A family of matings between transcendental entire functions and a Fuchsian group

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For Noel Baker

August 25, 2005

Abstract

Let Γ denote the subgroup of the modular group generated by $z \to -1/z$ and $z \to z+2$. We present a family of holomorphic correspondences $\mathcal{F}$ on the Riemann sphere which realise "matings" between maps of the form $z \to \mu sin^2(\pi\sqrt{z})$ and Γ, in the sense that the Riemann sphere is partitioned into dynamically invariant regions Ω and $\Lambda = \Lambda_+ \cup \Lambda_-$ with the following properties:

(i) Ω is open, simply connected and the action of $\mathcal{F}$ on Ω is conjugate to that of a generating set of elements of Γ;

(ii) Λ_- and Λ_+ are each homeomorphic to a set obtained from the filled Julia set of $\mu sin^2(\pi\sqrt{z})$ by "splitting infinity and shrinking dynamical rays to points". $\mathcal{F}$ restricted to Λ_- as domain and range is single-valued and conjugate to $\mu sin^2(\pi\sqrt{z})$ on this modified filled Julia set. The action of $\mathcal{F}^{-1}$ restricted to Λ_+ is conjugate to that of $\mathcal{F}$ on Λ_-.

While we have investigated matings between polynomial maps and Kleinian groups in earlier work, the examples presented here are the first to involve transcendental entire functions. We think of our construction as "resolving" the essential singularity of $z \to \mu sin^2(\pi\sqrt{z})$ by replacing the point at infinity with a region on which the dynamics is group-like.

*Both authors thank the UK EPSRC for their support through grant no. GR/R73232/01.

1 The Group Γ

Let Γ be the subgroup of $PSL(2,\mathbb{Z})$ generated by the matrices:

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

The action of Γ on the upper half plane by Moebius transformations is proper and discontinuous, with limit set $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. As an abstract group Γ is the free product of two cyclic groups, one of infinite order, generated by

$$\rho = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix},$$

and the other of order two, generated by

$$\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

We view Γ as the limit as $n \to \infty$ of the *Hecke groups* H_n generated by the matrices

$$\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \rho_n = \begin{pmatrix} 0 & 1 \\ -1 & -2cos(\pi/n) \end{pmatrix}.$$

Let $\mathbb{R}_+$ denote the set of positive real numbers and let $\hat{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{0, \infty\}$. For each $j \in \mathbb{N}$ we have

$$\sigma\rho^j(\hat{\mathbb{R}}_+) = [\frac{j+1}{j}, \frac{j}{j-1}] \text{ and } \sigma\rho^{-j}(\hat{\mathbb{R}}_+) = [\frac{j-1}{j}, \frac{j}{j+1}].$$

To each $x \in \hat{\mathbb{R}}$, which is not in the orbit of 1, we associate an infinite sequence $s(x)$ of non-zero integers, which has initial segment $s_1 s_2 \dots s_j$ if $x \in \sigma\rho^{s_1}\sigma\rho^{s_2}\dots\sigma\rho^{s_j}(\hat{\mathbb{R}}_+)$. Since each $\sigma\rho^j$ is a contraction in the spherical metric, every sequence specifies a unique point. However, points in the orbit of 0 and ∞ have more than one associated sequence, while points in the orbit of 1 have no sequence at all.

We overcome this difficulty as follows. Let $\mathcal{S}$ denote the set of infinite sequences $(s_1, s_2, ...)$ with each $s_j \in \mathbb{Z} - \{0\}$, and let $\sim$ be the relation defined on $\mathcal{S}$ by

$$(s_1, ..., s_n, -1, -1, -1, ...) \sim (s_1, s_2, ..., s_n + 1, 1, 1, 1, ...)$$

for $s_n \neq -1$. Next, let $\mathcal{S}_{fin}$ denote the set of finite sequences $(s_1, ..., s_n)$. By convention, we shall write each such sequence as $(s_1, \dots, s_n, \infty)$. Equip the union $(\mathcal{S}/\sim) \cup \mathcal{S}_{fin}$ with the following topology: if s is a sequence in $(\mathcal{S}/\sim)$, then a sequence t is close to s if the entries of t and s agree to a large number of places, regardless of whether t is in $\mathcal{S}/\sim$, or in $\mathcal{S}_{fin}$. If $s \in \mathcal{S}_{fin}$, with $s = (s_1, ..., s_n, \infty)$, then t is close to s if t is of the form $(s_1, ..., s_n, s_{n+1}, ...)$ with $|s_{n+1}|$ large, again regardless of whether t is in $\mathcal{S}/\sim$ or in $\mathcal{S}_{fin}$.

Proposition 1 *With the topology defined above,* $(\mathcal{S}/\sim) \cup \mathcal{S}_{fin}$ *is homeomorphic to* $\hat{\mathbb{R}}_+$*.*

Proof This is a straightforward consequence of the action of Γ described above. Sequences in $\mathcal{S}$ correspond to points not in the orbit of 1, and the relation $\sim$ deals with the difficulty that each point in the orbit of 0 or ∞ has two associated sequences. Sequences in $\mathcal{S}_{fin}$ correspond to points in the orbit of 1: thus $(s_1, \ldots, s_n, \infty)$ represents the point $\sigma\rho^{s_1} \ldots \sigma\rho^{s_n}(1)$. $\Box$

With this representation, the action of $\sigma\rho^j$ on $\hat{\mathbb{R}}_+$ becomes "shift to the right and insert the non-zero integer j". The sequence (or sequences) $s(x)$ associated to a point $x \in \hat{\mathbb{R}}_+$ will be called the *itinerary* (or *itineraries*) of x.

2 Correspondences and Matings

A *correspondence* on the Riemann sphere $\hat{\mathbb{C}}$ is a multivalued map $\mathcal{F} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. The *graph* of $\mathcal{F}$ is the subset $\{(z, w) : w \in \mathcal{F}(z)\} \subset \hat{\mathbb{C}} \times \hat{\mathbb{C}}$. We allow the sets $\mathcal{F}(z)$ and $\mathcal{F}^{-1}(z)$ to have infinite cardinality. If this is the case for generic z, we describe the correspondence as having *bidegree* $(\infty : \infty)$.

Definition 1 *We say that an* $(\infty : \infty)$ *correspondence* $\mathcal{F}$ *is a mating between* Γ *and a holomorphic map if:*

- *there exists a conformal involution* J *on* $\hat{\mathbb{C}}$ *conjugating* $\mathcal{F}$ *to* $\mathcal{F}^{-1}$*;*
- $\hat{\mathbb{C}}$ *is partitioned into sets* Ω *and* $\Lambda = \Lambda_- \cup \Lambda_+$*, where*

  (i) Ω *is open, simply-connected and invariant under both* $\mathcal{F}$ *and* J*; moreover there is a conformal homeomorphism* ϕ *from the open complex upper half-plane to* Ω*, conjugating the action of* σ *to that of* $J|_\Omega$*, and the action of* $\bigcup_{j \in \mathbb{Z} - \{0\}} \sigma\rho^j$ *to that of* $\mathcal{F}|_\Omega$*, and extending to a continuous map at* 0 *and* ∞*, sending both to a single fixed point* p *of* J*, which lies in* $\partial\Omega$*;*

  (ii) Λ_- *is closed and backward invariant under* $\mathcal{F}$*, and* $\mathcal{F}$ *restricted to* Λ_- *as domain and codomain is a (single-valued) holomorphic map;*

  (iii) $\Lambda_+ = J(\Lambda_-)$ *and* $\Lambda_- \cap \Lambda_+$ *consists of the fixed point* p *of* J*.*

Examples are plotted in Figures 1 and 2 (the figures are plotted in coordinates where J is the involution $z \leftrightarrow -z$). These examples lie in the complex one-parameter family of correspondences $\mathcal{F}_a$ defined below.

We say that a correspondence f is a *subcorrespondence* of the correspondence $\mathcal{F}$ if $f(z) \subset \mathcal{F}(z)$ for all $z \in \hat{\mathbb{C}}$. We say that f is *holomorphic* if it is defined by an equation

$$w \in f(z) \Leftrightarrow p(z, w) = 0$$

where $p(z, w)$ is a polynomial in z and w. The graph of f is a Riemann surface, the algebraic curve defined by the equation $p(z, w) = 0$.

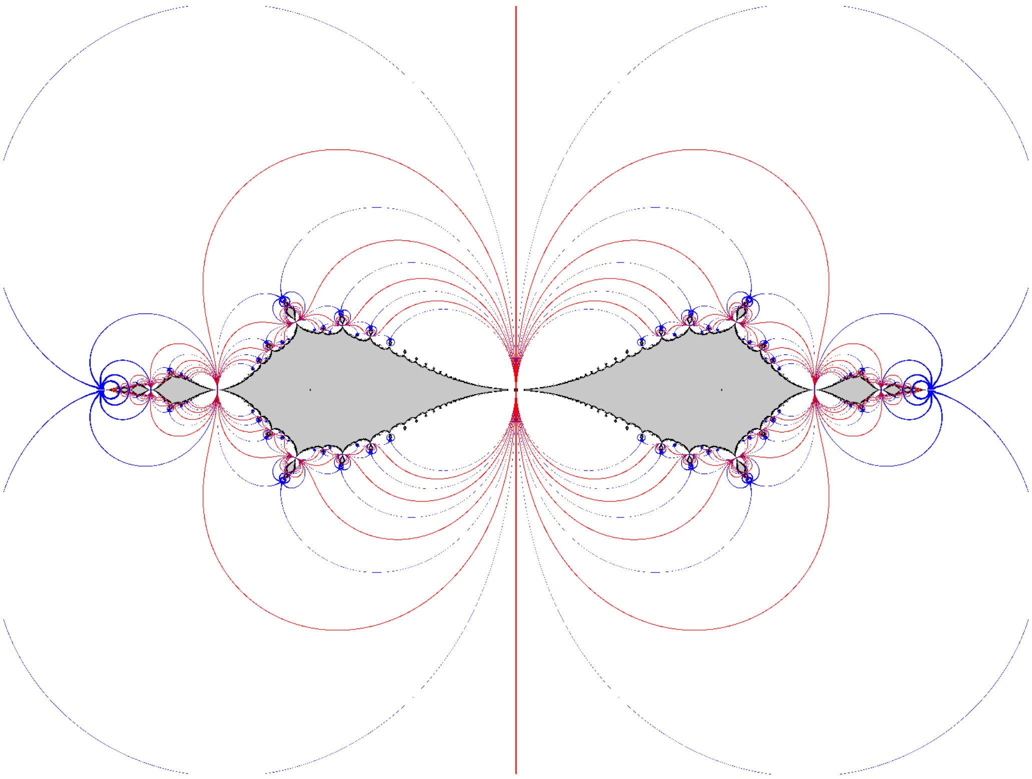

Figure 1: A mating between Γ and a holomorphic map. The grey region is the set Λ. Its complement Ω is tiled by fundamental domains for Γ. See Remark 5 at the end of Section 4 for details of the correspondences illustrated in Figures 1 and 2.

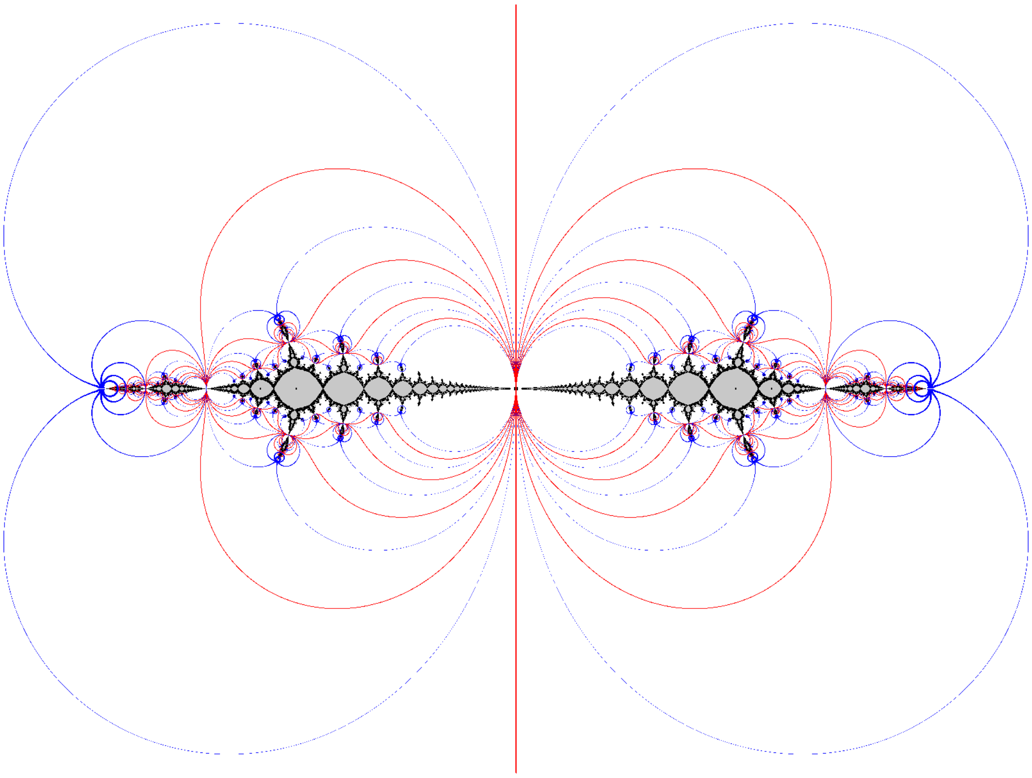

Figure 2: A mating between Γ and another holomorphic map.

As our candidates for matings we shall identify a complex one-parameter family of correspondences $\mathcal{F} = \mathcal{F}_a$, each of which splits into a countable union of holomorphic $(2:2)$ sub-correspondences $\{f_j\}_{j\geq 1}$ having $f_j|_\Omega$ conjugate to $\{\sigma\rho^j, \sigma\rho^{-j}\}$. We shall show that when a lies in a certain subset of the parameter space, $\mathcal{F}$ is a mating between Γ and a holomorphic map $\mathcal{F}|_{\Lambda_-}$, and that the holomorphic map $\mathcal{F}|_{\Lambda_-}$ has the combinatorics of a scalar multiple μR of the transcendental entire function

$$R(z) = \sin^2(\pi\sqrt{z})$$

For particular classes of such examples we prove (and for wider classes we conjecture) that for an appropriate value of μ there is a conformal conjugacy between the maps $\mathcal{F}|_{\Lambda_-}$ and μR on their respective Fatou sets. This conjugacy extends to a continuous bijection associating certain points in the limit set of Γ to "dynamical rays" in the Julia set of the entire function μR. For the particular values of μ we consider, the map μR has a repelling fixed point and a superattracting fixed point. The closure of the set of points with bounded orbits under μR, which we will call the filled Julia set of μR, is the whole of $\hat{\mathbb{C}}$. We picture the action of $\mathcal{F}$ on $\hat{\mathbb{C}}$ as obtained from two copies of the action of μR on $\mathbb{C}$ by cutting along the dynamical ray from infinity to the repelling fixed point in each copy to give two half planes which can be glued to form a single copy of $\mathbb{C}$. We then retract the two filled Julia sets to a bounded region of $\mathbb{C}$, at the same time contracting dynamical rays to points. Finally we glue into the newly created space in $\hat{\mathbb{C}}$ the action of Γ on the upper half plane.

Remark 1 *Since $z \to \lambda^2 \sin^2(\pi\sqrt{z})$ is double-covered by $\zeta \to \lambda \sin(\pi\zeta)$, via $z = \zeta^2$, the Fatou and Julia sets of the former are double-covered by the Fatou and Julia sets of the latter. Thus our descriptions below of the Fatou and Julia sets of the functions $\mu R(z)$ are easily translatable into descriptions of those of the functions $\lambda \sin(\pi\zeta)$, and vice versa, where $\mu = \lambda^2$.*

Remark 2 *In [2], we constructed matings between Hecke groups H_n and scalar multiples of the degree $n-1$ polynomial R_{n-1} defined by*

$$R_{n-1} = 1/2(1 - T_{n-1}(1-2z)),$$

where T_{n-1} is the $(n-1)$st Chebyshev polynomial (the polynomial which expresses $\cos((n-1)\theta)$ in terms of $\cos\theta$). The polynomial R_n is both conjugate to T_n and double-covered by it, in the sense that $R_n(z) = T_n^2(\sqrt{z})$. When we make the analogous construction in the infinite degree case (see Section 3), the polynomial T_n is replaced by the function $T(z) = \sin(\pi z)$, and the polynomial R_n is replaced by the function $R(z) = \sin^2(\pi\sqrt{z})$, which is double-covered by $T(z)$, The group Γ is the algebraic limit of the Hecke groups H_n. The current paper is independent of [2] in the sense that we shall present a direct construction of the $(\infty : \infty)$ correspondences. However we shall omit details of some proofs and refer the reader to the proofs in [2] of analogous results for the finite case.

3 A Family of Correspondences

The correspondences constructed in [2] had the form $J \circ Cov_0^Q$ (this notation will be explained later), where the rational map Q is the factored quotient map

$$\hat{\mathbb{C}} = \hat{\mathbb{C}}/S \to \hat{\mathbb{C}}/D_{2n} = \hat{\mathbb{C}}$$

obtained from the standard action of the dihedral group D_{2n} on $\hat{\mathbb{C}}$, and S is one of the pair of involutions generating this group. In suitable coordinates, Q is the Chebyshev polynomial T_n. Below we make the analogous construction in the case $n = \infty$.

Consider the action of the infinite dihedral group D_∞ on $\hat{\mathbb{C}}$ generated by the involution $S : \zeta \to -\zeta$ and the translation $T : \zeta \to \zeta + 1$. The limit set of this action consists of the single point ∞.

For the quotient map $\mathbb{C} \to \mathbb{C}/D_\infty$ we may take the map $\zeta \to \cos(2\pi\zeta)$, or, more conveniently, the map

$$\zeta \to \frac{1}{2}(1 - \cos(2\pi\zeta)) = \sin^2(\pi\zeta)$$

since, writing $z = \zeta^2$, this gives us the explicit form

$$R(z) = \sin^2(\pi\sqrt{z})$$

for the factored quotient map

$$\mathbb{C}/S \to \mathbb{C}/D_\infty.$$

The *covering correspondence* Cov^R, defined by

$$z \to w \quad \Leftrightarrow \quad R(z) = R(w),$$

splits into $(2:2)$ correspondences on $\mathbb{C}/S$, each coming from a pair of maps $\zeta \to \zeta \pm j$ on $\mathbb{C}$ (where j is a positive integer), together with a $(1:1)$ correspondence coming from $\zeta \to \zeta$. It is an elementary calculation (substitute z for ζ^2) that these $(2:2)$ correspondences have the form

$$g_j : z \to w \quad \Leftrightarrow \quad w^2 - 2(z + j^2)w + (z - j^2)^2 = 0.$$

Thus

$$w \in Cov^R(z) \quad \Leftrightarrow \quad z = w \text{ or } g_j(z, w) = 0 \text{ for some } j.$$

Despite the fact that $R(z)$ is undefined at $z = \infty$, each of the correspondences g_j has $g_j(\infty) = \{\infty\}$. Thus we may adopt the convention that $Cov^R(\infty) = \{\infty\}$ and think of Cov^R as a correspondence defined on the whole of $\hat{\mathbb{C}}$, with graph the union of the graphs of the $(2:2)$ holomorphic sub-correspondences g_j and that of the identity. We denote by Cov_0^R (the *deleted covering correspondence* *of R*) the correspondence obtained from Cov^R by deleting the graph of the identity.

We note that g_1 is a *symmetric* $(2:2)$ correspondence ($w \in g_1(z) \Leftrightarrow z \in g_1(w)$). It follows that for g_1 one can define "iteration without backtracking", under which if z maps to w we do not allow w to map directly back to z unless z is the *unique* element of $g_1(w)$. Denote the jth iterate of g_1 without backtracking by $g_1^{[j]}$.

Lemma 1 $g_j = g_1^{[j]}$.

Proof. Direct calculation (or the observation that g_j is $\{T^j, T^{-j}\}$ "pushed down from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}/S$"). $\square$

Given an involution J we use the notation $J \circ Cov_0^R$ to denote the correspondence obtained by composing the involution J with Cov_0^R, namely

$$w \in J \circ Cov_0^R(z) \quad \Leftrightarrow \quad (Jw)^2 - 2(z+j^2)(Jw) + (z-j^2)^2 = 0 \text{ for some } j \geq 1.$$

Thus the graph of $J \circ Cov_0^R$ is the union of the graphs of the holomorphic $(2:2)$ sub-correspondences f_j defined by

$$w \in f_j(z) \quad \Leftrightarrow \quad (Jw)^2 - 2(z+j^2)(Jw) + (z-j^2)^2 = 0.$$

Writing $\mathcal{F}$ for $J \circ Cov_0^R$ we see that $\{f_1, J\}$ *generates* $\{\mathcal{F}, J\}$ in the following sense.

Lemma 2 *The grand orbits obtained by mixed iteration of f_1, ${f_1}^{-1}$ and J are identical to those obtained by mixed iteration of $\mathcal{F}$, $\mathcal{F}^{-1}$ and J.*

Proof. This follows at once from the previous lemma and the fact that J is an involution. $\square$

Remark 3 *Our candidates for matings will be among correspondences $\mathcal{F}$ of the form $J \circ Cov_0^R$. We note that if such an $\mathcal{F}$ is a mating then restricted to Ω each $g_j|_\Omega$ is conjugate to $\{\rho^j, \rho^{-j}\}$ on the upper half-plane (by Lemma 1) and hence each $f_j|_\Omega$ is conjugate to $\{\sigma\rho^j, \sigma\rho^{-j}\}$.*

3.1 Fundamental Domains and Singular Points

The behaviour of the points $z = 0$ and $z = 1/4$ plays an important part in the dynamics of correspondences $\mathcal{F}$. The point 0 is a *singular* point for each f_j in the sense that $f_j(0)$ is the single point $J(j^2)$. The point $1/4$ has the property that for each j the intersection $f_j(1/4) \cap f_{j+1}(1/4)$ is non-empty. In fact, this intersection is the single point $J((j+1/2)^2)$. We also note here that the point $1/4$ is fixed by g_1, and sent to $J(1/4)$ by f_1.

Definition 2 *A transversal D_R for the function R is a maximal open domain in $\mathbb{C}$ on which R is injective.*

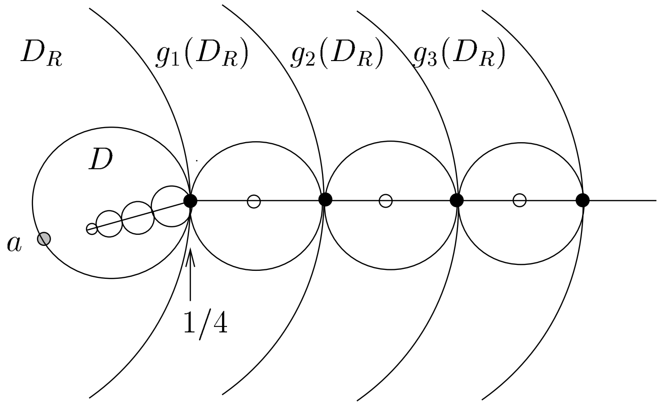

Figure 3: D_R is a maximal set on which R is injective; when it is symmetric about the real axis (as pictured) it is bounded by the line $z = \{(1/2+it)^2 \mid t \in \mathbb{R}\}$, which crosses the real axis at $1/4$, and the images of D_R under Cov_0^R are the"strips" $g_j(D_R)$ bounded by the lines $z = \{(j \pm 1/2 + it)^2 \mid t \in \mathbb{R}\}$. The disc D is a fundamental domain for J (which has fixed points $1/4$ and a). The discs shown within D are the images of D under $\mathcal{F}$. The discs shown outside D are the images of D under $Cov_0^R = J \circ \mathcal{F}$. The black points are $1/4$ and its images under Cov_0^R and the white points are the images of 0 under Cov_0^R.

A transversal plays the same role for the correspondence Cov_0^R as does a fundamental domain for a Kleinian group. Any transversal for our function R is an unbounded subset of $\mathbb{C}$, or equivalently it contains ∞ on its boundary. See Figure 3.

Proposition 2 *If $\mathcal{F}$, of the form $J \circ Cov_0^R$ with R as above, is a mating between Γ and a map, then one of the fixed points of J coincides with the fixed point $1/4$ of $g_1 = J \circ f_1$, and the singular point 0 lies in Λ_+. Moreover, there exists a closed disc D such that*

- *D is a fundamental domain of the involution J*
- *D is contained in the closure of a transversal D_R of R with $\partial D \cap \partial D_R = \{1/4\}$;*
- *$f_1(D) \subset D$ with $\partial f_1(D) \cap \partial(D) = \{1/4\}$;*
- *$f_j(D)$ is properly contained in D for all $j > 1$, and $f_j(D) \cap f_{j+1}(D) = \{f_{j+1}(1/4)\}$;*

See Figure 3.

Proof By Definition 1 and Remark 3, there exists a homeomorphism ϕ from the upper half plane to the set Ω, conjugating $\{\sigma\rho, \sigma\rho^{-1}\}$ to f_1 and σ to J. Moreover, ϕ extends continuously to 0 and ∞, sending both to a fixed point p

of J, which lies in $\partial\Omega$. The parabolic map ρ sends ∞ to 0. Since ϕ extends to these points and since ρ is conjugate to one of the branches of g_1, we deduce that p is the unique fixed point 1/4 of g_1.

We next show that the singular point 0 lies in Λ_+. The point 0 has only one image v_1 under g_1, but v_1 has two distinct inverse images under g_1. As there is no point in the upper-half plane which has this property under $\{\rho, \rho^{-1}\}$, it follows that 0 does not lie in Ω. But neither can 0 lie in Λ_-, for then at least one of the $f_j(0) = J(v_j)$ would also lie in Λ_- (since $\mathcal{F}$ restricted to Λ_- as domain and range is a well-defined map), but as points sufficiently close to 0 have two distinct but nearby images under f_j this would contradict the single-valuedness of $\mathcal{F}|\Lambda_-$. The only remaining possibility is that $0 \in \Lambda_+$.

Now consider the imaginary axis $\mathcal{I}$ in the upper half plane. Since $\phi(0) = \phi(\infty)$, the image $\phi(\mathcal{I})$ of $\mathcal{I}$ is a closed curve in Ω separating Λ_+ and Λ_- and dividing the sphere into a fundamental domain for J and its image. Let D denote the component of $\hat{\mathbb{C}} - \phi(\mathcal{I})$ containing $\phi(\mathcal{R})$, where $\mathcal{R} = \{z = x + iy \mid x, y > 0\}$, so $D - (\Omega \cap D)$ is either Λ_+ or Λ_-. In fact $D - (\Omega \cap D) = \Lambda_+$, as we shall now see. Any point $x \in \hat{\mathbb{R}}$ which is of the form $\sigma\rho^j(0)$ or $\sigma\rho^j(\infty)$, for $j \in \mathbb{Z} - \{0\}$, lies on the positive real line, which is part of the boundary of $\mathcal{R}$. Since the conjugacy ϕ extends to these points, it follows that $f_j(\phi(x))$ lies in D. Hence f_j is not single-valued as a map from $D - (\Omega \cap D)$ to itself, and so $D - (\Omega \cap D)$ cannot be Λ_-.

For a fundamental domain for the action of ρ on the upper half-plane we may take the region D_ρ bounded by the geodesic L_1 from -1 to ∞, together with the geodesic L_2 from -1 to 0 and the positive real axis. Since ϕ identifies 0 with infinity, $\phi(L_1) \cup \phi(L_2)$ forms a closed Jordan curve. Now D_ρ contains $\mathcal{R}$, so $D_R = \Lambda_+ \cup \phi(D_\rho)$ is a topological disc containing D and Λ_+. Note that $\partial D \cap \partial D_R = \{1/4\}$. Since Cov_0^R acts on $\phi(D_\rho)$ as $\{\phi\rho^j\phi^{-1}\}_{j\in\mathbb{Z}-\{0\}}$ and also sends Λ_+ to Λ_-, we deduce that Cov_0^R maps D_R outside itself, and hence that D_R is contained in a transversal for R. But L_1 is sent to L_2 by ρ, so D_R is a *maximal* set on which R is injective. Moreover, since $D \subset D_R$ and D is a fundamental domain for J, we deduce that f_j (which is the composition $J \circ g_j$) sends D into itself for each integer $j \geq 1$.

It only remains to show that $\partial f_1(D) \cap \partial(D) = \{1/4\}$, that $\partial f_j(D) \cap \partial D = \emptyset$ for each $j > 1$, and that $f_j(D)$ meets $f_{j+1}(D)$ in the single point $f_{j+1}(1/4)$ for each $j \geq 1$. First observe that from the conjugacy between g_j on Ω and $\{\rho^j, \rho^{-j}\}$ on the upper half-plane and the fact that ϕ identifies 0 with ∞, we may deduce that ϕ identifies $\rho^j(0)$ with $\rho^{-j}(\infty)$ for each j. Next, from the same conjugacy, we may deduce that for each $j \geq 1$ the set $g_j(D_R)$ is a pinched annulus, bounded by the circles $\phi(\rho^{1-j}L_1) \cup \phi(\rho^{j-1}L_2)$ and $\phi(\rho^{-j}L_1) \cup \phi(\rho^j(L_2)$ (which touch at $\phi(-1)$, which is the point at infinity on the Riemann sphere $\hat{\mathbb{C}}$), and that these pinched annuli ("strips" in Figure 3), together with the disc D_R, partition $\hat{\mathbb{C}}$. The set $g_j(D) \subset g_j(D_R)$ is a disc, bounded by a circle $\rho^{-j}(\mathcal{I}) \cup \rho^j(\mathcal{I})$. Since this circle touches ∂D_R only in the case $j = 1$, and then at the point 1/4, and since it touches $\partial g_j(D_R)$ only at the points $g_j(1/4) = (j - 1/2)^2$ and $g_{j+1}(1/4) =$

$(j+1/2)^2$, we deduce that $\partial g_1(D) \cap \partial D = \{1/4\}$, that $\partial g_j(D) \cap \partial D = \emptyset$ for each $j > 1$, and that $g_j(D)$ meets $g_{j+1}(D)$ in the single point $g_{j+1}(1/4)$ for each $j \geq 1$. Composing with the involution J we obtain the corresponding results for $\{f_j\}_{j\geq 1}$.

$\square$

As we shall see below, the converse of this Proposition also holds.

Definition 3 *We define J_a to be the involution of $\hat{\mathbb{C}}$ which has fixed points $1/4$ and a, and $\mathcal{M}$ to be the set of parameters $a \in \mathbb{C}$ such that the correspondence $\mathcal{F}_a = J_a \circ Cov_0^R$ satisfies the conclusions of Proposition 2.*

The set $\mathcal{M}$ is non-empty. For example, if a is such that $J_a(j^2) = 0$ for some integer $j \geq 1$, then the point 0 is fixed by f_j, and one can show that J_a satisfies the conclusions of the Proposition.

Now, let $\mathcal{F} = \mathcal{F}_a \in \mathcal{M}$ and define

$$\Lambda_+ = \overline{\bigcap_{n=0}^{\infty} \mathcal{F}^n(D)}, \quad \Lambda_- = J(\Lambda_+), \quad \Omega = \hat{\mathbb{C}} - (\Lambda_- \cup \Lambda_+).$$

Lemma 3 Λ_+ *is simply-connected.*

Proof Since $0 \in \Lambda_+ \in D$, we have that $f_j(D)$ is a topological disc for each j, and each $f_j(D) \cap f_{j+1}(D)$ consists of exactly one point in the orbit of $1/4$. In particular, $\overline{\mathcal{F}(D)}$ is simply-connected (see figure 3). Using an induction one can show that since $0 \in \mathcal{F}^n(D)$ for all n, $\overline{\mathcal{F}^n(D)}$ is simply-connected for all n. It follows that Λ_+ is simply-connected. $\square$

We can now prove:

Theorem 1 $\mathcal{F}_a = J_a \circ Cov_0^R$ *is a mating between Γ and a map if and only if $a \in \mathcal{M}$.*

Proof One implication is given by Proposition 2. The proof of the other implication is very similar to the proof of Proposition 7.1 in [2], and we shall only give an outline here.

Suppose that $a \in \mathcal{M}$. First we note that since Λ_+ is simply-connected, so are $\Lambda_+ \cup \Lambda_-$, and Ω. It follows that there exists a Riemann map ϕ from the upper half plane to Ω. Clearly, $J_a|_\Omega$ is conjugate via ϕ to an involution, which we may take to be $\sigma : z \to -1/z$, on the upper half-plane.

By Lemma 1 we know that $J_a \circ \mathcal{F}_a|_\Omega$ is conjugate via ϕ to a cyclic group generated by an element ρ_a of $PSL(2,\mathbb{R})$ which has infinite order, and that $\phi^{-1}(D_R \cap \Omega)$ is a fundamental domain for ρ_a. Moreover, ∂D_R lies in Ω, by the same argument as used in the proof of Proposition 2, and contains ∞. Thus, ∞ is accessible via a path in Ω, and so there exists a unique point q in the real line such that ϕ extends continuously to q and sends it to ∞. Now $q = \phi^{-1}(\infty)$ is a fixed point of ρ_a. Examining the behaviour of Cov_0^R, we see that the images of $D_R - \Lambda_+$ under branches of Cov_0^R accumulate at ∞. Thus, the images of the fundamental domain of ρ_a under all the powers of ρ and ρ^{-1} accumulate at the fixed point $\phi^{-1}(\infty)$, and hence ρ_a is parabolic.

To show that $\rho_a = \rho$, we first notice that the point $1/4$ is accessible from Ω, via the curve ∂D_R. A small neighbourhood of $1/4$ intersects Ω in two connected components. Thus, there exist two points a and b in the real line, such that ϕ continuously extends to both and sends them both to $1/4$. Moreover, a and b are end-points of the boundary of a fundamental domain of ρ_a. Similarly, there exist two points c and d in the real line, such that ϕ continuously extends to both and sends them both to $1/4$, and which are end-points of the boundary of a fundamental domain of σ. Using an argument explained in detail in [2], Proposition 7.1, one can show that $a = b$ and $b = c$. This is sufficient to prove that $\rho = \rho_a$. $\square$

4 Transcendental Matings

We claim that correspondences of the form $\mathcal{F}_a = J_a \circ Cov_0^R$ with $a \in \mathcal{M}$ can be regarded as matings between Γ and transcendental entire functions of the form $\mu R(z)$ where $R(z) = sin^2(\pi\sqrt{z})$ and μ is some scalar. In order to justify this claim we shall show that the infinite-to-one map $\mathcal{F}^{-1} : \Lambda_+ \to \Lambda_+$ resembles μR (for an appropriate value of μ) in a very strong sense. We cannot hope for a conjugacy between $\mathcal{F}^{-1}$ on the whole of Λ_+, and μR on the whole of its filled Julia set, since the latter is the whole complex plane. However, we define below the notion of the *nth truncation* of $\mathcal{F}$, with associated limit set $\Lambda_+^n \subset \Lambda_+ = \overline{\bigcup_n \Lambda_+^n}$ (pictorially, Λ_+^n is obtained from Λ_+, as shown in Figure 1, by "cutting off the ends of branches"). We then show in the remaining sections of the paper that there is a conjugacy from $\mathcal{F}^{-1}$ on $\bigcup_n \Lambda_+^n$ (*not* its closure) to μR on the analogous subset of its filled Julia set, and that the remaining points in Λ_+ correspond either to "dynamical rays" of μR or to the point at infinity.

For simplicity, we shall restrict our attention to parameters a for which the singular point 0 is fixed by f_m for some $m \geq 1$, ie $0 = J_a(m^2)$ for some $m \geq 1$. We denote the correspondence satisfying this condition by $\mathcal{F}_m$.

We write Λ'_+ for the set $\bigcap_k \mathcal{F}^k(D)$. Let P denote the point on the real line where the images $f_j(D)$ accumulate. So P is ∞ in the parametrization $\mathcal{F}_m = J \circ Cov_0^R$ and a finite real point in the coordinates adopted in Figures 1 and 2.

Lemma 4 *$\Lambda_+ - \Lambda'_+$ consists of points in the orbit of P.*

Proof Clearly, $\Lambda_+ - \Lambda'_+$ contains all the points in the orbit of P. Now let $\mathcal{P}_k = \bigcup_{i=0}^k \mathcal{F}^i(P)$, for $k \geq 0$, so the orbit of P is equal to the union $\bigcup_k \mathcal{P}_k$. Then

$$\Lambda'_+ \cup \bigcup_k \mathcal{P} = \bigcap_k (\mathcal{F}^k(D) \cup \mathcal{P}_{k-1}) = \overline{\bigcap_k \mathcal{F}^k(D)} = \Lambda_+.$$

$\square$

The structure of Λ'_+ is easily understood: each point which is not in the orbit of $1/4$ lies in the intersection of a sequence of nested discs D_k, each D_k being a connected component of the interior of $(\mathcal{F}_m)^k(D)$. The point $1/4$ lies on the boundary of infinitely many such discs, and it is the only point at which the boundaries of these discs touch. The same is true for any point x in the orbit of $1/4$, and since $f_j(1/4) \cap f_{j+1}(1/4)$ is non-empty, we have two sequences of discs associated to such an x.

We see that if $m > 1$, the sub-correspondence $f_m^{-1} : f_m(D) \to D$ is quadratic-like and hybrid equivalent to $z \to z^2$. If $m = 1$, then this map is a *simply-pinched* quadratic-like map [2]. Thus, by the Straightening Theorem [6] (or an ad hoc argument when $m = 1$), the set $\mathcal{K} = \bigcap f_m^n(D)$ is homeomorphic to the filled Julia set of $z \to z^2$. The homeomorphism conjugates f_m to $z \to \pm\sqrt{z}$ and is conformal on the interior of $\mathcal{K}$.

Theorem 2 Λ_+ *is the closure of a countable union of copies of* $\mathcal{K}$.

Proof By Lemma 4 it is sufficient to show that each point in the interior of Λ'_+ lies in one of the images of $\mathcal{K}$ under forward iterates of $\mathcal{F}_m$. Suppose $x \in \Lambda'_+$ and suppose that x does not lie in one of the images of $\mathcal{K}$. Then x lies in the intersection $\mathcal{I}$ of a sequence of nested discs as described above. Let $\{D_n\}$ denote that sequence, and let $A_n = D_n - \overline{D_{n+1}}$, a sequence of nested annuli. Using techniques explained in detail in the proof of Lemma 7.1 in [2], one can show that there exists a subsequence $\{A_{n_k}\}$ of annuli with the property that some iterate of $\mathcal{F}^{-1}$ maps each A_{n_k} onto an annulus B_k, which is one of the sets $D - \overline{f_j(D)}$, where $j > 1$. Moreover, this iterate is either a degree one or a degree two map when restricted to A_{n_k}. Thus, the modulus of each A_{n_k} is either equal to that of B_k, or it is half the modulus of B_k. Now, the moduli of the annuli $D - \overline{f_j(D)}$, $j > 1$, are clearly bounded below, since the discs $f_j(D)$ accumulate at $J(\infty)$. Hence, the moduli of the A_{n_k} are bounded below and their sum diverges. Now the annulus $D_1 - \mathcal{I}$ contains all the annuli A_{n_k}. By the Groetzsch inequality, the modulus of $D_1 - \mathcal{I}$ is infinite, and hence $\mathcal{I}$ consists of a single point, namely our point x.
Each of the discs D_n is an image of D and therefore contains an image of $\mathcal{K}$, so x is an accumulation point of such images.

□

It will be useful to regard Λ_+ as a limit of sets Λ_+^n associated to *truncated correspondences*: for each positive integer n, we denote by $\mathcal{F}_m|_n$ the correspondence $f_1 \cup f_2 \cup \ldots \cup f_n$, and by Λ_+^n the set $\bigcap_k (\mathcal{F}_m|_n)^k(D)$. The configuration of the discs $(\mathcal{F}_m|_n)^k(D)$ depends on the position of 0 within $(\mathcal{F}_m|_n)^{k-1}(D)$. See figure 4. Clearly, $\Lambda_+ = \overline{\bigcup_n \Lambda_+^n}$.

Remark 4 *There are two sets of points that lie in* Λ_+ *but not in* $\bigcup_n \Lambda_+^n$. *The first set consists of points in the orbit of* P. *The second consists of points* x *of* Λ'_+ *with the property that the set* $\{k \mid \mathcal{F}_m^{-l}(x) \in f_k(D) \text{ for some } l\}$ *is not bounded above. A point of this type is said to have an* unbounded itinerary.

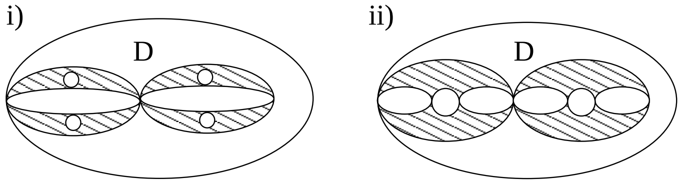

Figure 4: Figure (i) shows D and its images under the truncated map $(\mathcal{F}_1|_2)^i(D)$ for $i = 1, 2$. Figure (ii) shows D and its images under the truncated map $(\mathcal{F}_2|_2)^i(D)$ for $i = 1, 2$.

Remark 5 *Figure 1 shows $\mathcal{F}_1$, in other words with the parameter a chosen such that $J_a(1) = 0$. Figure 2 shows a correspondence with the parameter a chosen to make 0 an attracting point of period 2 under f_1^{-1} - a case not treated in the current paper but which could be analysed by generalising the methods presented here.*

5 Dynamics of the map $\mu_m R(z)$

In this section we describe the dynamics of $z \to \mu_m R(z) = \mu_m sin^2(\pi\sqrt{z})$, where $\mu_m = (m - 1/2)^2$ and m is a positive integer. This map has critical points j^2, $j \geq 1$ all mapping to the fixed point zero, and $(j - 1/2)^2$, $j \geq 1$, all mapping to the critical value μ_m. Thus, the point $(m - 1/2)^2$ is a super-attracting fixed point. Let $\lambda_m = m - 1/2$ and define $r_{\lambda_m}(z) = \lambda_m sin(\pi z)$. Then r_{λ_m} has critical points $j - 1/2$, $j \geq 1$ mapping to $\pm\lambda_m$.

Proposition 3 *Let m be an odd positive integer, and let D_+^t denote the set $\{z = x + iy \mid |z| < t, x > 0\}$. Then*

(i) For each odd positive integer $n > 1$, there exists a real number $t(n)$ such that for all $t \geq t(n)$ there is a disc U_n^t such that

- *The critical point $n - 1/2$ is contained in U_n^t,*
- *$\bar{U}_n^t \subset D_+^t$,*
- *$r_{\lambda_m}(U_n^t) = D_+^t$. Thus $r_{\lambda_m}|_{U_n^t}$ is quadratic-like.*

(ii) For each even positive integer the same is true with r_{λ_m} replaced by $-r_{\lambda_m}$.

(iii) For $n = 1$, there exists a real number $t(1)$ such that for all $t \geq t(1)$ there is a disc U_1^t such that

- *The critical point $1/2$ is contained in U_1^t,*
- *$\bar{U}_1^t \subset \bar{D}_+^t$, and the $\partial U_1^t \cap \partial D_+^t$ consists of a sub-interval of the imaginary axis,*
- *$r_{\lambda_m}(U_1^t) = D_+^t$.*

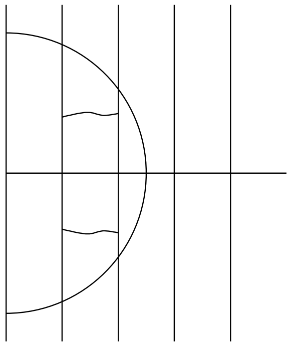

Figure 5: The discs D_+^t and U_2^t for some $t \geq t(2)$.

Proof Assume that n is odd (for n even just replace r_{λ_m} by $-r_{\lambda_m}$). For any $t \in \mathbb{R}$, the inverse image of D_+^t which contains $n-1/2$ consists of a quadrilateral region which has vertical segments at $n-1$ and n as two of its boundary components. A calculation shows that for large enough t, these vertical segments and the other two boundary components are properly contained in D_+^t. See Figure 5. $\square$

Remark 6 *If m is even, then the same result holds, but with r_{λ_m} replaced by $-r_{\lambda_m}$.*

Proposition 4 *Let m be a positive integer. Then for each positive integer n there exists a topological disc D_n such that the interior of $(\mu_m R)^{-1}(D_n)$ consists of infinitely many connected components V_j^n and such that:*

- *each V_j^n contains the critical point $(j-1/2)^2$;*
- *∂V_j^n and ∂V_{j+1}^n meet in the point j^2 which is an inverse image of 0;*
- *$\mu_m R$ restricted to each V_j^n is a two-to-one map. For $1 < j \leq n$ the disc V_j^n is properly contained in D_n and hence this restriction is quadratic-like. It is simply-pinched quadratic-like if $j = 1$.*

Proof Given m and n, let $t \geq t_n$ and consider the disc D_+^t. We note that $t(n) > t(1)$, so by the previous Proposition the inverse image of D_+^t contains the quadrilateral region U_1^t, which contains $1/2$ and has a subinterval I of the imaginary axis as a boundary component. Let W denote the image of $U_1^t \cup I$ under $z \to z^2$. Then the map $\mu_m R$ restricted to W is a quadratic-like map, 0 is a repelling fixed point and the landing point of the external ray of argument zero of this map. By "cutting out" the external ray of argument zero from W_n we can create a topological disc S such that $\mu_m R$ restricted to S is simply-pinched quadratic-like with contact point 0. Define $D_n = \mu_m R(S)$. Then D_n has the required properties. $\square$

6 The Fatou set of $\mu_m R$

We fix a positive integer m and continue to study the dynamics of the map $\mu_m R$. For any positive integer n, we define the *truncated map* $\mu_m R|_n$ to be the restriction of $\mu_m R$ to the discs V_i^n, $1 \le i \le n$, defined in the previous Proposition. Then each $\overline{V_i^n}$ is contained in $\overline{D_n}$, and the boundaries touch only if $i = 1$, in which case the contact point is 0. We define

$$K_n = \bigcap_k (\mu_m R|_n)^{-k}(D_n).$$

For n large enough, one of the V_i^n contains the critical point $(2m-1)^2/4$, which is a fixed point. Thus, by the same reasoning as for the truncated correspondences, the set K_n contains a copy $\mathcal{H}$ of the filled Julia set of $z \to z^2$ and the map restricted to this copy is conjugate to $z \to z^2$, the conjugacy being conformal on the interior. We note that $\mathcal{H}$ does not depend on n.

Theorem 3 *The union of the inverse images of $\mathcal{H}$ under $\mu_m R$ is the Fatou set of $\mu_m R$.*

Proof Each Fatou component of $\mu_m R$ is pre-periodic, and eventually lands on either an attracting or parabolic basin, or a Siegel disc (Herman rings, Baker domains and wandering domains do not occur since $(\mu_m R)^{-1}$ has finitely many singular points). Each of these Fatou components "uses up" a critical point of $\mu_m R$ (see [9], Theorems 2.4.1, 2.4.2 and 2.4.6.). However, all critical points map to the same critical value μ_m, so $\mathcal{H}$ is the only forward invariant Fatou component. $\square$

The similarities between the dynamics of the truncated map $\mu_m R|_n$ and the truncated correspondence $\mathcal{F}_m|_n$ are clear: for each k the sets $(\mu_m R|_n)^k(D_n)$ and $(\mathcal{F}_m|_n)^k(D)$ are topologically the same (see Figure 4). The points zero and μ_m in the dynamical plane of $\mu_m R$ play the same roles as the points $1/4$ and 0 (respectively) in the correspondence plane. We will see that there is a conformal conjugacy between the two maps restricted to their truncated filled Julia sets. The difference between the two situations is that this conjugacy does not extend to the sets $K = \overline{\bigcup_n K_n}$ and $\Lambda_+ = \overline{\bigcup_n \Lambda_+^n}$. In the $\mu_m R$ situation, the discs D_n vary with n. In fact these discs increase in size as n increases, and K is the whole of $\mathbb{C}$. $K - \bigcup_n K_n$ consists of a set of curves, so-called dynamical rays, while $\Lambda_+ - \bigcup_n \Lambda_+^n$ consists of a set of points (see Remark 4). In other words, while a sequence of points in $\bigcup_n \Lambda_+^n$ might accumulate at a point, the corresponding sequence in $\bigcup_n K_n$ might accumulate along the whole of a curve. We shall discuss this possibility further in Section 8. But first we shall establish a conjugacy between actions on truncated filled Julia sets.

7 Conjugacies on “truncated filled Julia Sets”

Consider the correspondence $\mathcal{F}_m$ and the map $\mu_m R$. We construct a bijection

$$\Psi : \bigcup_n \Lambda_+^n \to \bigcup_n K_n$$

which is a homeomorphism on each Λ_+^n, sending it onto K_n. We have a conjugacy $\Psi : \mathcal{K} \to \mathcal{H}$, which is conformal on the interior of $\mathcal{K}$, coming from the conjugacy to $z \to z^2$ of the relevant restrictions of the map and correspondence. We can extend Ψ to the inverse images of the closed Fatou component $\mathcal{K}$ to give a conjugacy on these. Now consider any point $x \in \bigcup_n \Lambda_+^n$ which does not lie in an image of $\mathcal{K}$ under iterates of $\mathcal{F}_m$. Then x lies in Λ_+^n for some n, and it is in the intersection of a nested sequence $\{D_k\}$ of discs, as described in section 4. To each disc D_k we can associate a unique disc D'_k in $(\mu_m R|_n)^k(D_n)$, as the dynamics of the truncated map and correspondence are topologically the same. Hence we obtain a sequence $\{D'_k\}$ of nested discs in the dynamical plane of $\mu_m R|_n$. By Theorem 2 we have $\bigcap_k D_k = \{x\}$. It is clear that $\bigcap_k D'_k$ is not an inverse image of the Fatou component $\mathcal{H}$. We can use the methods in the proof of Theorem 2 to prove that $\bigcap_k D'_k$ consists of a single point y. We define $\Psi(x) = y$.

It is easy to check that Ψ is a homeomorphism restricted to each Λ_+^n, that it is conformal on interiors, has image K_n and conjugates $\mathcal{F}_m$ to $\mu_m R$. Ψ restricted to the union of the Fatou components of Λ_+ is a conformal conjugacy between $\mathcal{F}_m$ and $\mu_m R$ restricted to its Fatou set. .

8 Dynamical Rays

8.1 $\partial\Lambda_+$ and the group

Recall that the correspondence $\mathcal{F}_m$ represents a mating between a map and the group Γ. Thus, there exists a conformal homeomorphism ϕ from the upper half plane to $\Omega = \hat{\mathbb{C}} - \Lambda$ which conjugates the action of the group to that of $\mathcal{F}_m|_\Omega$.

Theorem 4 *For each $m > 1$ the homeomorphism ϕ from the upper half plane to Ω can be extended to a continuous surjection $\phi : \hat{\mathbb{R}} \to \partial\Lambda$, with $\phi(\hat{\mathbb{R}}_+) = \partial\Lambda_+$. Hence $\partial\Lambda_+$ is locally connected.*

Proof We shall only give a very brief indication of the method here. The analogous result with the Hecke group H_4 in place of Γ is stated as Theorem 9.1 in [2] and proved in detail there.

Each point $x \in \hat{\mathbb{R}}_+$ is the unique point in the intersection of a nested sequence of closed discs in $\hat{\mathbb{C}}$ (these nested sequences correspond to the nested sequences of closed intervals on $\hat{\mathbb{R}}_+$ discussed in Section 1). The parts of the boundaries of these discs which lie in the upper half-plane are carried by ϕ to curves in Ω. By a combination of arguments involving nested discs in $\Omega \cup \Lambda$ and the hyperbolicity of the quadratic maps involved in the restrictions of $\mathcal{F}_m^{-1}$, it is possible to show that the end points of these curves converge to a unique point $y \in \partial\Lambda_+$ which we then define to be $\phi(x) = y$.

$\square$

Remark 7 *The restriction of this result to $m > 1$ is for technical reasons, involving the fact that for $m = 1$ the relevant quadratic map is only pinched-equivalent to z^2 rather than hybrid-equivalent to it. However we conjecture that the Proposition is equally valid in the case $m = 1$, and indeed we have an outline argument for a proof of this using pinching techniques.*

The homeomorphism ϕ gives rise to an equivalence relation $\sim$ on $\hat{\mathbb{R}}$ by $x \sim y$ if and only if $\phi(x) = \phi(y)$; thus $\hat{\mathbb{R}}/\sim$ is homeomorphic to $\partial\Lambda_+$. We give an explicit definition of ϕ:

Definition 4 *Define the* critical interval $\mathcal{C}$ *to be the interval with endpoints given by the itineraries $(-m, -m, -m, ...)$ and $(m, m, m, ...)$ (see Section 1).*

Definition 5 *Let $\sim$ be the closure of the following equivalence relation defined inductively on the orbit of 0 and ∞: let $0 \sim \infty$ and suppose $x \sim y$.*

- *If $y = 1/x$ and $x, y \notin \mathcal{C}$ define $\sigma\rho^i(x) \sim \sigma\rho^{-i}(y) = \frac{1}{\sigma\rho^i(x)}$ for any $i \neq 0$.*
- *Otherwise define $\sigma\rho^i(x) \sim \sigma\rho^i(y)$.*
- *For $x, y \in \mathbb{R}_-$ define $x \sim y$ if and only if $\sigma(x) \sim \sigma(y)$.*

Theorem 5 *For any two points x and y in $\hat{\mathbb{R}}$ we have $\phi(x) = \phi(y)$ if and only if $x \sim y$.*

Proof Once again we give only the briefest of indications of the proof: see [2] Lemmas 9.1 and 9.2 and Proposition 9.1 for details in the analogous case where H_4 plays the role of Γ here. Let $\approx$ be the equivalence relation $x \approx y$ if and only if $\phi(x) = \phi(y)$. The two branches of f_j correspond to the pair $\{\sigma\rho^j, \sigma\rho^{-j}\}$. From this one can show that points in ∂K map via ϕ^{-1} to the set T consisting of points whose itineraries are infinite sequences in m and $-m$ only. These are identified under $\approx$ to form a circle, since $\mathcal{K}$ is homeomorphic to the filled Julia set of $z \to z^2$. One can derive the definition of $\approx$ inductively on the orbit of 0 and ∞: if $x \approx y$ then either $\sigma\rho^i(x) \approx \sigma\rho^i(y)$ or $\sigma\rho^i(x) \approx \sigma\rho^{-i}(y)$. The fact that points in T get identified by $\approx$ to form a circle imposes combinatorial restrictions on which of the two possibilities can occur in each case. Closer inspection shows that the rule prescribing which possibility occurs is exactly that given in the definition of $\sim$.

$\square$

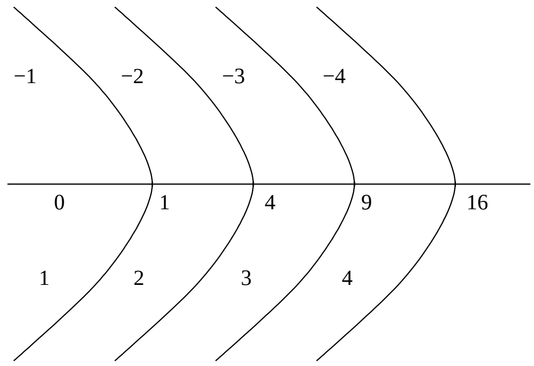

Figure 6:

8.2 The structure of $J(\mu_m R)$

By adapting [1] and [4] we assign an *itinerary* to each point in the Julia set $J(\mu_m R)$, and we show that $J(\mu_m R)$ is disjoint union of curves, each labelled by a sequence. First divide the plane into labelled strips as follows (Figure 6):

- Let $S(-n)$ denote the open region bounded by the real interval $[(n-1)^2, n^2]$ and by the lines $\{(x+iy)^2 \mid x = (n-1), y \geq 0\}$ and $\{(x+iy)^2 \mid x = n, y \geq 0\}$;
- Let $S(n)$ denote the open region bounded by the real interval $[(n-1)^2, n^2]$ and by the lines $\{(x+iy)^2 \mid x = (n-1), y \leq 0\}$ and $\{(x+iy)^2 \mid x = n, y \leq 0\}$.

Each strip is an inverse image of $\mathbb{C} - \{\mathbb{R}_- \cup [0, \mu_m]\}$, where $\mathbb{R}_-$ denotes the negative real line. The non-real boundary components of each strip map onto $\mathbb{R}_-$, and the real boundary component maps two-to-one onto the interval $[0, \mu_m]$.

If z is a point in the Julia set whose orbit never lands on $\mathbb{R}_- \cup [0, \mu_m]$ then each $z_k = (\mu_m R)^k(z)$ lies inside one of the open strips. We give z an *itinerary* $s(z) = \{s_n\}_{n \geq 0}$ by defining $s_i = n$ if and only if $(\mu_m R)^i(z)$ lies in the strip $S(n)$, for some $n \in \mathbb{Z} - \{0\}$.

Next consider $z \in \mathbb{R}_+$ whose orbit never lands on 0. Then all the images of z are real and positive and hence each lies on the common boundary of two strips. We give z two itineraries $s(z)$ and $s'(z)$ inductively as follows: if $z = z_0$ lies on the boundary of strips $S(n)$ and $S(-n)$ define $s_0 = n$ and $s'_0 = -n$. Now suppose that s_i and s'_i have been defined for all $i \leq k$ and that z_k and z_{k+1} lie on the common boundary of the strips $S(l)$ and $S(-l)$, and $S(m)$ and $S(-m)$ respectively. Then if z_k lies in the open interval $((l-1)^2, (l-1/2)^2)$ we define $s_{k+1} = -m$ and $s'_{k+1} = m$. Otherwise we define $s_{k+1} = m$ and $s'_{k+1} = -m$.

If z is a point whose orbit eventually lies on $\mathbb{R}_+$ but does not include 0, we define the itinerary accordingly. Now we have itineraries for all points in the Julia set which do not eventually land on $\mathbb{R}_- \cup \{0\}$.

Note that points in $\partial \bigcup_n K_n$ have bounded orbits, and therefore the itinerary of any such point has the property that the set of integers j such that j or $-j$ occurs in the itinerary of z is bounded above. We call such itineraries *bounded itineraries.*

Theorem 6 *For any sequence s of non-zero integers, the set h_s of points in the Julia set which have itinerary s is either empty or homeomorphic to a ray $[y_s, \infty]$. If s is bounded, then the end-point (the point corresponding to y_s) lies on $\partial \bigcup_n K_n$ and it is the only point in $h_s \cap \partial \bigcup_n K_n$.*

Proof We will only give an outline of the proof and refer the reader to [1] and [4] for details. Firstly we notice that the partition of the plane that gives rise to the sets h_s is double covered by a partition of the plane which respects the dynamics of the map $r_{\lambda_m}(z) = \lambda_m sin(\pi z)$, $\lambda_m = \sqrt{\mu_m}$, and that it is sufficient to prove an analogous result for $r_{\lambda_m}(z)$. Our choice of parameter makes it possible to show that the set $A = \{z \ : \ |(r_{\lambda_m})'(z)| \leq 1\}$ lies within the union of the Fatou components of r_{λ_m} which meet the real line: it consists of topological disks A_j, $j \in \mathbb{Z}$, each containing exactly one critical point $j-1/2$. A calculation shows that each A_j maps under r_{λ_m} either into the disc A_m containing the fixed critical point λ_m or into the disc A_{-m} containing the fixed critical point $-\lambda_m$. It follows that each A_j lies in the basin of attraction of either λ_m or $-\lambda_m$. This result enables us to use exactly the same methods as in [1] and [4] to prove that for each itinerary the set of points with that itinerary is either empty or homeomorphic to a ray. We transfer this result to the $\mu_m R$-plane to get the required result.

Now suppose that $z \in \partial \bigcup_n K_n$ has itinerary s, so the set h_s is non-empty. Suppose that z does not lie on the boundary of the Fatou component $\mathcal{H}$ or any of its pre-images under $\mu_m R$. Then it lies in the intersection of a sequence of nested discs as described earlier. Any other point in $\partial \bigcup_n K_n$ with the same itinerary s lies in the same intersection of nested discs. But this consists of a single point, so z is the unique point in $\partial \bigcup_n K_n$ with itinerary s.

Suppose that $z \in \partial\mathcal{H}$. Recall that $\mathcal{H}$ is the filled Julia set of a quadratic-like restriction of $\mu_m R$ to a disc D (this restriction is pinched if $m = 1$). Thus z lies in the intersection of infinitely many inverse images of the cut disc $D - [0, \mu_m]$. Again, one can check that any other point in $\partial \bigcup_n K_n$ with itinerary s lies in the same intersection. For $m > 1$, the quadratic-like restriction is hybrid-equivalent to $z \to z^2$ which is expanding on a neighbourhood of its Julia set. Therefore z is the only point in $\partial \bigcup_n K_n$ with itinerary s. If $m = 1$ we still have a topological conjugacy between $z \to z^2$ and $\mu_m R$ on $\partial\mathcal{H}$, so the result holds in this case as well. This implies the result for points lying in an inverse image of $\partial\mathcal{H}$. $\square$

So far, we have ignored points whose orbits land on $\mathbb{R}_- \cup \{0\}$. Clearly these lie on a set of curves, the inverse images of $\mathbb{R}_-$, each with end-point an inverse image of 0. The half-line $\mathbb{R}_-$ and its pre-images form the boundary components of adjacent strips, so we can give such points several itineraries, leading to an identification of sequences in non-zero integers. We note that we have encountered the same identification before, namely in the parametrisation of points in $\hat{\mathbb{R}}_+$ by infinite sequences, given in section 1.

We call each curve h_s a *dynamical ray* (or *hair*) of $\mu_m R$ and note that the Julia set is the union of the dynamical rays. Note that two dynamical rays might share a common end-point. Moreover, it is not true that there exists a dynamical ray h_s for each itinerary s: each dynamical ray is the intersection of infinitely many inverse images of $\mathbb{C} - [0, \mu_m]$ and this intersection might consist only of the point at infinity. Let us call a sequence *realisable* if it has a non-empty dynamical ray associated to it. A precise condition for realisability is given in [10], namely that a sequence is realisable if and only if it is *exponentially bounded* (see [10] for a definition of this term).

Thus we have a picture of $J(\mu_m R)$ as consisting of a Cantor set of dynamical rays corresponding to realisable sequences, each ray leading from a point on the boundary of the Fatou set to the point at infinity. Restricting attention at first to *bounded* sequences, and recalling the bijection

$$\Psi : \bigcup_n \Lambda_+^n \to \bigcup_n K_n$$

of Fatou sets defined in Section 7, a close look at the action of $\mathcal{F}_m^{-1}$ on $\partial\Lambda_+$ and that of $\mu_m R(z)$ on $\bigcup_n \partial K_n$ shows that for each $z \in \bigcup_n \partial\Lambda_+^n$, and each of the inverse images x of z under ϕ, the point $\Psi(z) = w$ is the end-point of a dynamical ray with itinerary $s(x)$.

It might be tempting to extend Ψ to a map $\partial\Lambda_+ \to \partial(\bigcup_n K) \cup \{\infty\}$ by assigning to each $z \in \partial\Lambda_+$ the end-point of a dynamical ray with itinerary $s(x)$, where $x \in \phi^{-1}(z)$, if such a ray exists, and assigning to z the value infinity if there is no such ray. But such a Ψ could not be continuous as it would send a dense set in $\partial\Lambda_+$ to ∞. Points $z_n \in \partial\Lambda_+$ which accumulate at some point z can have images $\Psi(z_n)$ accumulating not at $\Psi(z)$, but along a ray that lands at $\Psi(z)$. The best we can say formally is that if we give the space of dynamical rays the topology associated to the sequences labelling them, then there is a homeomorphism between points in $\partial\Lambda_+$ with realisable sequences, and the quotient of the space of realisable rays obtained by identifying rays which land at the same point.

Informally, we think of the action of $\mathcal{F}_m^{-1}$ on Λ_+ as a "compactified" version of the action of $\mu_m R$ on $\mathbb{C}$. Each dynamical ray gets shrunk to a point and the point at infinity is split into infinitely many points, one for each empty dynamical ray, and one for each approach to ∞ through inverse images of the real line (corresponding to points in the orbit of P). The essential singularity at ∞ has been "blown up" and in its place we have glued an open disc carrying the action of the group Γ.

9 Remarks and Generalisations

Firstly we remark that each truncated map $\mu R|_n$ has the combinatorics of a restriction of a scalar multiple of the Chebyshev polynomial of odd degree $2n+1$. For results on how the filled Julia sets K_n of these polynomials converge as $n \to \infty$ see [8]. For further properties of the Fatou and Julia sets of maps $z \to \lambda \sin(z)$ see [5] and [10].

Secondly, we remark that in this paper we have only considered the simplest matings involving the group Γ, namely those for which the point $(2m-1)^2/4$ is fixed by $\mathcal{F}^{-1}$. This means that the restriction f_m^{-1} of $\mathcal{F}^{-1}$ is hybrid-equivalent to $z \to z^2$. However, using the methods of [7], one can prove that for each real c such that $z \to z^2 + c$ has a connected filled Julia set, there exists a correspondence $\mathcal{F}_{a(c)}$ depending on a real parameter $a(c)$, such that the restriction f_m^{-1} of $\mathcal{F}_{a(c)}^{-1}$ is hybrid-equivalent to $z \to z^2 + c$. Similarly, it can be shown that there exists a parameter μ_c such that the relevant restriction of $\mu_c R$ is hybrid-equivalent to $z \to z^2 + c$. Provided $z \to z^2 + c$ is hyperbolic, the methods used in this paper can be adapted to show that $\mathcal{F}_c$ represents a mating between Γ and $\mu_c R$. Note that the parameters μ we consider here lie in the limb $\mathcal{L}$ of the connectedness locus of the family μR in which 0 is a non-attracting fixed point at which a single dynamical ray lands.

In general, we conjecture that the moduli space $\mathcal{M}$ of correspondences in our family is the closure of a countable union of copies of the quadratic Mandelbrot set: any composition $f_{\epsilon_1} \circ \ldots \circ f_{\epsilon_n}$ maps D onto a set of discs and its inverse restricted to each disc is quadratic-like. We conjecture that for each such restriction and for each $z \to z^2 + c$ with connected Julia set there exists $a \in \mathcal{M}$ such that the relevant restriction of $\mathcal{F}_a$ is hybrid-equivalent to $z \to z^2 + c$. Thus we obtain a copy of the Mandelbrot set for each such restriction: in the language of [2] we have a copy of the Mandelbrot set for each *critical address*.

Using exactly the same reasoning, we expect that the limb $\mathcal{L}$ of the connectedness locus of the family $\{\mu R\}$ is the closure of a countable union of copies of the Mandelbrot set, and that $\mathcal{M}$ and $\mathcal{L}$ are homeomorphic (it is known that $\mathcal{L}$ does contain infinitely many copies of the Mandelbrot set lying along the real axis). We conjecture that for each μ in the interior of $\mathcal{L}$ the map μR can be mated with the group Γ by the corresponding $\mathcal{F}_a$ with a in the interior of $\mathcal{M}$.

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