|_J \leq \left|\left| {\mathcal{M}_{\beta z}}^k 1 \right|\right|_J ||\psi||_J \leq \left|\left| {\mathcal{M}_{\beta z}}^k 1 \right|\right|_J ||\psi||_{BV(J)}$$

and one checks that

$$\mathrm{var}_J({\mathcal{M}_{\beta z}}^k \psi) \leq \left|\left| {\mathcal{M}_{\beta z}}^k 1 \right|\right|_J \mathrm{var}_J(\psi) + \sup_{\underline{n}_k} \left(z^{|\underline{n}_k|} \mathrm{var}_J(|{G_{\underline{n}_k}}'|^\beta) \right) ||\psi||_J .$$

By Lemma 4.1, $\mathrm{var}_J(|G_{\underline{n}_k}{}'|^\beta) \le C\left|\left||G_{\underline{n}_k}{}'|^\beta\right|\right|_J$, and thus

$$\left|\left|\mathcal{M}_{\beta z}{}^k\right|\right|_{BV(J)} \le \left|\left|\mathcal{M}_{\beta z}{}^k 1\right|\right|_J + C \sup_{\underline{n}_k} \left(z^{|\underline{n}_k|} \left|\left||G_{\underline{n}_k}{}'|^\beta\right|\right|_J \right).$$

Now, taking the limit of the k-th root, the second term of the r.h.s. converges to $\mathrm{r}_{\mathrm{ess}}(\mathcal{M}_{\beta z}) \le \mathrm{r}(\mathcal{M}_{\beta z})$, so that we are left with the upper bound $||\mathcal{M}_{\beta z}||_{BV(J)} \le \lim_k \left|\left|\mathcal{M}_{\beta z}{}^k 1\right|\right|_J^{1/k}$. Due to submultiplicativity, we can replace the limit over k by the infimum. □

Another way of showing the existence of a positive eigenfunction is to use additional smoothness of f'. Furthermore, smoothness of f' implies smoothness of this eigenfunction as well.

Lemma 4.5 *Suppose that*

(a) $0 \le z < 1$ *or*

(b) $z = 1$ *and* $\beta > \frac{r}{1+r}$.

Moreover, suppose that $F_1{}' \in C^\epsilon(I)$ *and* $x^{-r}(F_0{}'(x) - 1) \in C^\epsilon(I)$ *for some* $\epsilon \le \alpha$.

Then $\mathcal{M}_{\beta z}$ *has a positive eigenfunction* $\Psi_{\beta z}$ *with positive eigenvalue* $\lambda_{\beta z}$. *Moreover,* $\Psi_{\beta z}$ *is Hölder-continuous and satisfies*

$$|\log \Psi_{\beta z}(x) - \log \Psi_{\beta z}(y)| \le C|x-y|^\epsilon$$

for some $C > 0$.

PROOF:

This proof uses a method of [29]. In contrast to [29], our specific setting imposes bounds on the exponent ϵ of the Hölder-continuity.

Suppose

$$C_\epsilon = \sup_n \left|\log|(F_1{F_0}^{n-1})'|\right|_{\epsilon,J}$$

is finite. Define

$$\Lambda_\beta = \left\{\psi \in BV(J) \;\middle|\; 0 \le \psi, \quad ||\psi|| = 1, \quad \psi(x) \le \psi(y)\exp\left(\frac{|\beta|C_\epsilon}{1-\lambda_0^{-\epsilon}}|x-y|^\epsilon\right)\right\}.$$

Λ_β is equi-continuous, hence its closure in supremum norm, $\overline{\Lambda}_\beta$, is compact.

Now, for functions $\psi \in \overline{\Lambda}_\beta$ we have

$$\begin{aligned}
&\mathcal{M}_{\beta z}\psi(x) \le \\
&\le \sup_n \left\{\left|\frac{(F_1{F_0}^{n-1})'(x)}{(F_1{F_0}^{n-1})'(y)}\right|^\beta \frac{\psi\circ F_1{F_0}^{n-1}(x)}{\psi\circ F_1{F_0}^{n-1}(y)}\right\}\mathcal{M}_{\beta z}\psi(y) \\
&\le \exp\left(|\beta|C_\epsilon|x-y|^\epsilon\right)\sup_n\left\{\frac{\psi\circ F_1{F_0}^{n-1}(x)}{\psi\circ F_1{F_0}^{n-1}(y)}\right\}\mathcal{M}_{\beta z}\psi(y) \\
&\le \exp\left(|\beta|C_\epsilon|x-y|^\epsilon\right)\sup_n\left\{\exp\left(\frac{|\beta|C_\epsilon}{1-\lambda_0^{-\epsilon}}|F_1{F_0}^{n-1}(x)-F_1{F_0}^{n-1}(y)|^\epsilon\right)\right\}\mathcal{M}_{\beta z}\psi(y) \\
&\le \exp\left(|\beta|C_\epsilon|x-y|^\epsilon\right)\exp\left(\frac{|\beta|C_\epsilon}{1-\lambda_0^{-\epsilon}}\lambda_0^{-\epsilon}|x-y|^\epsilon\right)\mathcal{M}_{\beta z}\psi(y) \\
&= \exp\left(\frac{|\beta|C_\epsilon}{1-\lambda_0^{-\epsilon}}|x-y|^\epsilon\right)\mathcal{M}_{\beta z}\psi(y).
\end{aligned}$$

Define $\widetilde{\mathcal{M}}_{\beta z}$ via

$$\widetilde{\mathcal{M}}_{\beta z}\psi = ||\mathcal{M}_{\beta z}\psi||_J^{-1}\mathcal{M}_{\beta z}\psi.$$

$\widetilde{\mathcal{M}}_{\beta z}$ maps $\overline{\Lambda}_\beta$ into itself. By the Schauder Tychonoff theorem there exists a fixed point of $\widetilde{\mathcal{M}}_{\beta z}$ in $\Psi_{\beta z} \in \overline{\Lambda}_\beta$ such that $\widetilde{\mathcal{M}}_{\beta z}\Psi_{\beta z} = \Psi_{\beta z}$ and hence, with $\lambda_{\beta z} = ||\mathcal{M}_{\beta z}\Psi_{\beta z}||_J$,

$$\mathcal{M}_{\beta z}\Psi_{\beta z} = \lambda_{\beta z}\Psi_{\beta z}.$$

By being in $\overline{\Lambda}_\beta$, this eigenfunction naturally fulfills the smoothness condition stated in the Lemma with $C = \frac{|\beta|C_\epsilon}{1-\lambda_0^{-\epsilon}}$. We also have a lower bound on the eigenvalue $\lambda_{\beta z}$,

$$\lambda_{\beta z} = \frac{||\mathcal{M}_{\beta z}\Psi_{\beta z}||_J}{||\Psi_{\beta z}||_J} \ge \exp\left(\frac{-2|\beta|C_\epsilon}{1-\lambda_0^{-\epsilon}}\right).$$

Concluding the proof, we show that C_ϵ is indeed finite. We have

$$\begin{aligned}
&\left|\log|(F_1{F_0}^{n-1})'|\right|_{\epsilon,J} \leq \\
&\leq \left|\log|F_1'{F_0}^{n-1}|\right|_{\epsilon,J} + \sum_{i=1}^{n-1}\left|\log|F_0'{F_0}^{i-1}|\right|_{\epsilon,J} \\
&\leq |\log|x||_{1,F_1'{F_0}^{n-1}(J)}\left|F_1'{F_0}^{n-1}\right|_{\epsilon,J} + \sum_{i=1}^{n-1}|\log|x||_{1,F_0'{F_0}^{i-1}(J)}\left|F_1'{F_0}^{i-1}\right|_{\epsilon,J}.
\end{aligned}$$

Now, $|\log|x||_1 \leq ||1/x||$ and $||1/F_i'||_I \leq ||f'||_I$. Thus, we can continue

$$\begin{aligned}
&\left|\log|(F_1{F_0}^{n-1})'|\right|_{\epsilon,J} \leq \\
&\leq ||f'||_I\left\{\left|F_1'{F_0}^{n-1}\right|_{\epsilon,J} + \sum_{i=1}^{n-1}\left|F_0'{F_0}^{i-1}\right|_{\epsilon,J}\right\} \\
&\leq ||f'||_I\left\{|F_1'|_{\epsilon,{F_0}^{n-1}(J)}\left|{F_0}^{n-1}\right|_{1,J}^{\epsilon} + \sum_{i=1}^{n-1}|F_0'|_{\epsilon,{F_0}^{i-1}(J)}\left|{F_0}^{i-1}\right|_{1,J}^{\epsilon}\right\} \\
&\leq ||f'||_I\left\{|F_1'|_{\epsilon,{F_0}^{n-1}(J)}\left|\left|({F_0}^{n-1})'\right|\right|_J^{\epsilon} + \sum_{i=1}^{n-1}|F_0'|_{\epsilon,{F_0}^{i-1}(J)}\left|\left|({F_0}^{i-1})'\right|\right|_J^{\epsilon}\right\} \\
&\leq ||f'||_I C_0\left\{|F_1'|_{\epsilon,{F_0}^{n-1}(J)}\, n^{-\epsilon(1+\frac{1}{r})} + \sum_{i=1}^{n-1}|F_0'|_{\epsilon,{F_0}^{i-1}(J)}\, i^{-\epsilon(1+\frac{1}{r})}\right\}. \qquad (4.39)
\end{aligned}$$

The last step uses the asymptotic behavior of the iterates of F_0. Further, using boundedness of $|x^{-r}({F_0}'(x)-1)|_{\epsilon,I}$ and $||x^{-r}({F_0}'(x)-1)||_I$, we get

$$\begin{aligned}
&|F_0'|_{\epsilon,{F_0}^{i-1}(J)} \\
&\leq \left|(F_0'-1)x^{-r}\right|_{\epsilon,{F_0}^{i-1}(J)}||x^r||_{{F_0}^{i-1}(J)} + \left|\left|(F_0'-1)x^{-r}\right|\right|_{{F_0}^{i-1}(J)}|x^r|_{\epsilon,{F_0}^{i-1}(J)} \\
&\leq \left|(F_0'-1)x^{-r}\right|_{\epsilon,I}||x^r||_{{F_0}^{i-1}(J)} + \left|\left|(F_0'-1)x^{-r}\right|\right|_I\left|\left|rx^{r-1}\right|\right|_{{F_0}^{i-1}(J)}\left|{F_0}^{i-1}(J)\right|^{1-\epsilon} \\
&\leq C_1\left\{i^{-1} + i^{(1-r)/r-(1-\epsilon)(1+\frac{1}{r})}\right\} = C_1\left\{i^{-1} + i^{-2+\epsilon(1+\frac{1}{r})}\right\}.
\end{aligned}$$

Thus, the terms in the sum of (4.39) behave as

$$O(i^{-1-\epsilon(1+\frac{1}{r})}) + O(i^{-2})$$

whence the sum is uniformly bounded in n. $\square$

4.2.3 The Spectral Gap for $\mathcal{M}_{\beta z}$

Lemma 4.6 *Suppose that*

(a) $0 \leq z < 1$ *or*

(b) $z = 1$ *and* $\beta > \frac{r}{1+r}$.

Further assume that $\mathrm{r}_{\mathrm{ess}}(\mathcal{M}_{\beta z}) < \mathrm{r}(\mathcal{M}_{\beta z})$. *Then* $\mathcal{M}_{\beta z}$ *has a spectral gap, i.e. there is a* $\theta < \mathrm{r}(\mathcal{M}_{\beta z})$ *such that the only part of the spectrum outside the disk with radius* θ *is the leading eigenvalue* $\lambda_{\beta z} = \mathrm{r}(\mathcal{M}_{\beta z})$.

PROOF: Since $\mathrm{r}_{\mathrm{ess}}(\mathcal{M}_{\beta z}) < \mathrm{r}(\mathcal{M}_{\beta z})$, $\lambda_{\beta z}$ is a simple eigenvalue with positive eigenfunction $\Psi_{\beta z}$ bounded away from 0. Consider the normalized operator

$$\mathcal{N}_{\beta z}\psi = \frac{1}{\lambda_{\beta z}\Psi_{\beta z}}\mathcal{M}_{\beta z}(\Psi_{\beta z}\psi)$$

(i.e. $\mathcal{N}_{\beta z}1 = 1$). Then we need to show that 1 is the only eigenvalue of $\mathcal{N}_{\beta z}$ with magnitude equal to 1.

Suppose $\Psi \in BV(J)$ is eigenvector to an eigenvalue $|\gamma| = 1$ and choose $||\Psi|| = 1$. Then we have

$$\begin{aligned}
|\Psi(x)| &= |\mathcal{N}_{\beta z}\Psi(x)| \\
&= \left|\sum_{n=1}^{\infty} \frac{|G_n'(x)|^{\beta}\ \Psi_{\beta z} \circ G_n(x)}{\lambda_{\beta z}\Psi_{\beta z}(x)}\Psi \circ G_n(x)\right| \\
&\leq \sum_{n=1}^{\infty} \frac{|G_n'(x)|^{\beta}\ \Psi_{\beta z} \circ G_n(x)}{\lambda_{\beta z}\Psi_{\beta z}(x)}\,|\Psi \circ G_n(x)| \\
&\leq \sum_{n=1}^{\infty} \frac{|G_n'(x)|^{\beta}\ \Psi_{\beta z} \circ G_n(x)}{\lambda_{\beta z}\Psi_{\beta z}(x)} \\
&= 1.
\end{aligned}$$

There exists an $x' \in J$ such that $\Psi(x) \to \gamma'$ with $|\gamma'| = 1$ for $x \searrow x'$ or $x \nearrow x'$. Without loss of generality, assume that $\gamma' = 1$. Then, taking one of these limits in the above inequality, the left hand side converges to 1. This only happens if, for all n, $\Psi \circ G_n(x) \to \gamma$. Repeating the argument, we see that for all $k = 0, 1, 2, \ldots$ and for all $n_1, \ldots, n_k = 1, 2, \ldots$,

$$\Psi \circ G_{n_1} \circ G_{n_2} \circ \ldots \circ G_{n_k}(x) \to \gamma^k \qquad \text{for} \quad x \searrow x' \quad \text{or} \quad x \nearrow x'.$$

If $\gamma \neq 1$, this leads to an infinite variation $\mathrm{var}_J(\Psi)$, whence $\gamma = 1$. Thus, there is no other eigenvalue with magnitude equal to 1 and we have a spectral gap. $\square$

Remark 4.2 *If in the above proof the partition were finite, it could still suffice for γ to be a root of 1. Therefore, this proof is not applicable for $\mathcal{L}_\beta$ and we will have to argue differently, invoking the operator relations between $\mathcal{L}_\beta$ and $\mathcal{M}_{\beta z}$.*

4.2.4 Examples: Piecewise Linear Map and Farey Map

For the piecewise linear map $\tilde{f}$, we directly get

$$\mathrm{r}(\widetilde{\mathcal{M}}_{\beta z}) = \sum_{n=1}^{\infty} z^n q_n^\beta > \sup_n z^n q_n^\beta = \mathrm{r}_{\mathrm{ess}}(\widetilde{\mathcal{M}}_{\beta z}).$$

Thus, we have a leading eigenvalue and a spectral gap. (We already know that the eigenfunction was the constant function.)

For the Farey map f given by (1.19), we estimate

$$(1+n)^{-2} \leq |G_n{}'| \leq (1+n/2)^{-2}$$

so that

$$\mathrm{r}_{\mathrm{ess}}(\mathcal{M}_{\beta z}) \leq \sup_n z^n (1+n/2)^{-2\beta} < 1$$

and

$$\mathrm{r}(\mathcal{M}_{\beta z}) \geq \sum_{n=1}^{\infty} z^n (1+n)^{-2\beta}.$$

In particular, for z large enough and β close to 1, $\mathrm{r}_{\mathrm{ess}}(\mathcal{M}_{\beta z}) < \mathrm{r}(\mathcal{M}_{\beta z})$ and again we have existence of a leading eigenvalue and a spectral gap. As $|f'|$ is piecewise Lipshitz, the corresponding eigenfunction is Lipshitz as well.

4.2.5 Some Inequalities for $\mathcal{M}_{\beta z}$

Later, we will need relations between $\mathcal{M}_{\beta z'}$ and $\mathcal{M}_{\beta z}$ for z' complex with $|z'| = z$. These are given by the next Lemma.

Lemma 4.7 *Suppose that*

(a) $0 \leq z < 1$ *and* $\beta \geq 0$ *or*

(b) $z = 1$ *and* $\beta > \frac{r}{1+r}$.

If $|z'| = z$ *then*

$$\mathrm{r}_{\mathrm{ess}}(\mathcal{M}_{\beta z'}) = \mathrm{r}_{\mathrm{ess}}(\mathcal{M}_{\beta z}).$$

If, in addition, $z' \neq z$ *and* $\mathrm{r}_{\mathrm{ess}}(\mathcal{M}_{\beta z}) < \mathrm{r}(\mathcal{M}_{\beta z})$ *holds, then*

$$\mathrm{r}(\mathcal{M}_{\beta z'}) < \mathrm{r}(\mathcal{M}_{\beta z}).$$

PROOF: The first equality is straightforward, as the formula for the essential spectral radius (4.37) is independent of $\arg(z)$.

Now, if $\mathrm{r_{ess}}(\mathcal{M}_{\beta z}) < \mathrm{r}(\mathcal{M}_{\beta z})$ then $\mathrm{r}(\mathcal{M}_{\beta z})$ is eigenvalue of $\mathcal{M}_{\beta z}$. We only need to show that there is no eigenvalue of $\mathcal{M}_{\beta z'}$ with magnitude equal or larger to $\mathrm{r}(\mathcal{M}_{\beta z})$.

Consider the normalized operator

$$\mathcal{N}_{\beta z'}\psi = \frac{1}{\lambda_{\beta z}\Psi_{\beta z}}\mathcal{M}_{\beta z'}(\Psi_{\beta z}\psi)$$

where $\Psi_{\beta z}$ is eigenvector of $\mathcal{M}_{\beta z}$ with eigenvalue $\lambda_{\beta z}$. Clearly, $|\mathcal{N}_{\beta z'}\psi| \leq \mathcal{N}_{\beta z}1||\psi|| = ||\psi||$. Thus, there is no eigenvalue with magnitude larger than 1.

Assume that there is an eigenvalue $|\gamma| = 1$ of $\mathcal{N}_{\beta z'}$. Then, for an eigenvector $\Psi \in BV(J)$ with $||\Psi|| = 1$,

$$|\Psi(x)| = |\mathcal{N}_{\beta z'}\Psi(x)| \leq 1.$$

Arguing as in the previous proof, we see that for some $x' \in J$ and for all $n = 1, 2, \ldots$,

$$(z')^n \ \Psi \circ G_n(x) \to \gamma \qquad \text{for} \quad x \searrow x' \quad \text{or} \quad x \nearrow x'.$$

Now, $G_n(x)$ converges uniformly to 1 for $n \to \infty$. Pick a sequence $x_k \to x'$ such that $(z')^n \ \Psi \circ G_n(x_k) \to \gamma$ for $k \to \infty$. Then $G_n(x_n) \to 1$ and $\Psi \circ G_n(x_n)$ converges due to $\Psi \in BV(J)$, whence $(z')^{-n}\gamma$ must converge for $n \to \infty$. Therefore, $z' = z$ in contradiction to the above assumption.

Thus, no eigenvalue of $\mathcal{M}_{\beta z'}$ is in modulus equal to $\lambda_{\beta z} = \mathrm{r}(\mathcal{M}_{\beta z})$, and the spectral radius $\mathrm{r}(\mathcal{M}_{\beta z})$ has to be strictly smaller. $\square$

Moreover, the following inequalities hold.

Lemma 4.8 *Suppose that*

(a) $|z| < 1$ *and* $\beta \geq 0$ *or*

(b) $|z| = 1$ *and* $\beta > \frac{r}{1+r}$.

For $|z'| \leq |z|$ *we have*

$$\mathrm{r_{ess}}(\mathcal{M}_{\beta z'}) \leq \frac{|z'|}{|z|}\mathrm{r_{ess}}(\mathcal{M}_{\beta z}).$$

If, in addition, z, z' *are real then*

$$\mathrm{r}(\mathcal{M}_{\beta z'}) \leq \frac{z'}{z}\mathrm{r}(\mathcal{M}_{\beta z}).$$

Also, for $\beta' \leq \beta$ *we have*

$$\mathrm{r_{ess}}(\mathcal{M}_{\beta' z}) \geq \lambda_0^{\beta-\beta'}\mathrm{r_{ess}}(\mathcal{M}_{\beta z}).$$

If, in addition, z *is real then*

$$\mathrm{r}(\mathcal{M}_{\beta' z}) \geq \lambda_0^{\beta-\beta'}\mathrm{r}(\mathcal{M}_{\beta z}).$$

(Here, $\beta, \beta' > \frac{r}{1+r}$ *for* $|z| = 1$*.)*

PROOF: The inequalities follow directly from the formulas (4.37) and (4.38) for $\mathrm{r_{ess}}(\mathcal{M}_{\beta z})$ and $\mathrm{r}(\mathcal{M}_{\beta z})$, along with $|G_{\underline{n}_k}{}'| \leq \lambda_0^{-k}$. □

Remark 4.3 *Thus, we have shown that* $\mathrm{r}(\mathcal{M}_{\beta z})$ *is strictly increasing in* z *and strictly decreasing in* β.

We end this section with considering the special case $\mathcal{M}_{11}$.

Lemma 4.9 *We have* $\mathrm{r}(\mathcal{M}_{11}) = 1$*. 1 is a simple leading eigenvalue with positive eigenfunction and* $\mathcal{M}_{11}$ *has a spectral gap. The Lebesgue measure* μ_L *is an eigenmeasure of* $\mathcal{M}_{11}^*$ *with eigenvalue* 1.

PROOF: For each function $\phi \in BV(J)$,

$$\mu_L(\mathcal{M}_{11}\phi) = \int_J \sum_{n=1}^{\infty} |G_n{}'(x)| \, \phi \circ G_n(x) dx = \sum_{n=1}^{\infty} \int_{J_n} \phi(x) dx = \int_J \phi(x) dx = \mu_L(\phi).$$

Thus, μ_L is eigenmeasure of $\mathcal{M}_{11}$ with eigenvalue 1 and it follows that $\mathrm{r}(\mathcal{M}_{11}) = \mathrm{r}(\mathcal{M}_{11}^*) \geq 1$. Since $\mathrm{r}_{\mathrm{ess}}(\mathcal{M}_{11}) \leq 1/\lambda_0 < 1$, $\lambda_{11} = \mathrm{r}(\mathcal{M}_{11})$ is a simple eigenvalue with positive eigenfunction Ψ_{11}. It follows that

$$0 < \mu_L(\Psi_{11}) = \mu_L(\mathcal{M}_{11}\Psi_{11}) = \lambda_{11}\mu_L(\Psi_{11}),$$

whence $\mathrm{r}(\mathcal{M}_{11}) = \lambda_{11} = 1$. $\square$

4.3 The Spectrum of $\mathcal{L}_\beta$

4.3.1 The Essential Spectral Radius of $\mathcal{L}_\beta$

Lemma 4.10 *Suppose $\beta \geq 0$. The essential spectral radius $\mathrm{r}_{\mathrm{ess}}(\mathcal{L}_\beta)$ is equal to 1.*

PROOF: The conditions for the application of the theorem are fulfilled:

f is a piecewise monotone transformation of $[0, 1]$ with a finite generating partition

$$\mathcal{Z} = \{[0, a), [a, 1]\}.$$

$s(x)$ is given by

$$s(x) = |f'(x)|^{-\beta}$$

and is in $BV(I)$. The condition $\sum_{Z_i \in \mathcal{Z}} ||s||_{Z_i} < \infty$. is trivially fulfilled.

We need to compute the essential spectral radius. For $\beta \geq 0$, we have

$$\begin{aligned} s_n(x) &= |f'(f^{n-1}x)|^{-\beta} \cdot \ldots \cdot |f'(fx)|^{-\beta} |f'(x)|^{-\beta} \\ &= |(f^n)'(x)|^{-\beta} \\ &\leq 1, \end{aligned}$$

where equality holds at the indifferent fixed point $x = 0$. Thus,

$$\mathrm{r_{ess}}(\mathcal{L}_\beta) = \lim_{n \to \infty} ||s_n||^{1/n} = 1. \ \Box$$

4.3.2 The Existence of a Leading Eigenvalue λ_β and a Spectral Gap of $\mathcal{L}_\beta$ for $0 \leq \beta < 1$

Here, we apply the previously derived relations between $\mathcal{L}$ and $\mathcal{M}_z$. According to Theorem 2.7, eigenfunctions of $\mathcal{M}_{\beta z}$ with eigenvalue 1 are also eigenfunctions of $\mathcal{L}_\beta$ with eigenvalue $\lambda = 1/z$, provided that $|\lambda| > \mathrm{r}(\mathcal{L}_{0\beta}) = 1$. Now, the radius of convergence of $\mathcal{M}_{\beta z}$ as well as $\mathrm{r_{ess}}(\mathcal{L}_\beta)$ are equal to 1, so that we control the whole region $|\lambda| > 1$ through $\mathcal{M}_{\beta z}$. Figure 4.4 shows the relation between the spectra of $\mathcal{M}_{\beta z(\beta)}$ (a) and $\mathcal{L}_\beta$ (b) for $\beta < 1$:

(a) The essential spectral radius of $\mathcal{M}_{\beta z(\beta)}$ is strictly smaller than 1, with only isolated eigenvalues outside. The leading eigenvalue $\lambda_{\min(\mathcal{M}_{\beta z(\beta)})}$ is isolated from the rest of the spectrum by a spectral gap. Moreover, $z(\beta)$ is chosen such that this leading eigenvalue is equal to 1.

(b) Thus, the leading eigenvalue of $\mathcal{L}_\beta$ is equal to $1/z(\beta)$. Here, the essential spectral radius is equal to 1, with only isolated eigenvalues outside. Again, the leading eigenvalue is isolated from the rest of the spectrum by a spectral gap.

We will first show that

Theorem 4.2 *Let $0 \leq \beta < 1$. Then $\mathcal{L}_\beta$ has a leading eigenvalue $\lambda_\beta > 1$. This eigenvalue is simple and the corresponding eigenfunction positive.*

PROOF: We have $\mathrm{r}_{\mathrm{ess}}(\mathcal{M}_{\beta z}) < |z|\lambda_0^{-\beta}$ which is less than 1 unless $\beta = 0$ and $|z| = 1$. Now, by Lemma 4.8 we have $\mathrm{r}(\mathcal{M}_{\beta 1}) > \mathrm{r}(\mathcal{M}_{11}) = 1$ for $\beta < 1$. Using monotonicity of $\mathcal{M}_{\beta z}$ in z, for each $\beta < 1$ we can choose a $z > 0$ such that $\mathrm{r}(\mathcal{M}_{\beta z}) = 1$. Then, $\mathcal{M}_{\beta z}$ has a simple eigenvalue 1 with positive eigenfunction, which implies that $\lambda = z^{-1}$ is a simple eigenvalue of $\mathcal{L}_\beta$ with positive eigenfunction.

Moreover, since we have $\mathrm{r}(\mathcal{M}_{\beta z'}) < \mathrm{r}(\mathcal{M}_{\beta z})$ for $|z'| < z$, there is no eigenvalue λ of $\mathcal{L}_\beta$ with $|\lambda| > z^{-1}$. $\square$

Finally, we get

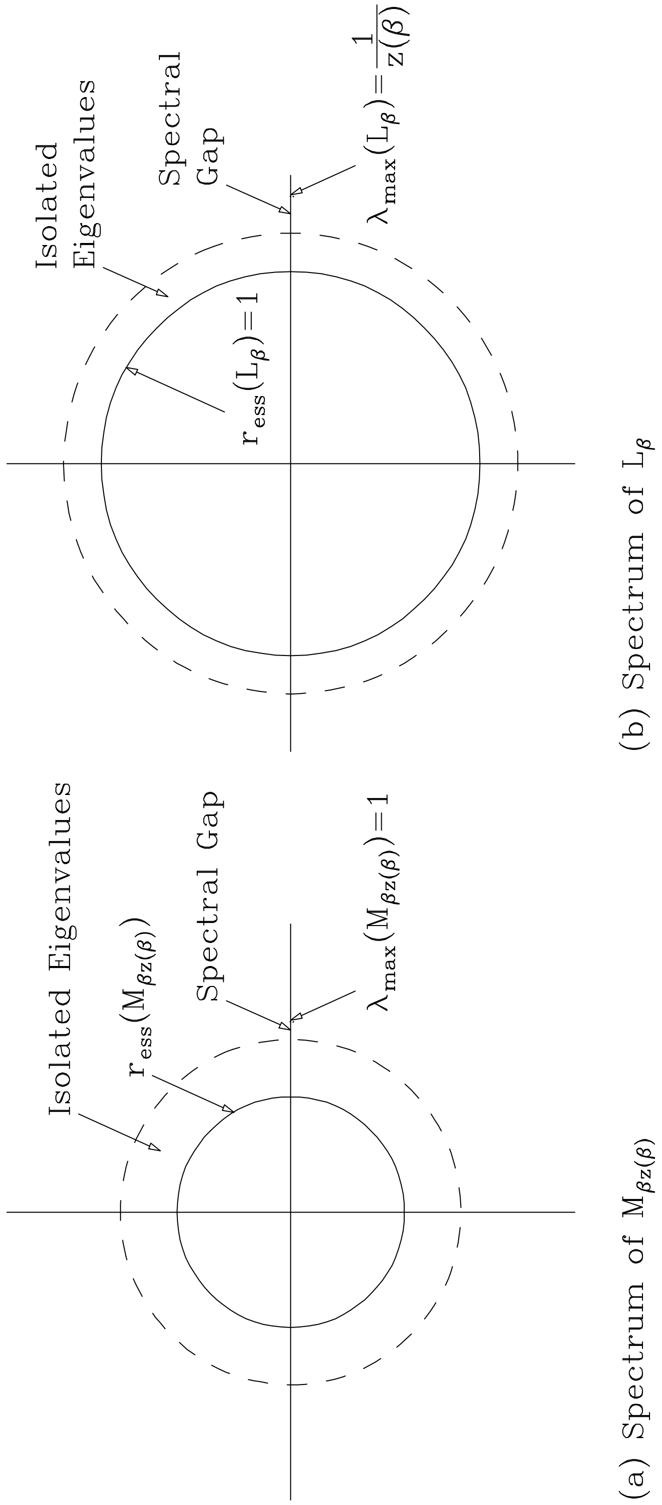
Lemma 4.11 *$\mathcal{L}_\beta$ has a spectral gap for $\beta < 1$, i.e. there is a $\theta < \mathrm{r}(\mathcal{L}_\beta)$ such that the only part of the spectrum outside the disk with radius θ is the leading eigenvalue $\lambda_\beta = \mathrm{r}(\mathcal{L}_\beta)$.*

PROOF: Let $z = \lambda_\beta^{-1}$. Then, we have by Lemma 4.7 $\mathrm{r}(\mathcal{M}_{\beta z'}) < \mathrm{r}(\mathcal{M}_{\beta z})$ for $|z'| = z$, $z' \neq z$, so that there is no other eigenvalue λ of $\mathcal{L}_\beta$ with $|\lambda| = \lambda_\beta$.

Moreover, as $1 = \mathrm{r}_{\mathrm{ess}}(\mathcal{L}_\beta) < \lambda_\beta$, all other eigenvalues of $\mathcal{L}_\beta$ have to be in a disk with radius strictly smaller than λ_β. $\square$

4.3.3 Analyticity of λ_β for $0 \leq \beta < 1$

Lemma 4.12 *The leading eigenvalue λ_β of $\mathcal{L}_\beta$ is real-analytic in $\beta > 1$.*



The spectra of the modified transfer operator $M_{\beta z(\beta)}$ (a) and the transfer operator L_β (b) for $\beta<1$

Figure 4.4:

PROOF: We have to show that $z(\beta)$, defined by $\mathrm{r}(\mathcal{M}_{\beta z(\beta)}) = 1$, is analytic. As $\mathrm{r}_{\mathrm{ess}}(\mathcal{M}_{\beta z}) < 1$, this is a simple eigenvalue. Since $\mathcal{M}_{\beta z}$ is jointly analytic in $|z| < 1$ and β, standard perturbation theory of simple eigenvalues ([22], Chapter VII, Theorem 1.9) applies, by which there are at most finitely many solutions to $\mathrm{r}(\mathcal{M}_{\beta z}) = 1$. Thus, $z(\beta)$ is unique in a small neighborhood of in a small neighborhood of $\{\beta, z(\beta)\}$ and, by [22], Chapter VII, Theorem 1.8, consists of a branch of an algebraic function. Thus, $z(\beta)$ is analytic. □

For the piecewise linear map, $z(\beta)$ can be estimated in a particularly simple way. It is given implicitly by the equation

$$\sum_{n=1}^{\infty} z^n q_n^\beta = 1.$$

4.3.4 The Phase Transition at $\beta = 1$

Lemma 4.13 *For $\beta \geq 1$, $\mathrm{r}(\mathcal{L}_\beta) = 1$*

PROOF: $\mathrm{r}(\mathcal{L}_\beta)$ is decreasing in β. Furthermore, $\mathrm{r}(\mathcal{L}_1) = 1$ and $\mathrm{r}_{\mathrm{ess}}(\mathcal{L}_\beta) = 1$. □

Thus, it follows immediately that

Corollary 4.2 $\mathrm{r}(\mathcal{L}_\beta)$ *is analytic in $\beta < 1$ and $\beta > 1$ with a non-analyticity at $\beta = 1$*

Thus, we have accomplished a complete description of the pressure function, as depicted in Figure 4.1.

At the phase transition $\beta = 1$, we still have an eigenfunction ϕ of $\mathcal{M}_{11}$. However, the extension $\mathcal{M}^+_{\beta z}$ is unbounded and computing $\mathcal{M}_{11}\phi$ leads to a singularity at the origin. Using the asymptotics of ${G_n}'$ along with the positivity of ϕ in J, we can show that

Lemma 4.14

$$\mathcal{M}_{11}^{+}\phi(x) \sim x^{-r}, \quad x \to 0$$

PROOF: Theorem (3.1) implies

$$\begin{aligned}
\sum_{n=1}^{\infty} |{G_n}'(x)|\phi \circ G_n(x) &= \sum_{n=1}^{\infty} |{F_1}'(0)|(g^n)'(x)(1+o(1))(\psi(1)+o(1)) \\
&= |{F_1}'(0)|\phi(1)(1+o(1)) \sum_{n=1}^{\infty}(1+nrcx^r)^{-1-\frac{1}{r}}
\end{aligned}$$

Estimating the sum by the corresponding integral, we see that it diverges as x^{-r} for $x \to 0$. $\square$

Chapter 5

Asymptotic Behavior at the Phase Transition

Now, we compute the asymptotic behavior at the phase transition $\beta = 1$. In the first section, perturbation theory is used to get the expansion. This expansion will be done in terms of a function $\eta_r(z)$. The second section provides the asymptotics for $\eta_r(z)$ and, combining these expansions, we get the desired result.

In the following estimations, we need Hölder-continuity of the leading eigenfunction of $\mathcal{M}_{\beta z}$. Thus, we suppose throughout this chapter that $f \in \mathcal{C}_r$ is Hölder-continuous as in Theorem 4.5, i.e. ${F_1}' \in C^\epsilon(I)$ and $x^{-r}({F_0}'(x) - 1) \in C^\epsilon(I)$ for some $\epsilon \leq \alpha$.

5.1 Perturbation Expansion at the Phase Transition

First, we summarize our knowledge of $\mathcal{M}_{\beta z}$. For β close to 1, the map

$$(\beta, z) \mapsto \mathcal{M}_{\beta z}$$

is continuous in $0 \leq z \leq 1$ and β, and analytic in β. For z close to 1, we have an isolated simple eigenvalue $\lambda_{\beta z} = \mathrm{r}(\mathcal{M}_{\beta z})$ with a positive Hölder-continuous eigenfunction $\Psi_{\beta z}$, both being continuous in $z \leq 1$ and analytic in β. Moreover, $\lambda_{11} = 1$, and the Lebesgue measure $\mu_{11} = \mu_L$ is an eigenvector of $\mathcal{M}_{11}^*$ with eigenvalue 1, i.e. for all $\psi \in BV(I)$,

$$\mu_{11}(\mathcal{M}_{11}\psi) = \mu_{11}(\psi).$$

Also, $\lambda_{\beta z}$ is strictly increasing in z and strictly decreasing in β with a unique solution $z(\beta)$ of $\lambda_{\beta z(\beta)} = 1$ which is analytic for $\beta < 1$ and strictly increasing.

Let

$$\lambda_{\beta z} = \sum_{n=0}^{\infty} \lambda_n(z)(1-\beta)^n \quad \text{and} \quad \mathcal{M}_{\beta z} = \sum_{n=0}^{\infty} (1-\beta)^n \mathcal{M}_z{}^{(n)}. \tag{5.1}$$

be the expansions of $\lambda_{\beta z}$ and $\mathcal{M}_{\beta z}$ for β close to 1 and $z \leq 1$. $z \mapsto \lambda_n(z)$ is continuous and $\lambda_0(1) = 1$. Analyzing the perturbation series, we see that $\lambda_0(z) = \lambda_{1z}$ is the largest eigenvalue of

$$\mathcal{M}_z{}^{(0)} = \mathcal{M}_{1z}.$$

With P_{1z} denoting the spectral projection of $\mathcal{M}_{1z}$ corresponding to the eigenvalue λ_{1z} and Ψ_{11} being the positive eigenvector of $\mathcal{M}_{11}$ normalized to

$$\mu_{11}(\Psi_{11}) = 1, \tag{5.2}$$

we choose the left and right positive eigenvectors μ_{1z} and Ψ_{1z} of $\mathcal{M}_{1z}$ as

$$\Psi_{1z} = P_{1z}\Psi_{11} \quad \text{and} \quad \mu_{1z} = \mu_{11}P_{1z}. \tag{5.3}$$

Then, $\lambda_1(z)$ is given by

$$\lambda_1(z)\mu_{1z}(\Psi_{1z}) = \mu_{1z}(\mathcal{M}_z{}^{(1)}\Psi_{1z}). \tag{5.4}$$

We note that since

$$\mathcal{M}_z{}^{(1)}\Psi(x) = \sum_{n=1}^{\infty} z^n |G_n'(x)|(-\log|G_n'(x)|)\Psi \circ G_n(x), \tag{5.5}$$

the finite limit $\lambda_1(1)$ of $\lambda_1(z)$ as $z \nearrow 1$ is strictly positive:

$$\lambda_1(1) = \mu_{11}(\mathcal{M}_1^{(1)}\Psi_{11}) = \mu_L(\log|g'|\Psi_{11}) > 0. \tag{5.6}$$

In order to describe the asymptotics of λ_{1z} and of $z(\beta)$, we define for $r > 0$

$$\eta_r(z) = \sum_{n=1}^{\infty} \frac{1 - z^n}{n^{1+\frac{1}{r}}}. \tag{5.7}$$

$\eta_r(z) \searrow 0$ as $z \nearrow 1$. A more precise description of the behaviour of $\eta_r(z)$ will be given in the next section.

Now we can proceed to show

Lemma 5.1 *We have the asymptotic expression*

$$\lambda_{1z} = 1 - c_r\eta_r(z) + O(\eta_{r/(1+\epsilon)}(z))$$

with

$$c_r = \frac{c}{(rc)^{1+1/r}}|F_1{}'(0)|\Psi_{11}(1).$$

Proof: We note that since

$$\mathcal{M}_{1z}\Psi_{1z} = \lambda_{1z}\Psi_{1z} \quad \text{and} \quad \mu_{11}\mathcal{M}_{11} = \mu_{11}$$

we have

$$(\lambda_{1z} - 1)\mu_{11}(\Psi_{1z}) = \mu_{11}((\mathcal{M}_{1z} - \mathcal{M}_{11})\Psi_{1z}). \tag{5.8}$$

We will show that Ψ_{1z} can be replaced by Ψ_{11} to leading order and that taking into account the normalization (5.2) one has

$$\lambda_{1z} = 1 + \mu_{11}((\mathcal{M}_{1z} - \mathcal{M}_{11})\Psi_{11}) + O(\eta_r(z)^2). \tag{5.9}$$

To prove (5.9) we first note that

$$\begin{aligned} ||\mathcal{M}_z - \mathcal{M}_1||_{BV(J)} &\leq \sum_{n=1}^{\infty}(1 - z^n)||\mathcal{M}_n||_{BV(J)} \\ &\leq C\sum_{n=1}^{\infty}(1 - z^n)|||G_n|'|| \leq C'\sum_{n=1}^{\infty}\frac{1 - z^n}{n^{1+1/r}} = O(\eta_r(z)) \end{aligned}$$

for some $C, C' > 0$. By the spectral properties of $\mathcal{M}_{11}$ this implies that for the spectral projections

$$||P_{1z} - P_{11}||_{BV(J)} = O(\eta_r(z))$$

also holds and that therefore $||\Psi_{1z} - \Psi_{11}||_{BV(J)} = O(\eta_r(z))$. This together with (5.8) leads to

$$\begin{aligned} \lambda_{1z} &= 1 + \frac{\mu_{11}((\mathcal{M}_{1z} - \mathcal{M}_{11})\Psi_{11}) + \mu_{11}((\mathcal{M}_{1z} - \mathcal{M}_{11})(\Psi_{1z} - \Psi_{11}))}{1 + \mu_{11}(\Psi_{1z} - \Psi_{11}))} \\ &= 1 + \frac{\mu_{11}((\mathcal{M}_{1z} - \mathcal{M}_{11})\Psi_{11}) + O(\eta_r(z)^2)}{1 + O(\eta_r(z))} \end{aligned}$$

which in turn implies (5.9), since also $\mu_{11}((\mathcal{M}_{1z} - \mathcal{M}_{11})\Psi_{11}) = O(\eta_r(z))$.

Next, we estimate $\delta = \mu_{11}((\mathcal{M}_{11} - \mathcal{M}_{1z})\Psi_{11})$. One gets

$$\delta = \int_J \sum_{n=1}^{\infty}(1 - z^n)|G_n'(x)|\Psi_{11} \circ G_n(x)dx$$

$$= \sum_{n=1}^{\infty}(1-z^n)\int_{J_n}\Psi_{11}(x)dx, \tag{5.10}$$

where $J_n = G_n(J)$ has boundary points $F_1{F_0}^n(1)$ and $F_1{F_0}^{n-1}(1)$. Denoting a primitive of Ψ_{11} by $\widetilde{\Psi}$, one can write (5.10) as

$$\delta = \sum_{n=1}^{\infty}(1-z^n)\left|\widetilde{\Psi}\circ F_1{F_0}^n(1) - \widetilde{\Psi}\circ F_1{F_0}^{n-1}(1)\right|. \tag{5.11}$$

By Theorem 4.5, Ψ_{11} is C^ϵ, so that $\widetilde{\Psi}$ is $C^{1+\epsilon}$. Since F_1 is $C^{1+\epsilon}$ by assumption, we see that $\widetilde{\Psi}\circ F_1$ is $C^{1+\epsilon}$ as well.

Using $a_n = {F_0}^n(1)$, we have

$$\begin{aligned}\delta &= \sum_{n=1}^{\infty}(1-z^n)\left|\widetilde{\Psi}\circ F_1F_0(a_{n-1}) - \widetilde{\Psi}\circ F_1(a_{n-1})\right| \\ &= \sum_{n=1}^{\infty}(1-z^n)\left|(\widetilde{\Psi}\circ F_1)'(\xi_n)\right|[a_{n-1}-F_0(a_{n-1})],\end{aligned}$$

where $\xi_n \in [F_0(a_{n-1}), a_{n-1}]$. Using $R(x)$ of (1.11), we write

$$\delta = \sum_{n=1}^{\infty}(1-z^n)\ |(\widetilde{\Psi}\circ F_1)'(\xi_n)|\ c(a_{n-1})^{1+r}(1+R(a_{n-1})). \tag{5.12}$$

Hölder-continuity leads to the estimate

$$(\widetilde{\Psi}\circ F_1)'(x) = F_1'(x)\Psi_{11}\circ F_1(x) = F_1'(0)\Psi_{11}(1) + O(x^\epsilon),$$

and Theorem 3.1 implies

$$a_{n-1} = (nrc)^{-\frac{1}{r}}\left\{1 + O(n^{-\alpha/r})\right\}.$$

Inserting this and $R(x) = O(x^\alpha)$ into (5.12) we can write

$$\begin{aligned}\delta &= \sum_{n=1}^{\infty}(1-z^n)\left\{|F'(0)|\Psi_{11}(1) + O(n^{-\epsilon/r})\right\}c(nrc)^{-1-\frac{1}{r}}\left\{1+O(n^{-\alpha/r})\right\} \\ &= c_r\sum_{n=1}^{\infty}(1-z^n)\left\{n^{-1-\frac{1}{r}} + O(n^{-1-(1+\epsilon)/r}) + O(n^{-1-(1+\alpha)/r})\right\} \\ &= c_r\eta_r(z) + O(\eta_{r/(1+\alpha)}(z)) + O(\eta_{r/(1+\epsilon)}(z)), \qquad z\nearrow 1.\end{aligned}$$

Combining this with (5.9) and $\epsilon \leq \alpha$ we obtain the lemma, since

$$O(\eta_r(z)^2) = O(\eta_r(z)) = O(\eta_{r/(1+\alpha)}(z)) = O(\eta_{r/(1+\epsilon)}(z)), \qquad z \nearrow 1. \ \square$$

From this lemma, the desired result follows immediately.

Theorem 5.1 *As* $\beta \nearrow 1$

$$z(\beta) = \eta_r^{-1}\left(\frac{\lambda_1(1)}{c_r}(1-\beta)\right)[1+o(1)].$$

Proof: Combining Lemma 5.1 with the expansion (5.1) of $\lambda_{\beta z}$ gives

$$1 = \lambda_{\beta z} = 1 - c_r\eta_r(z) + O(\eta_{r/(1+\epsilon)}(z)) + (1-\beta)\lambda_1(1)\{1 + o((1-z)^0)\}\{1 + O(1-\beta)\},$$

whence

$$\eta_r(z) + o(\eta_r(z)) = \frac{\lambda_1(1)}{c_r}(1-\beta) + O((1-\beta)^2).$$

Hence, $\eta_r(z) = O(1-\beta)$ and, thus,

$$\eta_r(z) = \frac{\lambda_1(1)}{c_r}(1-\beta) + o(1-\beta).$$

Applying η_r^{-1} on both sides implies the theorem. $\square$

5.2 Asymptotic Estimation

We now investigate the asymptotics of $\eta_r(z)$ as $z \nearrow 1$. In order to obtain an integral representation of $\eta_r(z)$, we use

$$\int_0^\infty t^{\frac{1}{r}} e^{-nt} dt = \frac{\Gamma(1+\frac{1}{r})}{n^{1+\frac{1}{r}}}.$$

Then, for $z \leq 1$ one obtains

$$\sum_{n=1}^{\infty} \frac{z^n}{n^{1+\frac{1}{r}}} = \frac{1}{\Gamma(1+\frac{1}{r})} \int_0^\infty t^{\frac{1}{r}} \frac{ze^{-t}}{1-ze^{-t}} dt,$$

and hence,

$$\begin{aligned}
\eta_r(z) &= \frac{1}{\Gamma(1+\frac{1}{r})} \int_0^\infty t^{\frac{1}{r}} e^{-t} \left[\frac{1}{1-e^{-t}} - \frac{z}{1-ze^{-t}} \right] dt \\
&= \frac{1}{\Gamma(1+\frac{1}{r})} \int_0^\infty t^{\frac{1}{r}} e^{-t} \frac{1-z}{(1-e^{-t})(1-ze^{-t})} dt \\
&= \frac{1}{\Gamma(1+\frac{1}{r})} \int_0^\infty t^{\frac{1}{r}-1} \frac{e^{-t}}{\phi(t)} \frac{1}{1+\zeta t \phi(t)} dt \qquad (5.13)
\end{aligned}$$

where

$$\phi(t) = \frac{1-e^{-t}}{t} \quad \text{and} \quad \zeta = \frac{z}{1-z}.$$

We will investigate the asymptotic behavior of (5.13) as $\zeta \to \infty$. It will be convenient to split up the integral as follows: using the identity

$$\frac{1}{1+x} = \frac{1}{x} - \frac{1}{x}\frac{1}{1+x}, \quad x > 0, \qquad (5.14)$$

and setting

$$\psi(t) = \frac{e^{-t}}{\phi(t)^2} \qquad \left(= \frac{4t^2}{\sinh(\frac{t}{2})^2} \right)$$

we can write

$$\begin{aligned}
\eta_r(z) &= \frac{1}{\Gamma(1+\frac{1}{r})} \int_0^1 t^{\frac{1}{r}-1} \frac{e^{-t}}{\phi(t)} \frac{1}{1+\zeta t \phi(t)} dt + \\
&\quad + \frac{1}{\Gamma(1+\frac{1}{r})} \zeta^{-1} \left[\int_1^\infty t^{\frac{1}{r}-2} \psi(t) dt - \int_1^\infty t^{\frac{1}{r}-2} \psi(t) \frac{1}{1+\zeta(1-e^{-t})} dt \right].
\end{aligned}$$

The first integral in the square brackets is finite for all r and the last one can be majorized by

$$\frac{e}{\zeta(e-1)} \int_1^\infty t^{\frac{1}{r}-2} \psi(t) dt$$

so that we are left with estimating the integral

$$\vartheta_r(z) = \int_0^1 t^{\frac{1}{r}-1} \frac{e^{-t}}{\phi(t)} \frac{1}{1+\zeta t\phi(t)} dt. \tag{5.15}$$

We first consider the case of $0 < r < 1$. Here, we again apply (5.14) and get

$$\vartheta_r(z) = \zeta^{-1} \left[\int_0^1 t^{\frac{1}{r}-2} \psi(t) dt - \int_0^1 t^{\frac{1}{r}-2} \psi(t) \frac{1}{1+\zeta(1-e^{-t})} dt \right].$$

The first integral is bounded and the second one is majorized by

$$\begin{aligned} \int_0^1 t^{\frac{1}{r}-2} \frac{4}{e} \frac{1}{1+\zeta t(1-e)/e} dt &= \frac{4}{e} \zeta^{1-\frac{1}{r}} \int_0^\zeta \tau^{\frac{1}{r}-2} \frac{1}{1+\tau(1-e)/e} d\tau \\ &= O(\zeta^{1-\frac{1}{r}}) + O(\zeta^{-1}). \end{aligned}$$

Here we performed the change of variables

$$\tau = \zeta t. \tag{5.16}$$

Hence,

$$\eta_r(z) = \frac{1}{\Gamma(1+\frac{1}{r})} \zeta^{-1} \int_0^\infty t^{\frac{1}{r}-2} \psi(t) dt + O(\zeta^{-\min\{\frac{1}{r},2\}}).$$

In the case of $r = 1$ we write

$$\begin{aligned} \vartheta_1(z) &= \int_0^1 \frac{e^{-t}}{\phi(t)} \frac{1}{1+\zeta t\phi(t)} dt \\ &= \zeta^{-1} \left[\int_0^1 \frac{e^{-\tau/\zeta}}{\phi(\tau/\zeta)} \frac{1}{1+\tau\phi(\tau/\zeta)} d\tau + \int_1^\zeta \frac{e^{-\tau/\zeta}}{\phi(\tau/\zeta)} \frac{1}{1+\tau\phi(\tau/\zeta)} d\tau \right]. \end{aligned}$$

The first integral converges to $\int_0^1 \frac{dt}{1+t} = \log 2$ as $\zeta \to \infty$. Applying (5.14) to the second integral, we write it as

$$\int_1^\zeta \frac{\psi(\tau/\zeta)}{\tau} d\tau - \int_1^\zeta \frac{\psi(\tau/\zeta)}{\tau} \frac{1}{1+\tau\phi(\tau/\zeta)} d\tau. \tag{5.17}$$

The second integral in (5.17) is $O(1)$. Performing the change of variables (5.16) in the first integral, we can write it as

$$\int_{1/\zeta}^{1} \frac{\psi(t)}{t} dt.$$

Since ψ is analytic at the origin and $\psi(0) = 1$, then $\psi(t) = 1 + t\psi_1(t)$ with ψ_1 bounded at the origin. Therefore, for (5.17) we obtain

$$\log \zeta + \int_{1/\zeta}^{1} \psi_1(t) dt + O(1)$$

and thus,

$$\eta_1(z) = \frac{1}{\Gamma(1+\frac{1}{r})} \zeta^{-1} \log \zeta + O(\zeta^{-1}).$$

For $r > 1$, we use the change of variables (5.16) in (5.15) and obtain

$$\vartheta_r(z) = \zeta^{-\frac{1}{r}} \int_0^{\zeta} \tau^{\frac{1}{r}-1} \frac{e^{-\tau/\zeta}}{\phi(\tau/\zeta)} \frac{1}{1+\tau\phi(\tau/\zeta)} d\tau.$$

The integrand can be written as

$$\tau^{\frac{1}{r}-1} \frac{1}{1+\tau} \left(1 + \frac{\tau}{\zeta}\psi_2(\tau, \tau/\zeta)\right)$$

where $|\psi_2(\tau, \tau/\zeta)|$ is uniformly bounded for $\tau \in [0, \zeta]$. Since $\int_0^{\zeta} \frac{\tau^{\frac{1}{r}}}{1+\tau} d\tau = O(\zeta^{\frac{1}{r}})$, we deduce

$$\vartheta_r(z) = \zeta^{-\frac{1}{r}} \int_0^{\infty} \tau^{\frac{1}{r}-1} \frac{1}{1+\tau} d\tau + O(\zeta^{-1})$$

as $\zeta \to \infty$. Thus,

$$\eta_r(z) = \frac{1}{\Gamma(1+\frac{1}{r})} \zeta^{-\frac{1}{r}} \int_0^{\infty} \tau^{\frac{1}{r}-1} \frac{1}{1+\tau} d\tau + O(\zeta^{-1}).$$

Summarizing, we obtain that as $\zeta \to \infty$

$$\eta_r(z) = \begin{cases} \frac{1}{\Gamma(1+\frac{1}{r})} \zeta^{-1} \int_0^{\infty} t^{\frac{1}{r}-2} \psi(t) dt + O(\zeta^{-\min\{\frac{1}{r}, 2\}}), & 0 < r < 1 \\ \frac{1}{\Gamma(1+\frac{1}{r})} \zeta^{-1} \log \zeta + O(\zeta^{-1}), & r = 1 \\ \frac{1}{\Gamma(1+\frac{1}{r})} \zeta^{-\frac{1}{r}} \int_0^{\infty} \frac{\tau^{\frac{1}{r}-1}}{1+\tau} d\tau + O(\zeta^{-1}), & r > 1. \end{cases} \tag{5.18}$$

This leads to the final result on the asymptotic behavior of λ_β for $\beta \nearrow 1$ and, thus, for the pressure $P(\beta \log |f'|)$.

Theorem 5.2 *The asymptotic behavior of the pressure $P(\beta \log |f'|)$ for functions $f \in \mathcal{C}_r$ is given as*

$$P(\beta \log |f'|) = \begin{cases} d_r(1-\beta)[1+o(1)], & 0 < r < 1 \\ d_1 \frac{1-\beta}{-\log(1-\beta)}[1+o(1)], & r = 1 \\ d_r(1-\beta)^r[1+o(1)], & r > 1 \end{cases}$$

for $\beta \nearrow 1$. Here,

$$d_r = \begin{cases} \Gamma(1+\frac{1}{r})\frac{\lambda_1(1)}{c_r}\left(\int_0^\infty t^{\frac{1}{r}}(\frac{1}{2}\sinh\frac{t}{2})^{-2}dt\right)^{-1}, & 0 < r < 1 \\ \frac{\lambda_1(1)}{c_1}, & r = 1 \\ \left(\Gamma(1+\frac{1}{r})\frac{\lambda_1(1)}{c_r}\right)^r \left(\int_0^\infty \frac{\tau^{\frac{1}{r}-1}}{1+\tau}d\tau\right)^{-r}, & r > 1 \end{cases}$$

with

$$c_r = \frac{c}{(rc)^{1+1/r}}|F_1{}'(0)|\Psi_{11}(1)$$

and

$$\lambda_1(1) = \int_J \Psi_{11}(x)\log|g'(x)|dx.$$

Proof: This follows from (5.18) along with Theorem 5.1, using $P(\beta \log |f'|) = -\log z(\beta)$ and $-\log z = \zeta^{-1} + O(\zeta^{-2})$. Inverting $\eta_r(z)$ for $r = 1$, we use that $-y \log y = x$ implies $y = \frac{x}{-\log x}(1 + O(\frac{\log(-\log x)}{(\log x)^2}))$. $\square$

Appendix A

Notation and General Definitions

Here, we list some notation and general definitions for reference.

A.1 Notation

We use the following symbols:

- $\mathbf{Z}$ the set of integers $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$
- $\mathbf{N_0}$ the set of non-negative integers $\{0, 1, 2, \ldots\}$
- $\mathbf{N}$ the set of positive integers $\{1, 2, \ldots\}$
- $\mathbf{R}$ the set of real numbers
- $\mathbf{C}$ the set of complex numbers
- $\mathrm{int}D$ the interior of the set D

$\mathrm{cl}D$ the closure of the set D

∂D the boundary of the set D

D^c the complement of the set D

$f = O(g)$ means that $\frac{f(t)}{g(t)}$ is bounded

$f = o(g)$ means that $\lim_t \frac{f(t)}{g(t)} = 0$

$f \sim g$ means that $\lim_t \frac{f(t)}{g(t)} = 1$, i. e. $f(t) = g(t) + o(g(t))$

A.2 General Definitions

For a complex-valued function Ψ on the interval $I = [a, b]$, let $||\Psi||_I$ denote the usual supremum norm, i. e.

$$||\Psi||_I = \sup_{x \in I} |\Psi(x)|.$$

The space of continuous functions, equipped with the supremum norm, is denoted by $C(I)$. We further denote

$$\begin{aligned} \mathrm{var}_I(\Psi) &= \sup\{\sum_{i=1}^{n} |\Psi(a_i) - \Psi(a_{i-1})| : n \geq 1, \;\; a_0 < a_1 < \cdots < a_n, \;\; a_i \in I\}, \\ ||\Psi||_{BV(I)} &= \mathrm{var}_I(\Psi) + ||\Psi||_I, \end{aligned}$$

and define

$$BV(I) = \left\{ \Psi : I \to \mathbf{C} : \;\; ||\Psi||_{BV(I)} < \infty \right\},$$

the space of functions with bounded variation on the interval I.

We note that for functions of bounded variation the one-sided limits exist, i. e. $\lim_{x \searrow y} \Psi(x)$ for $y \in [a, b[$ and $\lim_{x \nearrow y} \Psi(x)$ for $y \in]a, b]$.

Moreover, for functions f and g on I,

$$\begin{aligned} \mathrm{var}_I(|f|) &\leq \mathrm{var}_I(f) \\ \mathrm{var}_I(f+g) &\leq \mathrm{var}_I(f) + \mathrm{var}_I(g) \\ \mathrm{var}_I(f \cdot g) &\leq ||f||_I \mathrm{var}_I(g) + \mathrm{var}_I(f)||g||_I, \end{aligned}$$

and, for functions f on I and g on $J = f(I)$,

$$\mathrm{var}_I(g \circ f) = \mathrm{var}_J(g).$$

We also need the notion of Hölder-continuous functions. For a complex-valued function Ψ on I, denote for some $0 < \epsilon \leq 1$

$$|\Psi|_{\epsilon,I} = \sup \left\{ \frac{|\Psi(x) - \Psi(y)|}{|x-y|^\epsilon} : \ \ x, y \in I, \ \ x \neq y \right\}.$$

Naturally, $|\Psi|_{1,I} \leq ||\Psi'||_I$. For reference, we also note that for functions f and g on I,

$$|f+g|_{\epsilon,I} \leq |f|_{\epsilon,I} + |g|_{\epsilon,I}, \qquad |f \cdot g|_{\epsilon,I} \leq ||f||_I |g|_{\epsilon,I} + |f|_{\epsilon,I} ||g||_I,$$

and, for functions f on I and g on $J = f(I)$,

$$|g \circ f|_{\epsilon,I} \leq |g|_{\delta,J} (|f|_{\frac{\epsilon}{\delta},I})^\epsilon, \qquad \epsilon \leq \delta \leq 1.$$

In particular, choosing δ to be ϵ or 1, we get

$$|g \circ f|_{\epsilon,I} \leq |g|_{\epsilon,J} (|f|_{1,I})^\epsilon, \qquad |g \circ f|_{\epsilon,I} \leq |g|_{1,J} |f|_{\epsilon,I}.$$

Denoting

$$||\Psi||_{\epsilon,I} = \max\{||\Psi||_I, |\Psi|_{\epsilon,I}\},$$

the space of Hölder-continuous functions is given by

$$C^{\epsilon}(I) = \left\{ \Psi : I \to \mathbf{C} : \ \ ||\Psi||_{\epsilon} < \infty \right\}.$$

In particular, we will also deal with the space $C^{1+\epsilon}(I)$ of functions on I whose derivative is in $C^{\epsilon}(I)$, equipped with the norm $||\Psi||_{1+\epsilon,I} = \max\{||\Psi||_I, ||\Psi'||_{\epsilon.I}\}$.

We omit the reference to the interval I whenever the choice of I is clear out of the context.

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