6 ${}^{(\infty)}V^*(-)$ and its Z/p Formal Group Law

6.1 Problems in using the exact triangle

An obvious approach to ${}^{(n)}V^*$ and thus to ${}^{(\infty)}V^*$ is to use the exact triangle of 4.7 :

$$\begin{tikzcd} {}^{(n-1)}V^* \arrow[rr, "i"] & & {}^{(n)}V^* \arrow[dl, "j"] \\ & \pi_*(\underset{\sim}{MU}\wedge_{\Sigma_2} B(Z/p\wr\Sigma_n)^+) \arrow[ul, "k"] & \end{tikzcd}$$

The major difficulty is that for general n I do not know the structure of $\pi_*(\underset{\sim}{MU}\wedge_{\Sigma_2} B(Z/p\wr\Sigma_n)^+)$. It is closely related to $\pi_*(\underset{\sim}{MU}\wedge B(Z/p\wr\Sigma_n)^+) \cong U_*(B(Z/p\wr\Sigma_n))$, and as $B(Z/p\wr\Sigma_n)$ is $(\underbrace{BZ/p\times\ldots\times BZ/p}_{n})\times_{\Sigma_n} E\Sigma_n$, for a start we should like to know $U_*(B\Sigma_n)$. For $n=2$ ($B\Sigma_2 = RP^\infty$) this has been determined by Conner & Floyd (11) but for higher n $U_*(B\Sigma_n)$ remains unknown to me

One might hope for a relation of $U_*(B\Sigma_n)$ to $U_*(QS^0)$ like that of Kahn & Priddy (31) for ordinary homology, and further hope that it would extend to relate $U_*(B(Z/p\wr\Sigma_n))$ to $U_*(QBZ/p)$. However we have as yet no general theorem linking $U_*(QX)$ to $U_*(X)$ - it is known how to build $H_*(QX;Z/p)$ from $H_*(X;Z/p)$ (Dyer & Lashof (17), May (26)) but there are no results in other homology theories.(Indeed Hodgkin (20) shows that the situation is quite different in $K_*(QX;Z/p)$.)

Hodgkin (20) calculates $K_*(QS^0;Z/p)$ by using representations of Σ_n and one would hope that a similar approach to $U_*(QS^0)$, combined with the work of tom Dieck (15) using localisations to study $U_*(BG)$, would yield results. Also relevant is the work of Landweber ((23) & (24)) on the Thom map $U_*(BG) \to H_*(BG;Z)$ and Kunneth formulae for $U^*(BG\times BH)$.

In the rest of this chapter we shall calculate various details of ${}^{(\infty)}V^*(-)$ by using certain exact triangles which are related to the one above, but which split into short exact sequences, simplifying calculations.

N.B. ${}^{(\infty)}V^*(-)$, being a representative Z/p-theory, has ${}^{(\infty)}V_*(QS^0) \cong {}^{(\infty)}V^* \otimes H_*(QS^0;Z/p)$ etc. so has a "Kahn Priddy theorem". One would hope that ${}^{(\infty)}V^*(-)$, which is so close to $U^*(-)$ geometrically, might in due course aid us in investigating $U_*(QS^0)$.

6.2 The "Z/p-Homology of Spectra" Exact Triangle

Let $K(Z/p)$ denote the Eilenberg-Maclane spectrum for $H^*(-;Z/p)$ and let $M^{(n)}V$ denote a spectrum for the theory ${}^{(n)}V^*(-)$. (We have defined ${}^{(n)}V^*(X)$ for finite CW X and so the spectrum $M^{(n)}V$ is well-defined up to "weak equivalence" (see Adams (2)).)

$$\text{Then}\quad H_*(M^{(n)}V;Z/p) = \{ S,\ K(Z/p) \wedge M^{(n)}V \}_{\text{Stable}} = {}^{(n)}V_*(K(Z/p))$$

(So $H_*(M^{(n)}V;Z/p)$ is well-defined.)

Since homology theories commute with $\varinjlim$ the exact triangle of 4.7 may be applied with $X = K(Z/p)$ to give:

$$\begin{tikzcd} {}^{(n-1)}V_*(K(Z/p)) \arrow[rr, "i_{n*}"] & & {}^{(n)}V_*(K(Z/p)) \arrow[dl, "j_{n*}"] \\ & \pi_*(K(Z/p)^+ \wedge (MU \wedge_{\Sigma_2} B(Z/p \wr \Sigma_n)^+)) \arrow[ul, "k_{n*}"] & \end{tikzcd}$$

i_{n*} degree 0

j_{n*} degree $-n$

k_{n*} degree $n-1$

By the comment above this triangle is the same as:

$$\begin{tikzcd} H_*(M^{(n-1)}V;Z/p) \arrow[rr, "i_{n*}"'] & & H_*(M^{(n)}V;Z/p) \arrow[dl, "j_{n*}"] \\ & H_*(MU \wedge_{\Sigma_2} B(Z/p \wr \Sigma_n)^+;Z/p) \arrow[ul, "k_{n*}"] & \end{tikzcd} \qquad \cdots\cdots\cdots\cdots (1)$$

What is useful about this approach is that Nakaoka's results ((29) & (30)) on $H_*(B\Sigma_n;Z/p)$ tell us about $H_*(\underset{\sim}{MU}\wedge_{\Sigma_2}B(Z/p\wr\Sigma_n)^+;Z/p)$ and thus, by using the triangle, about $H_*(\underset{\sim}{M}^{(\infty)}V;Z/p)$. As ${}^{(\infty)}V^*(-)$ is a representative Z/p-theory we know that $H^*(\underset{\sim}{M}^{(\infty)}V;Z/p)$ is free over the Steenrod algebra $\mathcal{A}_p$, and this gives us information about ${}^{(\infty)}V^*$. As a by-product of this investigation we shall find the Z/p formal group structure of ${}^{(\infty)}V^*(BZ/p)$ given by the classes $\binom{\alpha_V}{\beta_V}$ of 4.2.

<u>Remark 6.2.1</u> The exact triangle (1) shows inductively that $H_q(\underset{\sim}{M}^{(n)}V;Z/p)$ is a finite dimensional Z/p-vector space for each q. Thus for $H^q(\underset{\sim}{M}^{(n)}V;Z/p)$ we have an isomorphic exact triangle dual (as vector spaces) to (1) :-

$$\begin{tikzcd}
H^*(\underset{\sim}{M}^{(n-1)}V;Z/p) \arrow[dr, "k_n^*"'] & & H^*(\underset{\sim}{M}^{(n)}V;Z/p) \arrow[ll, "i_n^*"'] \\
& H^*(\underset{\sim}{MU}\wedge_{\Sigma_2}B(Z/p\wr\Sigma_n)^+;Z/p) \arrow[ur, "j_n^*"'] &
\end{tikzcd} \qquad \cdots\cdots\cdots\cdots (2)$$

<u>Remark 6.2.2</u> The spectra $\underset{\sim}{M}^{(n)}V$ are defined up to "weak equivalence" [Spectra $\underset{\sim}{E},\underset{\sim}{F}$ are "weakly equivalent" if and only if $\{K,\underset{\sim}{E}\}=\{K,\underset{\sim}{F}\}$ for all finite CW K (Adams (2)).]; but, as shown above, $H_*(\underset{\sim}{M}^{(n)}V;Z/p)$ is well-defined. In Chapter 6 we shall need to examine natural transformations of theories, e.g. ${}^{(n)}V^*(-)\longrightarrow{}^{(n+1)}V^*(-)$. We are faced with the technical question as to whether these are necessarily induced by maps of spectra $\underset{\sim}{M}^{(n)}V\longrightarrow\underset{\sim}{M}^{(n+1)}V$ if the spectra are only defined up to weak equivalence; Adams (2) constructs representing objects for cohomology theories defined on finite CW complexes and shows that all natural transformations give maps of representing objects (though these maps are not necessarily unique). Thus for instance, there is a map $\underset{\sim}{M}^{(n)}V\longrightarrow\underset{\sim}{M}^{(n+1)}V$ inducing ${}^{(n)}V^*(-)\longrightarrow{}^{(n+1)}V^*(-)$

(If we used mock-bundles to define our theories (35) then we should find that they have canonical Δ-spectra defined geometric--ally and all our natural transformations geometrically induce canonical maps of Δ-spectra).

Alternatively one may observe that all our proofs of theorems in Chapter 6 will deal only with $H_*(-;Z/p)$ applied to spectra, and as we have seen $H_*(\underset{\sim}{M}^{(n)}V;Z/p) \cong {}^{(n)}V_*(K(\underset{\sim}{Z/p}))$ so that a natural transformation ${}^{(n)}V^*(-) \longrightarrow {}^{(n+1)}V^*(-)$ induces a well-defined map $H_*(\underset{\sim}{M}^{(n)}V;Z/p) \rightarrow H_*(\underset{\sim}{M}^{(n+1)}V;Z/p)$ (by regarding it as ${}^{(n)}V_*(K(\underset{\sim}{Z/p})) \longrightarrow {}^{(n+1)}V_*(K(\underset{\sim}{Z/p})))$ which is all that is really needed in the proofs. So, while for clarity of exposition in chapter 6 natural transformations will be assumed to have associated maps of spectra (which they do have by Adams' results), one may read the chapter with "H (-;Z/p)-spectacles" and not worry about the problem.

6.3 $H^*(M^{(\infty)}V;Z/p)$ as an Abelian Group

In this section we shall show that the maps k_{n*} in the triangle (1) of 6.2 are all zero, and thus that the triangle breaks up into short exact sequences.

First we examine k_{1*}. This is induced by the map of spectra $k_1 : \underset{\sim}{MU} \wedge BZ/p^+ \longrightarrow \underset{\sim}{MU}$ taking a Z/p-bundle over a U-manifold to its total space (see 4.7). k_1 commutes with the U^*-module structure, so it suffices to evaluate it on the sphere spectrum $\underset{\sim}{S} \hookrightarrow \underset{\sim}{MU}$.

Thus k_1 is "multiplication" by a stable map $g : \underset{\sim}{S}BZ/p \rightarrow \underset{\sim}{MU}$ i.e. k_1 factorises:

$$\underset{\sim}{MU} \wedge \underset{\sim}{S}BZ/p^+ \xrightarrow{1\wedge g} \underset{\sim}{MU} \wedge \underset{\sim}{MU} \xrightarrow{m} \underset{\sim}{MU} \quad \ldots\ldots\ldots\ldots\ldots (3)$$

(g represents an element $g \in U^0(BZ/p)$; as a U-mock-bundle over BZ/p, g is represented by: $\begin{array}{c} EZ/p \\ \downarrow \\ BZ/p \end{array}$)

For each inclusion of groups $H \hookrightarrow G$ tom Dieck (15) defines a transfer map $t : U^*(BH) \rightarrow U^*(BG)$. By following through his definition with $H = \{e\}$ and $G = Z/p$ we find that:-

$$t : U^* \longrightarrow U^*(BZ/p) \quad \text{sends} \quad 1 \longmapsto g.$$

Lemma 6.3.1 (tom Dieck (14), Novikov (8)etc.)

Let $i : Z/p \hookrightarrow S^1$. Then $i^* : U^*(CP^\infty) \twoheadrightarrow U^*(BZ/p)$ is epimorphic and $U^*(BZ/p) = U^*[[C]] \big/ e_{u*}(\underbrace{\xi\otimes\ldots\otimes\xi}_{p})$ ($C = e_u(\xi)$; ξ the canonical C-bundle over CP^∞, so $e_u(\underbrace{\xi\otimes\ldots\otimes\xi}_{p}) = F_u(\underbrace{C, F_u(C, \ldots}_{p}))$ F_u formal group law on $U^*(CP^\infty$

Proof tom Dieck (14). ∎

Corollary 6.3.2 $\quad g = t(1) = F_u(\underbrace{C, F_u(C, \ldots}_{p}))/_C = p + \ldots \in U^0(BZ/p)$

Proof Immediate from tom Dieck (15). ∎

Proposition 6.3.3 The map $k_1^*: H^*(\underset{\sim}{MU}; Z/p) \longrightarrow H^*(\underset{\sim}{MU} \wedge BZ/p^+; Z/p)$ in (2) is zero.

Proof By (3) it is sufficient to show that the map $g^*: H^*(\underset{\sim}{MU}; Z/p) \longrightarrow H^*(BZ/p; Z/p)$ is zero:-

Consider the Boardman map:

$$U^*(BZ/p) \xrightarrow{B} \mathrm{Hom}_{Z/p}[H_*(BZ/p;Z/p), H_*(\underset{\sim}{MU};Z/p)] \cong H_*(\underset{\sim}{MU};Z/p) \hat{\otimes} H^*(BZ/p;Z/p)$$

This sends $g \longmapsto g_*$

6.3.2 tells us that g is a power series with all its coefficients in I(p) = the ideal of all classes in U^* with all Chern numbers divisible by p. (This follows from $g = \log_u^{-1}(p \log_u C)$.)

Hence, under $B: U^* \longrightarrow H_*(\underset{\sim}{MU}; Z/p)$ all these coefficients go to zero, and as B is both natural and multiplicative we may deduce that $g_* = 0$. ▮

Corollary 6.3.4 $H^*(M^{(1)}V; Z/p) \underset{Z/p\,v.s.}{\cong} H^*(\underset{\sim}{MU}; Z/p) \oplus H^{*-1}(\underset{\sim}{MU} \wedge BZ/p^+; Z/p)$

Proof Immediate.

We generalise 6.3.3 :-

Proposition 6.3.5 The map $k_n^*: H^*(\underset{\sim}{M^{(n-1)}V}; Z/p) \longrightarrow H^{*-n+1}(\underset{\sim}{MU} \wedge_{\Sigma_2} B(Z/p \wr \Sigma_n)^+; Z/p)$ is zero.

Proof k_n^* is induced by the map of spectra (see 6.2.2):-

$$k_n : S^{n-1} \underset{\sim}{MU} \wedge_{\Sigma_2} B(Z/p \wr \Sigma_n)^+ \longrightarrow \underset{\sim}{M^{(n-1)}V}$$

which takes a U-manifold which is a $Z/p \wr \Sigma_n$-bundle to the total space of the associated $\underbrace{Z/p * \ldots * Z/p}_{n\text{ fold join}}$-bundle (which is an ${}^{(n-1)}V$--manifold). (see 4.7).

Now a $\underbrace{Z/p * \ldots * Z/p}_{n}$-bundle has a free Z/p-action on it (given by the diagonal Z/p-action) and the quotient is still an ${}^{(n-1)}V$-manifold, so we may factorise k_n as the composition:-

$$S^{n-1} \underset{\sim}{MU} \wedge_{\Sigma_2} B(Z/p \wr \Sigma_n)^+ \longrightarrow \underset{\sim}{M^{(n-1)}V} \wedge BZ/p^+ \longrightarrow \underset{\sim}{M^{(n-1)}V}$$

[total space of $\underbrace{Z/p * \ldots * Z/p}_{n}$-bundle, regarded as Z/p-bundle] [total space of Z/p-bundle]

The map $\underset{\sim}{M}^{(n-1)}V\wedge BZ/p^+ \rightarrow \underset{\sim}{M}^{(n-1)}V$ is again "multiplication" by $g\in U^\circ(BZ/p)$ (c.f. (3)) i.e. it is:

$$\underset{\sim}{M}^{(n-1)}V\wedge BZ/p^+ \xrightarrow{1\wedge g} \underset{\sim}{M}^{(n-1)}V\wedge \underset{\sim}{MU} \xrightarrow{m} \underset{\sim}{M}^{(n-1)}V$$

where m gives the U^*-module structure of ${}^{(n-1)}V^*(-)$.

In 6.3.3 it was proved that $g^*:H^*(\underset{\sim}{MU};Z/p) \rightarrow H^*(BZ/p;Z/p)$ is zero, and therefore k_n^* is zero. ▮

<u>Corollary 6.3.6</u> There is a Z/p-vector space isomorphism:

$H_*(\underset{\sim}{M}^{(n)}V;Z/p) \cong H_*(\underset{\sim}{M}^{(n-1)}V;Z/p) \oplus H_{*-n}(\underset{\sim}{MU}\wedge_{\Sigma_2} B(Z/p\wr\Sigma_n)^+;Z/p)$

<u>Proof</u> Immediate. (N.B. This isomorphism is not canonical; it depends on the choice θ_n of a vector space splitting of $j_{n_*}:H_*(\underset{\sim}{M}^{(n)}V;Z/p) \twoheadrightarrow H_{*-n}(\underset{\sim}{MU}\wedge_{\Sigma_2} B(Z/p\wr\Sigma_n)^+;Z/p)$.) ▮

Now we know that:

$$H_*(\underset{\sim}{MU}\wedge_{\Sigma_2} B(Z/p\wr\Sigma_n)^+;Z/p) \cong H_*(B(Z/p\wr\Sigma_n);H_*(\underset{\sim}{MU};Z/p)^{(1)})$$ where $Z/p^{(1)}$ denotes the sign action: $Z/p\wr\Sigma_n \rightarrow \Sigma_n \rightarrow \Sigma_2$ on Z/p. (This isomorphism is just the Künneth formula for $H_*(-;Z/p)$ twisted by Σ_2; as p and 2 are co-prime this formula is true by consideration of the double cover of $\underset{\sim}{MU}\wedge_{\Sigma_2} B(Z/p\wr\Sigma_n)^+$.)

Thus $H_{*-n}(\underset{\sim}{MU}\wedge_{\Sigma_2} B(Z/p\wr\Sigma_n)^+;Z/p) = \bigoplus_s \left\{ H_s(\underset{\sim}{MU};Z/p) \otimes H_{*-s-n}(B(Z/p\wr\Sigma_n);Z/p^{(1)}) \right.$

<u>Corollary 6.3.7</u> $H_*(\underset{\sim}{M}^{(n)}V;Z/p)$ is a free $H_*(\underset{\sim}{MU};Z/p)$-module.

<u>Proof</u> This follows at once from the vanishing of k_{n_*} in (1) and the remark above which implies that $H_{*-n}(\underset{\sim}{MU}\wedge_{\Sigma_2} B(Z/p\wr\Sigma_n)^+;Z/p)$ is a free $H_*(\underset{\sim}{MU};Z/p)$-module. ▮

<u>Corollary 6.3.8</u> There is a vector space isomorphism θ (dependent on the choice of θ_n):

$$\theta : \bigoplus_s \left\{ [H_s(\underset{\sim}{MU};Z/p)] \otimes \left[\bigoplus_m H_{*-s-m}(B(Z/p\wr\Sigma_m);Z/p^{(1)})\right] \right\} \cong H_*(\underset{\sim}{M}^{(\infty)}V;Z/p)$$

<u>Proof</u> Immediate from 6.3.6 and the remark following it. ▮

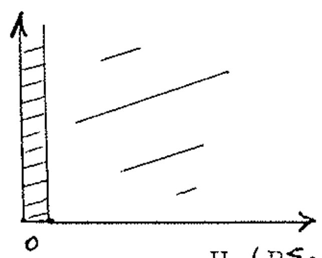
Proposition 6.3.9

$$H_*(B(Z/p\wr\Sigma_n);Z/p^{(1)}) \cong H_*(B\Sigma_n;(H_*(BZ/p;Z/p))^n \otimes Z/p^{(1)})$$

Proof Nakaoka (29) proves this for the trivial action of Σ_n on Z/p. The same proof works for the non-trivial action. See e.g. Bröcker (5). ▌

This means that for the fibration $(BZ/p)^n \longrightarrow B(Z/p\wr\Sigma_n) \longrightarrow B\Sigma_n$ the $H_*(-;Z/p)$ spectral sequence collapses (and dually the $H^*(-;Z/p)$ spectral sequence collapses).

$(H_*(BZ/p;Z/p))^n$



$H_*(B\Sigma_n;Z/p^{(1)})$

In the cohomology case the "first column" consists of the antisymmetric elements $\subset (H^*(BZ/p;Z/p))^n$. These are precisely the image of the map $i^*:H^*(B(Z/p\wr\Sigma_n);Z/p^{(1)}) \longrightarrow H^*(B(Z/p)^n;Z/p)$. The other columns are induced by the higher Z/p-homology of Σ_n (determined by Nakaoka (29)).

6.3.8 gave us the vector space structure of $H_*(\underset{\sim}{M}^{(\infty)}V;Z/p)$, and it is an immediate consequence of Corollary 6.3.7 that $H_*(\underset{\sim}{M}^{(\infty)}V;Z/p)$ is a free $H_*(\underset{\sim}{MU};Z/p)$-module. There is one more useful piece of information on $H_*(\underset{\sim}{M}^{(\infty)}V;Z/p)$ we shall need:

Proposition 6.3.10 $H_*(\underset{\sim}{M}^{(\infty)}V;Z/p)$ is a commutative (in Milnor's sense (27)) Z/p-algebra.

Proof $H_*(\underset{\sim}{M}^{(\infty)}V;Z/p) \cong {}^{(\infty)}V_*(K(\underset{\sim}{Z/p}))$ as rings and ${}^{(\infty)}V_*(-)$ is a commutative ring theory (4.6.6). ▌

6.4 $H^*(\underset{\sim}{M}^{(1)}V;Z/p)$ as a module over $\mathcal{Q}_p$

We determine the structure of $H^*(\underset{\sim}{M}^{(1)}V;Z/p)$ by showing how it injects into a known $\mathcal{Q}_p$-module, $H^*(\underset{\sim}{MU}\wedge BZ/p^+;Z/p)$; the technique will generalise to give information about $H^*(\underset{\sim}{M}^{(\infty)}V;Z/p)$.

Definition 6.4.1 Let c_N be the class of the identity map in $\{MU(N),MU(N)\}$ as an element of $U^{2N}(MU(N))$. Let $\alpha_V \in {}^{(1)}V'(BZ/p)$ be as defined in 4.2. Then $c_N.\alpha_V \in {}^{(1)}V^{2N+1}(MU(N)\wedge BZ/p^+)$ and by the definition of $\underset{\sim}{MU}$ the element $\underset{\sim}{\alpha}_V = \lim c_N.\alpha_V$ is a well-defined element of ${}^{(1)}V'(\underset{\sim}{MU}\wedge BZ/p^+)$.

Proposition 6.4.2 The class $\underset{\sim}{\alpha}_V \in {}^{(1)}V'(\underset{\sim}{MU}\wedge BZ/p^+)$ induces an injection $H^*(\underset{\sim}{M}^{(1)}V;Z/p) \hookrightarrow H^{*+1}(\underset{\sim}{MU}\wedge BZ/p^+;Z/p)$ with image consisting of "all classes divisible by α or $\beta \in H^*(BZ/p;Z/p)$" i.e. the image is $\{H^*(\underset{\sim}{MU};Z/p)\times\alpha\}\oplus\{H^{*-1}(\underset{\sim}{MU}\wedge BZ/p^+;Z/p).\beta\}$.

Proof

Case (i) N = 0 Let $f_0 : S^{-1}BZ/p \to \underset{\sim}{M}^{(1)}V$ be a (stable) map classifying α_V. From the exact triangle of theories we may picture f_0 :-

$$\begin{array}{ccc} & & \underset{\sim}{MU} \\ & & \downarrow i \\ S^{-1}BZ/p & \xrightarrow{f_0} & \underset{\sim}{M}^{(1)}V \\ & & \downarrow j \\ & & S\underset{\sim}{MU}\wedge BZ/p^+ \end{array}$$

f_0 classifies α_V so $f_0^* : \underset{H^0(\underset{\sim}{M}^{(1)}V;Z/p)}{1} \longmapsto \underset{H'(BZ/p;Z/p)}{\alpha}$ ....(I)

$\left[\text{since } f_0^* : \underset{{}^{(1)}V^0(M^{(1)}V)}{1} \longmapsto \underset{{}^{(1)}V'(BZ/p)}{\alpha_V}\right]$

jf_0 classifies the singularity set of α_V (i.e. β_V) and its normal Z/p-bundle. We may therefore factorise jf_0 :-

$$jf_0 : BZ/p \to MU(1)\wedge BZ/p^+ \to S^2\underset{\sim}{MU}\wedge BZ/p^+$$

On a filtration $S^{2n+1}/Z/p$ of BZ/p we may look at jf_0 as the map of Thom spaces:

$$\begin{array}{ccc} S^{2n+1}/Z/p & \longrightarrow & MU(1)\wedge S^{2n-1}/Z/p^+ \left(\longrightarrow S^2\underset{\sim}{MU}\wedge BZ/p^+\right) \\ \uparrow & & \uparrow \\ S^{2n-1}/Z/p & \xrightarrow{\ \omega\ } & BU(1)\times S^{2n-1}/Z/p \end{array}$$

where ω is the map $\left\{\begin{array}{l}\text{classification of } D^2\text{-bundle} \\ \text{assoc. to } Z/p\text{-bundle}\end{array}\right\}\times(\text{identity})$.

Applying $H^*(-;Z/p)$ to this we have:-

$$\begin{array}{ccccc} H^*(S^{2n+1}/Z/p;Z/p) & \longleftarrow & H^*(MU(1)\wedge S^{2n+1}/Z/p^+;Z/p) & \longleftarrow & H^{*-2}(\underset{\sim}{MU}\wedge BZ/p^+;Z/p) \\ \downarrow & & \downarrow & & \\ H^*(S^{2n-1}/Z/p;Z/p) & \xleftarrow[\omega^*]{} & H^*(BU(1)\times S^{2n-1}/Z/p^+;Z/p) & & \end{array}$$

By the definition of ω, ω^* sends:-

$H^*(BU(1)\times S^{2n-1}/Z/p;Z/p) \longrightarrow H^*(S^{2n-1}/Z/p;Z/p)$

$$t\otimes 1 \longmapsto \beta \qquad (\ t\in H^2(BU(1);Z/p)\ )$$

$$1\otimes\alpha^r\beta^s \longmapsto \alpha^r\beta^s \qquad (\ r,s\geq 0\ )$$

so $$t\otimes\alpha^r\beta^s \longmapsto \alpha^r\beta^{s+1}$$

But $H^{*-2}(MU;Z/p)\longrightarrow H^*(MU(1);Z/p)$ sends $1\longmapsto t$ so we have:-

$$(jf_0)^*:\ 1\otimes\alpha^r\beta^s \longmapsto \alpha^r\beta^{s+1} \quad \ldots\ldots\ldots\ldots\ldots\ldots(\mathrm{II})$$

<u>Case (ii) $N>0$</u> Let $f_N: S^{-(2N+1)}MU(N)\wedge BZ/p^+ \longrightarrow \underset{\sim}{M}^{(1)}V$ be a (stable) map classifying $c_N\cdot\alpha_V$. The exact triangle gives us:-

$$\begin{array}{ccc} & & \underset{\sim}{MU} \\ & & \downarrow i \\ S^{-(2N+1)}MU(N)\wedge BZ/p^+ & \xrightarrow{f_N} & \underset{\sim}{M}^{(1)}V \\ & & \downarrow j \\ & & S\underset{\sim}{MU}\wedge BZ/p^+ \end{array}$$

As f_N classifies the product $c_N\cdot\alpha_V$ we may write it:

$$S^{-2N}MU(N)\wedge S^{-1}BZ/p^+ \xrightarrow{g_N\wedge f_0} \underset{\sim}{MU}\wedge\underset{\sim}{M}^{(1)}V \xrightarrow{m} \underset{\sim}{M}^{(1)}V$$

where g_N classifies $c_N\in U^{2N}(MU(N))$; as $N\to\infty$ g_N becomes the identity map $\underset{\sim}{MU}\longrightarrow\underset{\sim}{MU}$ and so f_∞ is the composition:

$$\begin{tikzcd}
& \underset{\sim}{MU}\wedge\underset{\sim}{MU} \arrow[r, dashed] \arrow[d, dashed] & \underset{\sim}{MU} \arrow[d, dashed] \\
\underset{\sim}{MU}\wedge S^{-1}BZ/p^{+} \arrow[r, "id\wedge f_0"] & \underset{\sim}{MU}\wedge \underset{\sim}{M}^{(1)}V \arrow[r, "m"] \arrow[d, dashed] & \underset{\sim}{M}^{(1)}V \arrow[d, dashed] \\
& \underset{\sim}{MU}\wedge S\underset{\sim}{MU}\wedge BZ/p^{+} \arrow[r, dashed] & S\underset{\sim}{MU}\wedge BZ/p^{+}
\end{tikzcd} \qquad \ldots\ldots\ldots\ldots \text{(III)}$$

Let w be any class $\in H^*(\underset{\sim}{MU};Z/p)$. $m^*:H^*(\underset{\sim}{MU};Z/p)\longrightarrow H^*(\underset{\sim}{MU}\wedge\underset{\sim}{MU};Z/p)$ sends $w \longmapsto w\otimes 1 + w'\otimes\ldots + w''\otimes\ldots+\ldots$ (where w', w'' etc. have degree less than that of w.)

So, taking any lift of $w \in H^*(\underset{\sim}{MU};Z/p)$ to $\bar{w} \in H^*(\underset{\sim}{M}^{(1)}V;Z/p)$

$$m^*:H^*(\underset{\sim}{M}^{(1)}V;Z/p) \longrightarrow H^*(\underset{\sim}{MU};Z/p)\otimes H^*(\underset{\sim}{M}^{(1)}V;Z/p) \quad \text{sends:-}$$

$$\bar{w} \longmapsto w\otimes 1 + w'\otimes\ldots + w''\otimes\ldots+\ldots$$

($1 \in H^0(\underset{\sim}{MU};Z/p)$ has unique lift to $1 \in H^0(\underset{\sim}{M}^{(1)}V;Z/p)$ as both these groups are $\cong Z/p$. Again w', w'' etc. have deg. less than that of w.)

Using this and the result (I), in (III) we find that:-

$$f_\infty^* \text{ sends } \bar{w} \longmapsto w\otimes\alpha + w'\otimes\ldots+\ldots \qquad \ldots\ldots\ldots\ldots\text{(IV)}$$

By (III) jf_∞ is the composition:-

$$\underset{\sim}{MU}\wedge BZ/p^{+} \xrightarrow[id\wedge jf_0]{} \underset{\sim}{MU}\wedge S^2\underset{\sim}{MU}\wedge BZ/p^{+} \xrightarrow[m\wedge id]{} S^2\underset{\sim}{MU}\wedge BZ/p^{+}$$

So, using the result (II), we find that:

$$(jf_\infty)^*:H^{*-2}(\underset{\sim}{MU}\wedge BZ/p^{+};Z/p) \longrightarrow H^{*-2}(\underset{\sim}{MU}\wedge\underset{\sim}{MU}\wedge BZ/p^{+};Z/p) \longrightarrow H^*(\underset{\sim}{MU}\wedge BZ/p^{+};Z/p)$$

sends:- $w\otimes\alpha^\tau\beta^s \longmapsto (w\otimes 1 + w'\otimes\ldots+\ldots)\otimes\alpha^\tau\beta^s \longmapsto w\otimes\alpha^\tau\beta^{s+1} + w'\otimes\ldots+\ldots$

$$\ldots\ldots\ldots\ldots\text{(V)}$$

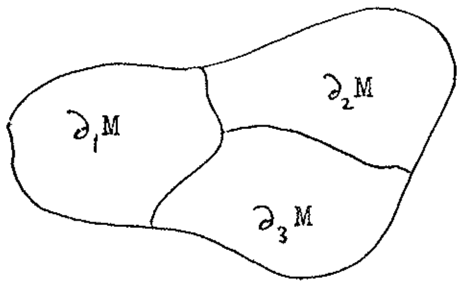
By (IV) and (V) Image f_∞^* is generated by certain classes $w\otimes\alpha+w'\otimes\ldots+\ldots$ and $w\otimes\alpha^\tau\beta^{s+1}+w'\otimes\ldots+\ldots$, existing for each w; but w', w'' etc. all have degree less than that of w, so the classes generate the image claimed. ∎

Remark 6.4.3 For $\underset{\sim}{MU}$ we have the "splitting principle": $H^*(MU(N);Z/p) \underset{i^*}{\hookrightarrow} H^*(BU(N);Z/p) \hookrightarrow H^*(\underbrace{BU(1)\times\ldots\times BU(1)}_{n};Z/p)$ where i is the map classifying $c'_N \in U^{2N}(BU(N))$. We shall use 6.4.2 in a similar way to "split" $H^*(\underset{\sim}{M}^{(1)}V;Z/p)$.

6.5 The Cobordism Theories ${}^{(n)}W^*(-)$

The technique of 6.4 does not apply directly to $H^*(\underset{\sim}{M}^{(n)}V;Z/p)$. For this reason we introduce yet another series of geometric cobordism theories ${}^{(n)}W^*(-)$. ${}^{(n)}W$-manifolds are U-manifolds with labelled singularities of type $Z/p, Z/p*Z/p, \ldots, \underbrace{Z/p*\ldots*Z/p}_{n \text{ fold join}}$ i.e. they are like ${}^{(n)}V$-manifolds but with the faces labelled, and the corners labelled by the faces meeting there. (One may think of this labelling as choosing a specific trivialisation of the Σ_n-data.)

Definition 6.5.1 An ${}^{(n)}W$-manifold is a U-manifold with faces and corners, M, where the boundary of M is a union of (labelled) faces $\partial_1 M \cup \partial_2 M \cup \ldots \cup \partial_n M$ with a free Z/p-action on each of these $\partial_i M$ such that these Z/p-actions are "orthogonal" where faces meet. (i.e. induce a free $(Z/p)^2$-action on $\partial_i M \cap \partial_j M$ etc.).



As M is a U-manifold with corners note that:

(1) Each $\partial_i M$ is a U-manifold of dimension m-1.

Each $\partial_i M \cap \partial_j M$ .. .. m-2. etc.

(2) A U-orientation is induced on $\partial_i M \cap \partial_j M \cap \partial_k M$ by regarding it as $\partial_{i_1}(\partial_{i_2}(\partial_{i_3} M))$ where i_1, i_2, i_3 is the permutation of i, j, k with $i_1 > i_2 > i_3$.

There is an obvious notion of cobordism of ${}^{(n)}W$-manifolds, to give a bordism theory ${}^{(n)}W_*(-)$. A similar argument to 4.3.6 shows that ${}^{(n)}W_*(-)$ is a generalised homology theory. ${}^{(n)}W_*(-)$ is naturally a U^*-module, but, as the singularities are labelled, Cartesian product of ${}^{(n)}W$-manifolds does not induce a commutative product in ${}^{(n)}W^*(-)$.

Theorem 6.5.2 There is an exact triangle of homology theories:

$$\begin{tikzcd} {}^{(n)}W_*(X) \arrow[rr, "i_{n+1}"] & & {}^{(n+1)}W_*(X) \arrow[dl, "\partial_{n+1}"] \\ & \pi_*(X^+\wedge(\underset{\sim}{M}{}^{(n)}W\wedge BZ/p^+)) \arrow[ul, "j_{n+1}"] & \end{tikzcd}$$

($\underset{\sim}{M}{}^{(n)}W$ denotes a spectrum for ${}^{(n)}W_*(-)$.)

where i_{n+1} is induced by "inclusion of manifolds with empty ∂_{n+1}".

j_{n+1} takes a Z/p-bundle over an ${}^{(n)}W$-manifold to its total space.

∂_{n+1} regards $\partial_{n+1}M$ as a Z/p-bundle (over an ${}^{(n)}W$-manifold).

Proof One just generalises the proof for the triangle:

$$\begin{tikzcd} U_*(X) \arrow[rr] & & {}^{(1)}V_*(X) \arrow[dl] \\ & \pi_*(X^+\wedge(\underset{\sim}{MU}\wedge BZ/p^+)) \arrow[ul] & \end{tikzcd} \qquad (4.7 \text{ with } n=1)$$

This is quite straightforward, so we omit the details. (Baas's methods (3) deal with triangles like that of 6.5.2.) ▌

Proposition 6.5.3

$j^*_{n+1}: H^*(\underset{\sim}{M}{}^{(n)}W;Z/p) \longrightarrow H^*(\underset{\sim}{M}{}^{(n)}W\wedge BZ/p^+;Z/p)$ is zero.

Proof As in 6.3.3 and 6.3.5 it is clear that j_{n+1} is multiplication by the element $g \in U^0(BZ/p)$. (Recall that ${}^{(n)}W_*$ is a U^*-module.) As we saw in 6.3.3 g_* is zero, so j^*_{n+1} is zero.

Putting $X = \underset{\sim}{K}(Z/p)$ in 6.5.2 and applying the result of 6.5.3 we obtain short exact sequences:

$$0 \to H^{*-1}(\underset{\sim}{M}{}^{(n)}W\wedge BZ/p^+;Z/p) \xrightarrow[\partial^*_{n+1}]{} H^*(\underset{\sim}{M}{}^{(n+1)}W;Z/p) \xrightarrow[i^*_{n+1}]{} H^*(\underset{\sim}{M}{}^{(n)}W;Z/p) \to 0 \;\ldots\ldots\; (1)$$

and dually:

$$0 \to H_*(\underset{\sim}{M}{}^{(n)}W;Z/p) \xrightarrow[i_{n+1*}]{} H_*(\underset{\sim}{M}{}^{(n+1)}W;Z/p) \xrightarrow[\partial_{n+1*}]{} H_{*-1}(\underset{\sim}{M}{}^{(n)}W\wedge BZ/p^+;Z/p) \to 0 \;\ldots\ldots\; (2)$$

The Cartesian product of a ${}^{(n)}W$-manifold with a ${}^{(1)}W$-manifold induces a multiplication: $\underset{\sim}{M}^{(n)}W \times \underset{\sim}{M}^{(1)}W \xrightarrow[m]{} \underset{\sim}{M}^{(n+1)}W$ (which is an $\underset{\sim}{MU}$-module map) and so we have maps:-

$$H_*(\underset{\sim}{M}^{(n)}W;Z/p)\otimes_{H_*(\underset{\sim}{MU};Z/p)} H_*(\underset{\sim}{M}^{(1)}W;Z/p) \xrightarrow[m_*]{} H_*(\underset{\sim}{M}^{(n+1)}W;Z/p)$$

Lemma 6.5.4 Such m_* are isomorphisms, and (2) is just $H_*(\underset{\sim}{M}^{(n)}W;Z/p)\otimes_{H_*(\underset{\sim}{MU};Z/p)}(---)$ applied to the sequence:

$$0 \to H_*(\underset{\sim}{MU};Z/p) \to H_*(\underset{\sim}{M}^{(1)}W;Z/p) \to H_{*-1}(\underset{\sim}{MU}\wedge BZ/p^+;Z/p) \to 0$$

Proof Everything in sight is (inductively) a free $H_*(\underset{\sim}{MU};Z/p)$-module so the lemma is proved by the diagram of short exact sequences (coefficients in Z/p):-

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & H_*(\underset{\sim}{M}^{(n)}W)\otimes_{H_*(MU)} H_*(\underset{\sim}{MU}) \arrow[r] \arrow[d, "m_*"', "\cong"] & H_*(\underset{\sim}{M}^{(n)}W)\otimes_{H_*(MU)} H_*(\underset{\sim}{M}^{(1)}W) \arrow[r] \arrow[d, "m_*"'] & H_*(\underset{\sim}{M}^{(n)}W)\otimes_{H_*(MU)} H_{*-1}(\underset{\sim}{MU}\wedge BZ/p^+) \arrow[r] \arrow[d, "m_*"', "\cong"] & 0 \\
0 \arrow[r] & H_*(\underset{\sim}{M}^{(n)}W) \arrow[r] & H_*(\underset{\sim}{M}^{(n+1)}W) \arrow[r] & H_{*-1}(\underset{\sim}{M}^{(n)}W\wedge BZ/p^+) \arrow[r] & 0
\end{tikzcd}$$

▌

We now proceed to generalise the "splitting principle" of 6.4.

Definition 6.5.5 Consider $\alpha_1 \ldots \alpha_n \in {}^{(n)}W^n(\underbrace{BZ/p\times\ldots\times BZ/p}_{n})$. This is just the ${}^{(n)}W$-manifold consisting of n copies of α_V (4.2), each mapping to the appropriate copy of BZ/p. Now define $\underset{\sim}{\alpha_1\ldots\alpha_n} \in {}^{(n)}W^n(\underset{\sim}{MU}\wedge(BZ/p\times\ldots\times BZ/p)^+)$ to be $\lim\limits_{N\to\infty} c_N.\alpha_1\ldots\alpha_n$ ($c_N \in U^{2N}(MU(N))$ as in 6.4.1.)

Proposition 6.5.6 The class $\underset{\sim}{\alpha_1\ldots\alpha_n} \in {}^{(n)}W^n(\underset{\sim}{MU}\wedge(\underbrace{BZ/p\times\ldots\times BZ/p}_{n})^+)$ induces an injection $H^*(\underset{\sim}{M}^{(n)}W;Z/p) \hookrightarrow H^{*+n}(\underset{\sim}{MU}\wedge(\underbrace{BZ/p\times\ldots\times BZ/p}_{n})^+;Z/p)$ with image consisting of "all those classes in $H^{*+n}(\underset{\sim}{MU}\wedge(\underbrace{BZ/p\times\ldots\times BZ/p}_{n})^+;Z/p)$ which are multiples of some $\delta_1\delta_2\ldots\delta_n \in H^*(\underbrace{BZ/p\times\ldots\times BZ/p}_{n};Z/p)$" (each δ_i being either α_i or β_i).

i.e the image consists of:

$$\{H^*(\underset{\sim}{MU};Z/p).\alpha_1\alpha_2\ldots\alpha_n\}$$

$$\oplus\{H^{*-1}(\underset{\sim}{MU}\wedge BZ/p^+;Z/p).\beta_1\alpha_2\ldots\alpha_n\}\oplus\{H^{*-1}(\underset{\sim}{MU}\wedge BZ/p^+;Z/p).\alpha_1\beta_2\alpha_3\ldots\alpha_n\}\oplus\ldots\oplus\ldots$$

labelled (1) labelled (2)

$$\oplus\{H^{*-2}(\underset{\sim}{MU}\wedge(BZ/p\times BZ/p)^+;Z/p).\beta_1\beta_2\alpha_3\ldots\alpha_n\}\oplus\ldots\oplus\ldots$$

labelled (1,2)

$\oplus\ \ldots$

$\oplus\ \ldots$

$$\oplus\{H^{*-n}(\underset{\sim}{MU}\wedge(\underbrace{BZ/p\times\ldots\times BZ/p}_{n})^+;Z/p).\beta_1\beta_2\ldots\beta_n\}$$

labelled (1,2,...,n)

(where the "labels" correspond to the copies of BZ/p in $H^{*+n}(\underset{\sim}{MU}\wedge(\underbrace{BZ/p\times\ldots\times BZ/p}_{n})^+;Z/p)$. Given an x divisible by some $\delta_1\ldots\delta_n$ it belongs in a unique line of this presentation by counting for how many i x is divisible by α_i but not β_i.) (Note that 6.5.6 agrees with 6.4.2 for n = 1 as ${}^{(1)}V^*(-)\cong{}^{(1)}W^*(-)$ by definition.)

<u>Proof</u> Let $c_N.\alpha_1\ldots\alpha_n$ be classified by $f_{N,n}:S^{-2N}MU(N)\wedge S^{-n}(\underbrace{BZ/p\times\ldots\times BZ/p}_{n}) \longrightarrow M^{(n)}W$

As in the proof of 6.4.2, $f_{\infty,n}$ is the composition:

$$\underset{\sim}{MU}\wedge S^{-n}(\underbrace{BZ/p\times\ldots\times BZ/p}_{n})\xrightarrow{id.\wedge f_{0,n}}\underset{\sim}{MU}\wedge\underset{\sim}{M^{(n)}}W\xrightarrow{m}\underset{\sim}{M^{(n)}}W$$

We may further factorise this as:

$$(\underset{\sim}{MU}\wedge S^{-(n-1)}(\underbrace{BZ/p\times\ldots\times BZ/p}_{n-1}))\wedge S^{-1}BZ/p\xrightarrow{(id\wedge f_{0,n-1})\wedge f_{0,1}}(\underset{\sim}{MU}\wedge\underset{\sim}{M^{(n-1)}}W)\wedge\underset{\sim}{M^{(1)}}W\xrightarrow{m\wedge id}\underset{\sim}{M^{(n-1)}}W\wedge\underset{\sim}{M^{(1)}}W\xrightarrow{m}\underset{\sim}{M^{(n)}}W$$

i.e. $f_{\infty,n}$ is the composition:

$$(\underset{\sim}{MU}\wedge S^{-(n-1)}(\underbrace{BZ/p\times\ldots\times BZ/p}_{n-1}))\wedge S^{-1}BZ/p\xrightarrow{f_{\infty,n-1}\wedge f_{0,1}}\underset{\sim}{M^{(n-1)}}W\wedge\underset{\sim}{M^{(1)}}W\xrightarrow{m}\underset{\sim}{M^{(n)}}W \quad\ldots\ldots\ (3)$$

By the proof of case (i) in 6.4.2 $f_{0,1*}:H_{*+1}(BZ/p;Z/p)\longrightarrow H_*(\underset{\sim}{M^{(1)}}W;Z/p)$ has image generating $H_*(\underset{\sim}{M^{(1)}}W;Z/p)$ as an $H_*(\underset{\sim}{MU};Z/p)$-module.

Now assume inductively that $f^*_{\infty,n-1}$ is injective (so $f_{\infty,n-1*}$ is onto).

But by 6.5.4 $H_*(\underset{\sim}{M}^{(n)}W;Z/p) \overset{m_*}{\cong} H_*(\underset{\sim}{M}^{(n-1)}W;Z/p)\otimes_{H_*(\underset{\sim}{MU};Z/p)} H_*(\underset{\sim}{M}^{(1)}W;Z/p)$ so we deduce that $f_{\infty,n\,*}$ is epimorphic (and dually that $f^*_{\infty,n}$ is injective) by inspecting the composition (3). Using 6.4.2 and (3) inductively it is also clear that $f^*_{\infty,n}$ has the stated image. ∎

6.6 The Relation of ${}^{(n)}W^*(-)$ to ${}^{(n)}V^*(-)$

An ${}^{(n)}W$-manifold may be regarded as an ${}^{(n)}V$-manifold simply by "forgetting" the labels on the faces and corners. This gives a natural transformation of theories which we denote by Φ_n: $\underset{\sim}{M}^{(n)}W \longrightarrow \underset{\sim}{M}^{(n)}V$ on spectra. (see 6.2.2).

Proposition 6.6.1 The following diagram commutes:-

$$\begin{tikzcd}
\underset{\sim}{M}^{(n-1)}W \arrow[r, "\Phi_{n-1}"] \arrow[d] & \underset{\sim}{M}^{(n-1)}V \arrow[d] \\
\underset{\sim}{M}^{(n)}W \arrow[r, "\Phi_n"] \arrow[d] & \underset{\sim}{M}^{(n)}V \arrow[d] \\
S\underset{\sim}{M}^{(n-1)}W\wedge BZ/p^+ \arrow[r, "\Psi_{n-1}"] & S^n\underset{\sim}{MU}\wedge_{\Sigma_n} B(Z/p\wr\Sigma_n)^+
\end{tikzcd}$$

Here Ψ_{n-1} is the map, defined on an ${}^{(n-1)}W$-manifold M with a Z/p-bundle ζ over it, which takes the depth n-1 singularity stratum of M and its normal $(Z/p)^{n-1}$-bundle, adds the restriction of ζ to get a $(Z/p)^n$-bundle over the stratum, and then classifies this as a $(Z/p\wr\Sigma_n)$-bundle over a U-manifold.

Proof It is immediate from the definition of Φ_n that this definition of Ψ_{n-1} makes the diagram commute. ∎

Since both sides of 6.6.1 give short exact sequences when $H^*(-;Z/p)$ is applied (by 6.3.5 & 6.5.3) we now know that $\Phi_n^*: H^*(\underset{\sim}{M}^{(n)}V;Z/p) \longrightarrow H^*(\underset{\sim}{M}^{(n)}W;Z/p)$ is determined by $\Psi_0^*, \Psi_1^*, \ldots, \Psi_{n-1}^*$.

Definition 6.6.2 Denote by A^*_m the antisymmetric elements $\hookrightarrow H^*((BZ/p)^m;Z/p)$ (i.e. the elements on which Σ_m has the sign action).

Proposition 6.6.3

The image of $\Phi^*_n : H^*(\underset{\sim}{M}^{(n)}V;Z/p) \longrightarrow H^*(\underset{\sim}{M}^{(n)}W;Z/p)$ is isomorphic (non-canonically) as a Z/p-vector space to:

$$\bigoplus_s \left\{ [H^s(\underset{\sim}{MU};Z/p)] \otimes [\bigoplus_{m\leq n} A^{*-s-n}_m] \right\}$$

Proof The definition of Ψ_{n-1} in 6.6.1 shows that it factorises:-

$$S\underset{\sim}{M}^{(n-1)}W \wedge BZ/p^+ \xrightarrow[\partial_{n-1}]{} S^2\underset{\sim}{M}^{(n-2)}W \wedge (BZ/p \times BZ/p)^+ \xrightarrow[\partial_{n-2}]{} \cdots \xrightarrow[\partial_1]{} S^n\underset{\sim}{MU} \wedge (\underbrace{BZ/p \times \cdots \times BZ/p}_{n})^+ \xrightarrow[i]{}$$
$$\xrightarrow[i]{} S^n\underset{\sim}{MU} \wedge_{\Sigma_2} B(Z/p \wr \Sigma_n)^+$$

(see 6.5.2 for definition of ∂_j)

Here i is induced by $(Z/p)^n \hookrightarrow (Z/p \wr \Sigma_n)$ and so we know that $i^*: H^{*-n}(\underset{\sim}{MU} \wedge_{\Sigma_2} B(Z/p \wr \Sigma_n)^+;Z/p) \longrightarrow H^{*-n}(\underset{\sim}{MU} \wedge (\underbrace{BZ/p \times \cdots \times BZ/p}_{n})^+;Z/p)$ has image $\bigoplus_s [H^s(\underset{\sim}{MU};Z/p) \otimes A^{*-s-n}_n]$ (see 6.3.9)

Now $\partial_1^*, \partial_2^*, \ldots, \partial_{n-1}^*$ are all injective (6.5.3) so we deduce:-

$$\text{Image } \Psi^*_{n-1} \cong \bigoplus_s [H^s(\underset{\sim}{MU};Z/p) \otimes A^{*-s-n}_n]$$

Summing $\Psi^*_0, \ldots, \Psi^*_{n-1}$ (which is the non-canonical step) we have the required result for the image of Φ^*_n. ∎

Let $g_{N,n} : S^{2N+n} MU(N) \wedge (BZ/p)^n \longrightarrow \underset{\sim}{M}^{(n)}V$ classify the element $c_N \cdot \alpha_1 \cdots \alpha_n \in {}^{(n)}V^{2N+n}(MU(N) \wedge (\underbrace{BZ/p \times \cdots \times BZ/p}_{n})^+)$. (See 6.5.6 to compare $g_{N,n}$ with $f_{N,n} : S^{2N+n} MU(N) \wedge (BZ/p)^n \longrightarrow \underset{\sim}{M}^{(n)}W$.)

Then, from the definitions of $f_{\infty,n}, g_{\infty,n}$ we have a commuting diagram:

$$\begin{array}{ccc} H^*(\underset{\sim}{M}^{(n)}V;Z/p) & \xrightarrow{\Phi^*_n} & H^*(\underset{\sim}{M}^{(n)}W;Z/p) \\ & \searrow^{g^*_{\infty,n}} & \downarrow f^*_{\infty,n} \\ & & H^{*+n}(\underset{\sim}{MU} \wedge (\underbrace{BZ/p \times \cdots \times BZ/p}_{n})^+;Z/p) \end{array}$$

$$\begin{tikzcd}
H^*(\underset{\sim}{M}^{(n)}V;Z/p) \arrow[r, "\Phi_n^*"] \arrow[dr, "g^*_{\infty,n}"'] & H^*(\underset{\sim}{M}^{(n)}W;Z/p) \arrow[d, "f^*_{\infty,n}"] \\
 & H^{*+n}(\underset{\sim}{MU}\wedge(\underbrace{BZ/p\times\dots\times BZ/p}_{n})^+;Z/p)
\end{tikzcd}$$

Proposition 6.6.4

(Image $g^*_{\infty,n}$) = (Image $f^*_{\infty,n}$) $\cap$ (Antisymmetric elements

$\hookrightarrow H^{*+n}(\underset{\sim}{MU}\wedge(\underbrace{BZ/p\times\dots\times BZ/p}_{n})^+;Z/p))$

Proof

The action of $\sigma\in\Sigma_n$ permuting the factors of $\underset{\sim}{\alpha_1\dots\alpha_n}\in{}^{(n)}V^n(\underset{\sim}{MU}\wedge(\underbrace{BZ/p\times\dots\times BZ}_{n}/p)^+)$ simply changes the sign of $\underset{\sim}{\alpha_1\dots\alpha_n}$ according as to whether σ is odd or even.

Thus (Image $g^*_{\infty,n}$) $\hookrightarrow$ (Antisym. els.) $\wedge$ (Image $f^*_{\infty,n}$)

We know (Image Φ_n^*) $\cong$ (Image $g^*_{\infty,n}$), since $f^*_{\infty,n}$ is injective so it now only remains to show that (Image Φ_n^*) has the same size as (Image $f^*_{\infty,n}$) $\wedge$ (Antisym. els.).

But consider the explicit form of (Image $f^*_{\infty,n}$) given in Proposition 6.5.6. Consider a particular antisymmetric element in (Image $f^*_{\infty,n}$); this will lie in some line of the expression of 6.5.6 and it is easy to see that the set (Antisym.) $\wedge$ (Image $f^*_{\infty,n}$ is in (1-1) correspondance with:

$$\bigoplus_{m\leq n}\{\text{Antisym. els.} \subset H^{*-m}(MU\wedge(BZ/p)^m;Z/p)\}$$

6.6.3 shows that (Image Φ_n^*) has precisely this size, concluding the proof of the proposition. ▌

Proposition 6.6.5 The maps $g_{\infty,n}$ are multiplicative, i.e. the following diagram commutes:

$$\begin{tikzcd}
\underset{\sim}{MU}\wedge(BZ/p)^m \arrow[d, "g_{\infty,m}"] & \wedge & \underset{\sim}{MU}\wedge(BZ/p)^n \arrow[d, "g_{\infty,n}"] \arrow[r] & \underset{\sim}{MU}\wedge(BZ/p)^{m+n} \arrow[d, "g_{\infty,m+n}"] \\
S^{-m}\underset{\sim}{M}^{(m)}V & \wedge & S^{-n}\underset{\sim}{M}^{(n)}V \arrow[r] & S^{-(m+n)}\underset{\sim}{M}^{(m+n)}V
\end{tikzcd}$$

where the various multiplications are the obvious ones.

Proof This is the limit of diagrams:-

$$\begin{array}{ccccc} MU(M)_\wedge(BZ/p)^m & \wedge & MU(N)_\wedge(BZ/p)^n & \longrightarrow & MU(M+N)_\wedge(BZ/p)^{m+n} \\ \downarrow g_{M,m} & & \downarrow g_{N,n} & & \downarrow g_{M+N,m+n} \\ S^{-2M-m}\underset{\sim}{M}^{(m)}V & \wedge & S^{-2N-n}\underset{\sim}{M}^{(n)}V & \longrightarrow & S^{-2(M+N)-(m+n)}\underset{\sim}{M}^{(m+n)}V \end{array}$$

But such diagrams commute, as $g_{M,m}$ classifies $c'_M\alpha'_1\ldots\alpha'_m$

$g_{N,n}$ classifies $c''_N\alpha''_1\ldots\alpha''_n$

and $g_{M+N,m+n}$ classifies $c_{M+N}\alpha_1\ldots\alpha_{m+n} = c'_M c''_N \alpha'_1\ldots\alpha'_m\alpha''_1\ldots\alpha''_n$ ▌

Definition 6.6.6

Let $u = (t_1 + a_2t_1^2 + a_4t_1^3 + \ldots)(t_2 + a_2t_2^2 + a_4t_2^3 + \ldots)(t_3 + \ldots)\ldots$

$\in H^*(\underset{\sim}{MU};Z/p)\hat{\otimes} P[a_2, a_4, \ldots] \hookrightarrow H^*(BU(1)\times BU(1)\times\ldots;Z/p)\hat{\otimes} P[a_2,\ldots]$

be Boardman's universal element (c.f. 2.1.10) giving a specific ring isomorphism: $H_*(\underset{\sim}{MU};Z/p)\cong P[a_2, a_4, \ldots]$

Let $v_n = u\otimes(\alpha_1 + \beta_1 b_1 + \alpha_1\beta_1 c_2 + \beta_1^2 b_3 + \ldots)(\alpha_2 + \beta_2 b_1 + \ldots)\ldots(\alpha_n + \beta_n b_1 + \ldots)$

$\in H^{*+n}(\underset{\sim}{MU}\wedge(\underbrace{BZ/p\times\ldots\times BZ/p}_{n})^+;Z/p)\otimes P[a_i]\otimes P[c_i]\otimes E[b_i]$ {with the a_i in dim. $i=-2j$, b_i in dim. $i=-(2j+1)$, c_i in dim. $i=-2j$ }

As $n\to\infty$ let $v_n\to v$.

Proposition 6.6.7 Regarded as a universal element v gives a duality between (Image $g^*_{\infty,n}$) $\hookrightarrow H^{*+n}(\underset{\sim}{MU}\wedge(\underbrace{BZ/p\ \ldots\ BZ/p}_{n})^+;Z/p)$ $(n\to\infty)$ and the algebra $P[a_i]\otimes P[c_i]\otimes E[b_i]$, under which the comultiplication $H^{*+m+n}(\underset{\sim}{MU}\wedge(BZ/p)^{m+n};Z/p)\longrightarrow H^{*+m}(\underset{\sim}{MU}\wedge(BZ/p)^m;Z/p)\otimes H^{*+n}(\underset{\sim}{MU}\wedge(BZ/p)^n;Z/p)$

corresponds to the multiplication in the algebra $P[a_i]\otimes P[c_i]\otimes E[b_i]$

Proof

In the expansion of v consider the term coefficient to b_1 :-

$$(\beta_1 b_1\alpha_2\ldots\alpha_n) + (\alpha_1\beta_2 b_1\alpha_3\ldots\alpha_n) + (\alpha_1\alpha_2\beta_3 b_1\alpha_4\ldots\alpha_n) + \ldots$$

$$= b_1\{(\beta_1\alpha_2\ldots\alpha_n) - (\alpha_1\beta_2\alpha_3\ldots\alpha_n) + (\alpha_1\alpha_2\beta_3\alpha_4\ldots\alpha_n) + \ldots\}$$

(Since b_1 is in odd dim. $\alpha_1 b_1 = -b_1\alpha_1$)

So in the duality b_1 corresponds to an antisymmetric element with each term divisible by some $\delta_1\ldots\delta_n$ (where δ_i is α_i or β_i).

In fact each element of $P[a_i]\otimes P[c_i]\otimes E[b_i]$ has coefficient in the expansion of v such an antisymmetric element, and vice versa. Perhaps the easiest way to see this is to observe that:

(i) Given an antisymmetric element in the α_i and β_i, if its leading term contains a factor $\beta_i^s\beta_j^s$ then it must also have a factor α_i or α_j (in order for the transposition (i,j) to reverse the sign.), so its coefficient in the expansion of v has no term divisible by b_s^2 i.e. its coefficient is a non-zero element of $P[a_i]\otimes P[c_i]\otimes E[b_i]$.

(ii) Given an element of $P[a_i]\otimes P[c_i]\otimes E[b_i]$ then its coefficient in the expansion of v must be antisymmetric, for the action of (1,2) on $(\alpha_1+\beta_1 b_1+\ldots)(\alpha_2+\beta_2 b_1+\ldots)$ changes its sign (as both factors are in dimension 1) and so changes the sign of each coefficient in its expansion; but such transpositions generate the action of Σ_n on $(\alpha_1+\beta_1 b_1+\ldots)(\alpha_2+\beta_2 b_1+\ldots)\ldots(\alpha_n+\beta_n b_1+\ldots)$.

Now 6.6.4 and 6.5.6 together tell us that the image of $g^*_{\infty,n}$ consists precisely of the antisymmetric elements having each term divisible by some $\delta_1\ldots\delta_n$. As v is the product $u\otimes(\alpha_1+\beta_1 b_1+\ldots)(\alpha_2+\beta_2 b_1+\ldots)\ldots$ the multiplication in $P[a_i]\otimes P[c_i]\otimes E[b_i]$ is dual to the comultiplication on $H^*(\underset{\sim}{MU}\wedge(BZ/p\times BZ/p\times\ldots)^+;Z/p)$. (c.f. 2.1.10) ∎

<u>Corollary 6.6.8</u> As v gives a duality between the quotient (Image $g^*_{\infty,\infty}$) of $H^*(\underset{\sim}{M}^{(\infty)}V;Z/p)$ and $P[a_i]\otimes P[c_i]\otimes E[b_i]$ we may regard it (dually) as giving an algebra isomorphism of the subalgebra (Image $g_{\infty,\infty *}$) of $H_*(\underset{\sim}{M}^{(\infty)}V;Z/p)$ to $P[a_i]\otimes P[c_i]\otimes E[b_i]$.
(Recall that by 6.6.5 $g_{\infty,\infty *}$ is an algebra map, so (Image $g_{\infty,\infty *}$) has the same algebra structure regarded as an image of $H_{*+\infty}(\underset{\sim}{MU}\wedge(BZ/p\times BZ/p\ \ldots)^+;Z/p)$ or as a subalgebra of $H_*(\underset{\sim}{M}^{(\infty)}V;Z/p)$.) ∎

Proposition 6.6.9

Let A^* denote (Image $g^*_{\infty,\infty}$). Then A^* is free as a module over the Steenrod algebra $\mathcal{Q}_p$.

Proof

$$(\text{Image } g^*_{N,n}) \hookrightarrow H^{*+2N+n}(MU(N) \wedge (BZ/p)^{n^+};Z/p)$$
$$\left[\hookrightarrow H^{*+2N+n}((BZ/p)^{N^+} \wedge (BZ/p)^{n^+};Z/p)\right]$$

In the limit (Image $g^*_{\infty,\infty}$) is a coalgebra with co-unit $\lim\limits_{\substack{N\to\infty \\ n\to\infty}} c_N\alpha_1\ldots\alpha_n$ (mapping to $\beta_1\ldots\beta_N\alpha_1\ldots\alpha_n \in H^{*+2N+n}((BZ/p)^{n+N};Z/p)$.)

The action of $\mathcal{Q}_p$ on $\beta_1\ldots\beta_N\alpha_1\ldots\alpha_n$ is free as n and $N\to\infty$. (See Steenrod & Epstein (37) Chapter VI Prop.2.4; this is a direct consequence of Milnor's theorem on the action of $\mathcal{Q}_p$ on $H^*(BZ/p;Z/p)$ (3.2.1).)

The proof of 6.6.9 is completed by the lemma of Milnor and Moore (28):-

Lemma 6.6.10 Let A be a connected Hopf algebra over a field F. Let M be a connected coalgebra over F with co-unit $1\in M_0$ and a left module over A such that the diagonal map is a map of A-modules. Suppose $\nu: A\to M$ $a\mapsto a.1$ is a monomorphism. Then M is a free left A-module.

Proof Stong (39) p94. ∎

6.7 The Z/p-Formal Group ${}^{(\infty)}V^*(BZ/p)$

As in the Z/2 case (2.1.9) we have a "Boardman map":

$$B: {}^{(\infty)}V^*(X) \xrightarrow{H} \mathrm{Hom}_{\mathcal{A}_p}[H^*(\underset{\sim}{M}^{(\infty)}V;Z/p), H^*(X;Z/p)]$$
$$\hookrightarrow \mathrm{Hom}_{Z/p}[H^*(\underset{\sim}{M}^{(\infty)}V;Z/p), H^*(X;Z/p)] \cong H^*(X;Z/p)\hat{\otimes} H_*(\underset{\sim}{M}^{(\infty)}V;Z/$$

By Rourke's result (3.1.2) we know that $H^*(\underset{\sim}{M}^{(\infty)}V;Z/p)$ is a free module over the Steenrod algebra $\mathcal{A}_p$, so H is an isomorphism and B is injective. It is an injection of algebras, where the multiplication on $H^*(X;Z/p)\hat{\otimes} H_*(\underset{\sim}{M}^{(\infty)}V;Z/p)$ is given by the multiplication maps $\underset{\sim}{M}^{(\infty)}V \wedge \underset{\sim}{M}^{(\infty)}V \longrightarrow \underset{\sim}{M}^{(\infty)}V$ together with the cup product on $H^*(X;Z/p)$.

Proposition 6.7.1 $B: {}^{(\infty)}V^* \hookrightarrow H_*(\underset{\sim}{M}^{(\infty)}V;Z/p)$ has as image those elements $x \in H_*(\underset{\sim}{M}^{(\infty)}V;Z/p)$ which are invariant under the coaction of the dual coalgebra ζ_p to the Steenrod algebra (see 3.2) i.e. the image consists of those x such that $x \longmapsto x\otimes 1$ under the coaction $H_*(\underset{\sim}{M}^{(\infty)}V;Z/p) \longrightarrow H_*(\underset{\sim}{M}^{(\infty)}V;Z/p)\otimes \zeta_p$.

Proof This follows at once from the definition of B. ∎

We now adapt 2.1.9 to determine the Z/p formal group ${}^{(\infty)}V^*(BZ/p)$ induced by the classes $\binom{\alpha_V}{\beta_V}$ (of 4.2).

Lemma 6.7.2 Applied to the space BZ/p the Boardman map $B: {}^{(\infty)}V^*(BZ/p) \hookrightarrow H^*(BZ/p;Z/p)\hat{\otimes} H_*(\underset{\sim}{M}^{(\infty)}V;Z/p)$ sends:

$$\alpha_V \longmapsto f_1\binom{\alpha}{\beta} = \alpha + \beta b_1 + \alpha\beta c_2 + \beta^2 b_3 + \dots$$
$$\beta_V \longmapsto f_2\binom{\alpha}{\beta} = \beta + \beta^2 a_2 + \beta^3 a_4 + \beta^4 a_6 + \dots$$

(Recall that 6.6.8 picks out a particular subalgebra $P[a_2, a_4, \dots] \otimes P[c_2, c_4, \dots] \otimes E[b_1, b_3, \dots]$ of $H_*(\underset{\sim}{M}^{(\infty)}V;Z/p)$.)

<u>Proof</u> Consider the identity map $\in \{\underset{\sim}{M}^{(1)}V, \underset{\sim}{M}^{(1)}V\}_{stable}$. This defines an element $1 \in {}^{(1)}V^{0}(\underset{\sim}{M}^{(1)}V)$, which maps to an element $1 \in {}^{(\infty)}V^{0}(M^{(1)}V)$ under the natural transformation ${}^{(1)}V^{*}(-) \longrightarrow {}^{(\infty)}V^{*}(-)$.

Under B, the element $1 \in {}^{(\infty)}V^{0}(\underset{\sim}{M}^{(1)}V)$ has image the natural homomorphism $\phi_* : H_*(\underset{\sim}{M}^{(1)}V;Z/p) \longrightarrow H_*(\underset{\sim}{M}^{(\infty)}V;Z/p)$, which, expressed as an element $w' \in H^*(\underset{\sim}{M}^{(1)}V;Z/p)\hat{\otimes} H_*(\underset{\sim}{M}^{(\infty)}V;Z/p)$ is just the restriction of the universal element $w \in H^*(\underset{\sim}{M}^{(\infty)}V;Z/p)\hat{\otimes} H_*(\underset{\sim}{M}^{(\infty)}V;Z/p)$.

Now ϕ_* has image $\hookrightarrow$ image $\Phi_* : H_*(\underset{\sim}{M}^{(\infty)}W;Z/p) \longrightarrow H_*(\underset{\sim}{M}^{(\infty)}V;Z/p)$ (because $\underset{\sim}{M}^{(1)}V = \underset{\sim}{M}^{(1)}W$) so $w' \in H^*(M^{(1)}V;Z/p)\hat{\otimes} A_*$ (where $A_* \equiv$ Image Φ_*
$\equiv$ Image $g_{\infty,\infty*}$)

Let $f_\infty : S^{-1}\underset{\sim}{MU}\wedge BZ/p^{+} \longrightarrow \underset{\sim}{M}^{(1)}V$ be a (stable) map classifying $\underset{\sim}{\alpha}_V$ (6.4.1). Then $f_\infty^* : H^*(\underset{\sim}{M}^{(1)}V;Z/p) \longrightarrow H^{*+1}(\underset{\sim}{MU}\wedge BZ/p^{+};Z/p)$ is an injection (6.4.2) and we have a commuting diagram:

$$\begin{tikzcd}
{}^{(\infty)}V^{0}(\underset{\sim}{M}^{(1)}V) \arrow[rrr, tail, "B"] \arrow[ddd, "f_\infty^*"'] & & & H^*(\underset{\sim}{M}^{(1)}V;Z/p)\hat{\otimes} H_*(\underset{\sim}{M}^{(\infty)}V;Z/p) \arrow[ddd, "f_\infty^*\otimes id."] \\
& 1 \arrow[r, mapsto] \arrow[d, mapsto] & w' \arrow[d, mapsto] & \\
& \underset{\sim}{\alpha}_V \arrow[r, mapsto] & \begin{array}{l}\text{restriction}\\ \text{of univ. el.}\end{array} & \\
{}^{(\infty)}V^{1}(\underset{\sim}{MU}\wedge BZ/p^{+}) \arrow[rrr, tail, "B"] & & & H^{*+1}(\underset{\sim}{MU}\wedge BZ/p^{+};Z/p)\hat{\otimes} H_*(\underset{\sim}{M}^{(\infty)}V;Z/p)
\end{tikzcd}$$

But the restriction of the universal element to $H^{*+1}(\underset{\sim}{MU}\wedge BZ/p^{+};Z/p)\hat{\otimes} A_*$ is just $u\otimes(\alpha+\beta b_1+\dots)$ (using the isomorphism $A_* \cong P[a_i]\otimes P[c_i]\otimes E[b_i]$ of 6.6.8.)

Thus $B : {}^{(\infty)}V^{1}(\underset{\sim}{MU}\wedge BZ/p) \longrightarrow H^{*+1}(\underset{\sim}{MU}\wedge BZ/p^{+};Z/p)\hat{\otimes} H_*(\underset{\sim}{M}^{(\infty)}V;Z/p)$ sends $\underset{\sim}{\alpha}_V$ to $u\otimes(\alpha+\beta b_1+\dots)$, and using the natural map $BZ/p \xrightarrow{\simeq} S^{0}\wedge BZ/p^{+} \longrightarrow \underset{\sim}{MU}\wedge BZ/p^{+}$ we get:

$$B : {}^{(\infty)}V^{1}(BZ/p) \xrightarrow{B} H^{*+1}(BZ/p;Z/p)\hat{\otimes} H_*(\underset{\sim}{M}^{(\infty)}V;Z/p)$$

$$\alpha_V \longmapsto \alpha + \beta b_1 + \alpha\beta c_2 + \beta^2 b_3 + \dots$$

Similarly, the identity class $\epsilon\{MU(1),MU(1)\}$ gives rise (via $\{MU(1),MU(1)\}\to\{MU(1),\underset{\sim}{MU}\}\to\{MU(1),\underset{\sim}{M}{}^{(\infty)}V\}$) to a class $\iota\,\epsilon\,{}^{(\infty)}V^2(MU(1))$ which, under the Boardman map, goes to the restriction of the universal element $t+t^2a_2+t^3a_4+\ldots$ $\epsilon\ H^{*+2}(MU(1);Z/p)\hat{\otimes}H_*(\underset{\sim}{M}{}^{(\infty)}V;Z/p)$.

The map $BZ/p\to MU(1)$ classifying β_V sends $\iota\,\epsilon\,{}^{(\infty)}V^2(MU(1))$ to $\beta_V\,\epsilon\,{}^{(\infty)}V^2(BZ/p)$ so we get:

$$B:{}^{(\infty)}V^2(BZ/p)\longrightarrow H^{*+2}(BZ/p;Z/p)\hat{\otimes}H_*(\underset{\sim}{M}{}^{(\infty)}V;Z/p)$$

$$\beta_V\longmapsto\beta+\beta^2a_2+\beta^3a_4+\beta^4a_6+\ldots \qquad \blacksquare$$

The classes $\binom{\alpha_V}{\beta_V}\epsilon\,{}^{(\infty)}V^*(BZ/p)$ induce a Z/p-formal group law. Thus there is a unique ring map $\phi:A_V\longrightarrow{}^{(\infty)}V^*$ inducing this law from the universal law (3.5 . 1).

<u>Theorem 6.7.3</u> ϕ is injective.(i.e. ${}^{(\infty)}V^*(-)$ "carries" the universal law).

<u>Proof</u> Exactly as in (2.1.9) we now know that the Z/p-formal group law induced by $\binom{\alpha_V}{\beta_V}$ is given by $f\left(f^{-1}\binom{\alpha'_V}{\beta'_V}+f^{-1}\binom{\alpha''_V}{\beta''_V}\right)$. $\left(f=\binom{f_1}{f_2}\right)$

Thus the law has logarithm f^{-1}. When this logarithm is put into canonical form (with the coefficients of $\beta_V^{p^j}$ zero), the coefficients are: a_i + decomposables $(i\neq 2(p^j-1))$

b_i + decomposables $(i\neq 2p^j-1)$

c_i + decomposables

These are all polynomial generators of ${}^{(\infty)}V_*$. By definition ϕ is the map sending the generators of A_V to these generators. Hence ϕ is injective. $\blacksquare$

6.8 Realisation of the Universal Z/p Formal Group $V^*(BZ/p)$

Recall 6.7: $H^*(M^{(\infty)}_{\sim}V;Z/p)$ is free over $\mathcal{Q}_p$ and so:

$$^{(\infty)}V^*(X) \cong \mathrm{Hom}_{\mathcal{Q}_p}\left[H^*(M^{(\infty)}_{\sim}V;Z/p), H^*(X;Z/p)\right]$$

Denote image $\Phi^*: H^*(M^{(\infty)}_{\sim}V;Z/p) \longrightarrow H^*(M^{(\infty)}_{\sim}W;Z/p)$ by A^*, and image Φ_* by A_*. A^* is free over $\mathcal{Q}_p$ (6.6.9) and so $\mathrm{Hom}_{\mathcal{Q}_p}\left[A^*, H^*(X;Z/p)\right]$ is a cohomology theory.

Definition 6.8.1 Define $V^*(X)$ to be $\mathrm{Hom}_{\mathcal{Q}_p}\left[A^*, H^*(X;Z/p)\right]$

Φ^* induces a natural inclusion:

$$\begin{array}{ccc} V^*(X) & \xhookrightarrow{\ i\ } & ^{(\infty)}V^*(X) \\ \wr\| & & \wr\| \\ \mathrm{Hom}_{\mathcal{Q}_p}\left[A^*, H^*(X;Z/p)\right] & \overset{i}{\hookrightarrow} & \mathrm{Hom}_{\mathcal{Q}_p}\left[H^*(M^{(\infty)}_{\sim}V;Z/p), H^*(X;Z/p)\right] \end{array}$$

$V^*(-)$ has an algebra structure induced from that on the subalgebra $A_* \hookrightarrow H^*(M^{(\infty)}_{\sim}V;Z/p)$ and thus it is a (commutative) multiplicative Z/p-theory. (3.1.1)

The argument used in the proof of 6.7.2 shows that $\binom{\alpha_V}{\beta_V} \in {}^{(\infty)}V^*(BZ/p)$ lie in the image of i: $V^*(BZ/p) \hookrightarrow {}^{(\infty)}V^*(BZ/p)$ (since they lie in $H^*(BZ/p;Z/p)\hat{\otimes} A_* \hookrightarrow H^*(BZ/p;Z/p)\hat{\otimes} H_*(M^{(\infty)}_{\sim}V;Z/p)$) Thus they give well-defined $\binom{\alpha_V}{\beta_V} \in V^*(BZ/p)$. This gives a formal group structure to $V^*(BZ/p)$, which clearly maps injectively to that given by 6.7.3 on $^{(\infty)}V^*(BZ/p)$.

Theorem 6.8.2 The universal map $\phi: A_V \longrightarrow V^*$ induced by the formal group law, is an isomorphism. i.e. $V^*(BZ/p)$ is a realisation of the universal Z/p-formal group.

Proof ϕ is injective by the remark above and 6.7.3. A^* free over $\mathcal{Q}_p$ implies that $A_* \cong V^* \otimes S_p$. But S_p is given by 3.2.1 , and $A_* \cong P[a_i] \otimes P[c_i] \otimes E[b_i]$ so V^* has the same size as A_V , and therefore ϕ is an isomorphism. ▮

Remark 6.8.3

Since $V^*(X) \hookrightarrow {}^{(\infty)}V^*(X)$ it would be useful to have some characterisation of which elements of ${}^{(\infty)}V^*(X)$ lie in $V^*(X)$, in order to get a complete geometric picture of $V^*(-)$. One might conjecture that $V^*(X) = \mathrm{Image}\left\{{}^{(\infty)}W^*(X) \xrightarrow{\Phi} {}^{(\infty)}V^*(X)\right\}$; this would be trivial to prove if $H^*(\underset{\sim}{M}^{(\infty)}W;Z/p)$ were free over $\mathcal{Q}_p$ [so that ${}^{(\infty)}W^*(X)$ would be isomorphic to $\mathrm{Hom}_{\mathcal{Q}_p}\left[H^*(\underset{\sim}{M}^{(\infty)}W;Z/p), H^*(X;Z/p)\right]$.] However, $H^*(\underset{\sim}{M}^{(\infty)}W;Z/p)$ is not free over $\mathcal{Q}_p$ (6.5.6), so there may be elements $g \in \mathrm{Hom}_{\mathcal{Q}_p}\left[H^*(\underset{\sim}{M}^{(\infty)}W;Z/p), H^*(X;Z/p)\right]$ which are not induced by any stable map $X \longrightarrow \underset{\sim}{M}^{(\infty)}W$, and thus elements of $V^*(X)$ not in the image of ${}^{(\infty)}W^*(X)$.

What we can say, however, is that V^* is generated by the coefficients of the Z/p formal group law on ${}^{(\infty)}V^*(-)$; one should be able to compute these coefficients as "Milnor manifolds" and their explicit form should help in the characterisation of $V^*(-) \hookrightarrow {}^{(\infty)}V^*(-)$.

Remark 6.8.4

Any $\mathcal{Q}_p$-module splitting $A^* \hookrightarrow H^*(\underset{\sim}{M}^{(\infty)}V;Z/p)$ of Φ^* gives a vector space splitting ${}^{(\infty)}V^*(-) \twoheadrightarrow V^*(-)$. One would hope for ring splittings; the following line of attack seems hopeful:-

(i) It should be possible to adapt Bröcker's proof (5) that $\bigoplus_n H_*(B\Sigma_n;Z/p^{(1)})$ is a free polynomial ring (up to sign) to show that $\bigoplus_n H_*(B(Z/p)\wr\Sigma_n);Z/p^{(1)})$ is also a free polynomial ring (up to sign).

(ii) Given (i) one can construct a θ (as in 6.3.8) inductively in such a way that it is a ring isomorphism (and not just a vector space isomorphism), which would prove that $H_*(\underset{\sim}{M}^{(\infty)}V;Z/p)$ is a free polynomial algebra.

(iii) ${}^{(\infty)}V^* \underset{B}{\hookrightarrow} H_*(M^{(\infty)}_{\sim}V;Z/p)$ so this would prove ${}^{(\infty)}V^*$ a free polynomial algebra over Z/p.

(iv) If one could then obtain a decomposition: ${}^{(\infty)}V^* \simeq V^* \otimes R^*$, R^* a free polynomial algebra, one could then get $V^*(-)$ from ${}^{(\infty)}V^*(-)$ geometrically, by "killing the polynomial generators of R^*" (using Baas's method (3) of killing free polynomial generators.)

The main work in such a program would be a deeper study of the ring and $\mathcal{Q}_p$-module structures of $H_*(B(Z/p \wr \Sigma_n);Z/p^n)$ than has been presented in this thesis.

7 Operations in $^{(\infty)}V^*(-)$ and Further Speculation

This chapter will outline how operations in $^{(\infty)}V^*(-)$ may be defined and indicate some other lines of development. It is intended merely as a sketch, and no attempt is made at rigour.

7.1 Steenrod Operations

The Steenrod operations $\mathcal{P}^i$ are geometrically constructed for $U^*(-)$ by tom Dieck (12). However, singularities must be introduced to construct any sort of Bockstein operation. On $^{(\infty)}V^*(-)$ we can analogously construct Steenrod operations, in this case getting the full algebra $\mathcal{A}_p$ of operations, not just $\mathcal{A}_p/(Q_0)$:-

Let $x \in {}^{(\infty)}V^n(X)$ (X a finite CW complex). We may represent x by a mock-bundle $M \to X$ with blocks $^{(\infty)}V$-manifolds of codimension n. Consider the equivariant p th power: $M^p \longrightarrow X^p$

$$\begin{tikzcd} {} \arrow[r, dashed] \arrow[d, dashed] & X \times BZ/p \arrow[d, "\Delta"] \\ M^p \times_{Z/p} EZ/p \arrow[r, "\text{codim. } np"'] & X^p \times_{Z/p} EZ/p \end{tikzcd}$$

The pull-back gives an element $y \in {}^{(\infty)}V^{np}(X \times BZ/p)$

But $^{(\infty)}V^*(X \times BZ/p)$ is canonically isomorphic to $^{(\infty)}V^*(X) \otimes P[\beta_V] \otimes E[\alpha_V]$ Taking the expansion of y and picking out the coefficient of $\alpha_V^s \beta_V^t$ (with $s+2t=r$), gives an element of $^{(\infty)}V^{np-r}(X)$ and this is defined to be the value of the Steenrod operation of degree $n(p-1)-r$ on x. (Proofs of well-definition etc. go just as tom Dieck's (12).)

The Bockstein $Q_0 : {}^{(\infty)}V^n(X) \to {}^{(\infty)}V^{n+1}(X)$ is given by the case $r = n(p-1)-1$. (Note that there is also a "geometric Bockstein" given by taking ∂' of the "manifold with corners" (4.5.5) and quotienting by the Z/p-action; these should be related.)

7.2 Landweber-Novikov Operations

Stable additive operations correspond to ${}^{(\infty)}V^*(\underset{\sim}{M}{}^{(\infty)}V)$ $\cong {}^{(\infty)}V^* \otimes H^*(\underset{\sim}{M}{}^{(\infty)}V;Z/p)$. One should be able to realise these operations geometrically, in the way Quillen does for $U^*(\underset{\sim}{MU})$ (33); here I can only indicate an approach to this.

Again, let $x \in {}^{(\infty)}V^n(X)$ be represented by the ${}^{(\infty)}V$-mock bundle $M \xrightarrow{f} X$. We may consider X a manifold (which is no loss of generality for X finite CW - see Quillen (33) or (4.1)) and M an ${}^{(\infty)}V$-manifold embedded in a stable complex bundle over X. Each singularity stratum M_i of M has normal bundle with structure group of the form $U(n)\times(Z/p \wr \Sigma_m)$. These bundles then have characteristic classes (in ${}^{(\infty)}V^*(M_i)$) given by the images of the universal classes $\in {}^{(\infty)}V^*(BU\times B(Z/p\wr\Sigma_m))$ via the classifying maps of the bundles. What remains unclear, however, is how to fit together these elements in the various ${}^{(\infty)}V^*(M_i)$ for the different strata, so as to give a class in ${}^{(\infty)}V^*(M)$ and thus, by the transfer map f_*, an element of ${}^{(\infty)}V^*(X)$.

If one could define Landweber-Novikov operations by this program it should be possible to identify the Steenrod operations (and the Adams operations) among them and so investigate ${}^{(\infty)}V^*(-)$ by using methods analogous to Quillen's for $U^*(-)$ (33).

7.3 Further Speculation

We know how $H_*(QX;Z/p)$ $(Q = \Omega^\infty S^\infty)$ is generated by $H_*(X;Z/p)$ (May (26)) and since ${}^{(\infty)}V_*(-) \cong {}^{(\infty)}V^*_* \otimes H_*(-;Z/p)$ the same is true for ${}^{(\infty)}V_*(QX)$; we should then be able to use the relation of $U_*(-)$ to ${}^{(\infty)}V_*(-)$ to give us information about $U_*(QX)$ and how it is related to $U_*(X)$. (see 6.1).

Another interesting area is to examine how far the theories ${}^{(n)}V^*(-)$ map onto $H^*(-;Z/p)$: consider the theories ${}^{(n)}U^*(-)$, defined to be the cohomology theories obtained from $U^*(-)$ by "killing" the Milnor polynomial generating elements $y_0, y_1, \ldots$ $\ldots, y_{n-1}$ (y_i dim. $2p^i - 2$) (using Baas's method (3) of introducing singularities to kill polynomial generators.) Thus ${}^{(n)}U^* \cong U^*/\langle y_0, \ldots, y_{n-1}\rangle$. I can show, by using the sort of geometric techniques employed to calculate ${}^{(1)}V^*$ in Chapter 5, that the natural maps $U^* \longrightarrow {}^{(n)}V$ have kernel $\supset I_n$ (see 5.2.3) and thus the natural transformations $U^*(-) \longrightarrow {}^{(n)}V^*(-)$ factor: $U^*(-) \longrightarrow {}^{(n)}U^*(-) \longrightarrow {}^{(n)}V^*(-)$. It is reasonable to conjecture that ${}^{(n)}U^*(-) \longrightarrow {}^{(n)}V^*(-)$ is injective; this is certainly true for $n = 1$, and one may use this fact to construct spaces with ${}^{(1)}V^*(X) \longrightarrow H^*(X;Z/p)$ not epimorphic, since ${}^{(1)}U^*(-)$ is easier to deal with than ${}^{(1)}V^*(-)$. The same technique should show that there exist X with ${}^{(n)}V^*(X) \longrightarrow H^*(X\,;Z/p)$ not epimorphic.

The ideas of the paragraph above should be linked with various known results: Quillen (33) proved that $U^*(CP^\infty)$ is the universal (Z-) formal group. The spectral sequence $H^*(X;U^*) \Longrightarrow U^*(X)$ does not in general collapse however, and $U^*(X) \longrightarrow H^*(X;Z)$ is not epimorphic, so we have no isomorphism of $U^*(X)$ to $U^* \otimes H^*(X;Z)$. However, he does give a nice result for $U^*(X)_{(p)}$ (i.e. localised at p); this is a ring isomorphism:

$$U^*(X)_{(p)} \cong U^*_{(p)} \otimes_{BP^*} BP^*(X) \qquad \text{(Quillen (32))}$$

($BP^*(-)$ is Brown-Peterson cohomology localised at p, with point ring $Z_{(p)}[y_1, y_2, \ldots$ y_i dim. $2p^i - 2$.)

He characterises the epimorphism $U^*(-)_{(p)} \xrightarrow{\mu} BP^*(-)$ by the fact that $U^*(CP^\infty)_{(p)} \xrightarrow[\mu]{} BP^*(CP^\infty)$ is the canonical map of the formal group law on $U^*_{(p)}$ to a "typical law" for p. [A "typical law" is a formal group law F with $F(\zeta_1 X^{1/q}, F(\zeta_2 X^{1/q}, F(\ldots, \zeta_q X^{1/q}))) = 0$ for all primes $q \neq p$ (ζ_i are q th roots of 1). Over a torsion-free ring R, such as $U^*_{(p)}$, a formal group law is typical if and only if its logarithm (over $R \otimes Q$) has the form $\sum l_i X^{p^i}$. The map $U^*_{(p)} \xrightarrow{\mu} BP^*$ sends $x_{p^j-1} \longmapsto y_j$ $\quad x_i \mapsto 0$ $(i \neq p^j - 1)$.]

We then have the factorisation of $\mu : U^*(X)_{(p)} \longrightarrow H^*(X;Z/p)$:-

$$U^*(X)_{(p)} \xrightarrow{\mu'} BP^*(X) \xrightarrow{\mu''} H^*(X;Z/p) \qquad (\mu'' \text{ is not epimorphic in general.})$$

Johnson and Wilson (21) examine μ'' by means of the tower of theories:

$$\begin{array}{l} BP^*(X) \to \ldots \to BP\langle n\rangle^*(X) \xrightarrow{\mu_n} BP\langle n-1\rangle^*(X) \to \ldots \to BP\langle 0\rangle^*(X) \xrightarrow{\mu_0} BP\langle -1\rangle^*(X) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \cong \qquad\qquad\qquad \cong \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad H^*(X;Z_{(p)}) \quad H\ (X;\ Z/p) \end{array}$$

where $BP\langle n\rangle^* = BP^*/\langle y_{n+1}, y_{n+2}, \ldots\rangle$ (the ideal is killed by Baas's method (3))

They show that for any finite complex X, there is an n such that $\mu_n, \mu_{n+1}, \ldots$ are all epimorphic and $\mu_{n-1}, \mu_{n-2}, \ldots$ are all not epimorphic, and further, that this n is hom $\dim_{BP^*}(BP^*(X))$, thus giving a great deal of information on the representability question.

There should be a connection with my results, using the short exact sequence: $BP\langle n-1\rangle^* \rightarrowtail U^*_{(p)} \twoheadrightarrow {}^{(n)}U^*_{(p)}$, but this remains to be explored.

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