Regular and Limit Sets for Holomorphic Correspondences

S Bullett and C Penrose,
School of Mathematical Sciences,
Queen Mary and Westfield College,
University of London,
Mile End Road, London E1 4NS, UK.

Abstract

Holomorphic correspondences are multivalued maps $f = \tilde{Q}_+\tilde{Q}_-^{-1} : Z \to W$ between Riemann surfaces Z and W, where $\tilde{Q}_-$ and $\tilde{Q}_+$ are (single-valued) holomorphic maps from another Riemann surface X onto Z and W respectively. When $Z = W$ one can iterate f forwards, backwards or globally (allowing arbitrarily many changes of direction from forwards to backwards and vice versa). Iterated holomorphic correspondences on the Riemann sphere display many of the features of the dynamics of Kleinian groups and rational maps, of which they are a generalisation. We lay the foundations for a systematic study of regular and limit sets for holomorphic correspondences, and prove theorems concerning the structure of these sets applicable to large classes of such correspondences.

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1 Introduction

A *holomorphic correspondence* is a multivalued map $f : Z \to W$ between Riemann surfaces Z and W, which has a factorisation $f = \tilde{Q}_+\tilde{Q}_-^{-1}$, where $\tilde{Q}_-$ and $\tilde{Q}_+$ are (single-valued) holomorphic maps from another Riemann surface X onto Z and W respectively. For example, when $Z = W = \bar{\mathbf{C}}$ (the Riemann sphere) any polynomial $P(z, w)$ determines a holomorphic correspondence $f : \bar{\mathbf{C}} \to \bar{\mathbf{C}}$ by $z \mapsto w \Leftrightarrow P(z, w) = 0$. Here X is the 'de-singularization' of the algebraic curve defined by the polynomial P. Conversely, any holomorphic correspondence $z \mapsto w$ on the Riemann sphere can be expressed in this form (by Chow's Theorem, see subsection 2.3).

When $Z = W$ one can iterate f forwards, backwards or globally (allowing arbitrarily many changes of direction from forwards to backwards and vice versa).

Iterated holomorphic correspondences [6, 7, 9] display many of the features of the dynamics of Kleinian groups and rational maps, of which they are a generalisation. Examples (plotted by computer) are illustrated in figures 2b,3b and 4c. In particular the Riemann sphere is partitioned into a 'regular set' and a 'limit set', though there are some choices to be made in definitions of these sets as we shall see. Below, we set out to lay the foundations for a systematic investigation of regular and limit sets for holomorphic correspondences. There are three distinct levels involved in this investigation: combinatorial, topological, and complex analytic. Wherever possible we try to isolate the most appropriate level for each concept that we introduce, but much of the richness of the study of algebraic functions arises from the subtle interplay between all three levels.

In Section 2 we consider relations on topological spaces in general and various notions of 'continuity' for multivalued maps, and we define what we mean by a *branched covering correspondence*. We next define operations combining correspondences and we introduce the idea of a 'diagram condition', an analogue of the notion of a 'relation' between generators of a group. The class of *separable* correspondences (those for which there are quotient maps $Q_+ : Z \to Y$ and $Q_- : W \to Y$ such that $f : z \to w$ if and only if $Q_+(z) = Q_-(w)$) is defined by a particularly simple 'diagram condition'. We conclude the section by introducing a class of correspondences we term 'Galois': for a rational function Q, the correspondence Gal^Q is that defined by sending each $z_0 \in \bar{\mathbf{C}}$ to the set of all other roots z of $Q(z) = Q(z_0)$, and its closure $\overline{\mathrm{Gal}}^Q$ is the correspondence obtained when we also allow z_0 to map to itself whenever $z = z_0$ is a repeated root. Galois correspondences are uninteresting from the dynamical point of view (all orbits are finite) but the class of correspondences which share the same diagram conditions as Galois correspondences plays a crucial role in our analysis later in the paper. We say that a correspondence f is 'off-separable' if there exists a homeomorphism M such that the join of f with M is separable. We show (Theorem 3) that any off-separable branched covering correspondence on a compact Hausdorff space can be written in the form $M \circ \overline{\mathrm{Gal}}^Q$. In particular any off-separable holomorphic correspondence on the Riemann sphere can be written in this form, with M Möbius and Q rational.

In Section 3 we are concerned with various notions of 'regular' and 'limit' sets for iterated holomorphic correspondences. In the case of a (non-elementary) Kleinian group G there are a number of equivalent definitions of the regular set (the domain where G acts properly discontinuously, the domain where the elements of G form a normal family etc.) and a number of equivalent definitions of the limit set. Similarly, in the case of a rational map there are a number of equivalent ways to define the 'Fatou' and 'Julia' sets. When we generalise these various definitions to correspondences we find they are no longer equivalent in general, but instead yield different notions of 'regular set' and 'limit set' suitable for different purposes. We concentrate our attention on one of these, the set $\Omega(f)$ which satisfies the strongest form of regularity, namely '$z_0 \in \Omega(f) \Leftrightarrow z_0$ has a neighbourhood U having only finitely many distinct returns under f^*'

(where f^* denotes the union of all forward, backward and mixed iterates of f). A more formal definition is given in the text. We call $\Omega(f)$ the 'regular set', and its complement $\Lambda(f)$ the 'non-regular limit set'. We prove that $\Omega(f)$ is completely invariant and that $\Omega(f)/f^*$ is Hausdorff, as is the case for the orbit space $\Omega(G)/G$ of the regular set of a Kleinian group. We also introduce the idea of the *equicontinuity set* for the branches of an iterated correspondence f, and we (tentatively) define the *Julia set* $\mathcal{J}(f)$ to be its complement.

It is a non-trivial problem to identify a 'fundamental domain' for the global iteration of f on $\Omega(f)$, in other words a transversal for the action of f^* on $\Omega(f)$, and much of the rest of the paper is concerned with resolving this problem for correspondences satisfying appropriate diagram conditions. In many situations our method also yields information concerning the various limits sets. In Section 4 we start our investigation by solving the easier problem of finding transversals for 'forward', 'backward' and 'bi-directional', rather than 'global', iteration.

In Section 5 we achieve our goal of constructing fundamental domains for global iteration, in the case that f is a reversible off-separable branched covering correspondence and hence of the form $f = J \circ \overline{\mathrm{Gal}}^Q$ (by results proved in Section 2). We define a (topological) directionality for a correspondence f on a topological space X to be a subspace S of X such that $f(\bar{S}) \subset S^\circ$ and we define a 'transversal directionality' for the (closed) correspondence $f = J \circ \overline{\mathrm{Gal}}^Q$ to be a (topological) directionality D for f which is also a transversal for Q. Various equivalent characterisations are listed in the text. We prove:

Theorem 7 *Let f be an $n:n$ J-off-separable branched covering correspondence on a compact Hausdorff space X, where $n \geq 2$ (almost everywhere) and J is an involution. If D is a transversal directionality for f then any transversal Δ for the action of J on $D \cap J(D)$ is a transversal for the (global) action of f on the complement $\Omega(f,D)$ of the global attractor $\omega(f,D) = \omega_+(f,D) \cup \omega_-(f,D)$ (where $\omega_+(f,D)$ denotes $\bigcap f^n(D)$ and $\omega_-(f,D)$ denotes $\bigcap f^{-n}(JD)$).*

If X is a Riemann surface and f is holomorphic then $\Omega(f,D)$ is contained in the regular set $\Omega(f)$ of f, and the various limit sets satisfy

$$\partial\omega(f,D) \subset \mathcal{J}(f) \subset \Lambda(f) \subset \omega(f,D) \ .$$

The existence of a transversal directionality D has consequences for the structure of the limit set of such a correspondence, as well as furnishing a transversal for the action on the subset $\Omega(f,D)$ of the regular set. If f is holomorphic and D is a topological disc, we can apply Douady and Hubbard polynomial-like mapping theory to show that under appropriate conditions the forward and backward 'limit sets' under bi-directional iteration, $\omega_+(f,D)$ and $\omega_-(f,D)$ respectively, are copies of filled Julia sets of polynomial maps. An example is plotted in figure 2b. Further examples of the class of 'reversible-off-separable' holomorphic correspondences are the *matings of quadratic maps with the modular group* discussed in [6]. These are a family of correspondences having actions which are conjugate on one part of the Riemann sphere to an action of $PSL(2,\mathbf{Z})$ and on other parts to forward and backward branches of maps of the form $q_c : z \mapsto z^2 + c$.

The definition of a 'J-off-separable holomorphic correspondence' might at first sight seem of rather limited applicability, chosen to capture the essential properties of the matings described above, but it turns out rather remarkably that the very much more natural class of all *separable* holomorphic correspondences fits naturally into the same framework. In Section 6 we introduce the strategy of replacing any given separable correspondence f, acting on a space X, by a new reversible-off-separable correspondence $\mathcal{F}$ which acts on a double cover of X and has as its bi-directional orbits the grand orbits of f. We then apply the methods and results of Section 5 to identify 'fundamental domains' for $\mathcal{F}$, and hence for f, and to show that in appropriate situations the limit set of f is an infinite unions of copies of polynomial Julia sets, as illustrated in figure 3b (and in [7]). In the special case that f is reversible we adapt the construction to obtain a reversible off-separable $\mathcal{F}$ acting on X itself rather than the double cover (an example is illustrated in figure 4c). Further examples to which the theory applies are the deformations of circle-packing representations of $C_2 * C_4$ considered in [9].

In a sequel to the present article we shall consider the conformal structure of limit sets of correspondences, in particular identifying conditions under which these limit sets are made up of copies of filled Julia sets of polynomial functions. The techniques involved for this analysis include Douady and Hubbard polynomial-like mapping theory and the combinatorial and metric properties of Yoccoz puzzle-pieces.

The subject of iterated correspondences is one in which computer experimentation and abstract theory have developed in parallel. For computer plotted illustrations of the results proved below, the reader is invited to look at the authors' paper [7] (which also contains the first announcements of some of these results).

The dynamical theory of holomorphic correspondences may be viewed as bridging the gap between the theory of Kleinian groups and that of rational maps, or perhaps more accurately it should be viewed as a third area of holomorphic dynamics needing considerable development to catch up with the first two. Sullivan very successfully applied methods and insight from Kleinian group theory to prove major results about rational maps, and vice versa [31], [32],[33] and his 'dictionary' between the two areas has been much expanded by McMullen and others [21], [22], [23]. The existence of this 'dictionary' has been a major motivation for our work, and our long-term objective is to extend it to include holomorphic correspondences.

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2 Relations and multi-valued maps

A *relation* between two spaces Z and W, that is a subset of $Z \times W$, determines a *multi-valued map*, or *correspondence* f between Z and W as follows. We say $f : z \mapsto w$ if and only if (z, w) lies in a prescribed subset, which we shall also denote by f, of $Z \times W$. Clearly the order of Z and W is important. The transpose relation, a subset of $W \times Z$, induces the *inverse* multi-valued map f^{-1} between W and Z. Thus

$$f^{-1} : w \mapsto z \;\Leftrightarrow\; (w, z) \in f^{-1} \;\Leftrightarrow\; (z, w) \in f \ .$$

We note the coordinate projections $\pi_- = \pi_-^{(Z,W)} : Z \times W \to Z$ (the *backward projection*) and $\pi_+ = \pi_+^{(Z,W)} : Z \times W \to W$ (the *forward projection*). When restricted to the subset f of $Z \times W$ these give an alternative description of the multi-valued maps:

$$\begin{aligned} f &= \pi_+ \circ (\pi_-|_f)^{-1} \ , \\ f^{-1} &= \pi_- \circ (\pi_+|_f)^{-1} \ . \end{aligned}$$

The image of a point z under f will be a subset of W:

$$f(z) := \{w \in W : (z, w) \in f\}$$

and the image of a subset S of Z will be

$$f(S) := \bigcup_{z \in S} f(z) = \pi_+ \left((\pi_-|_f)^{-1}(S)\right) = \pi_+ \left(\pi_-^{-1}(S) \cap f\right) \ .$$

Likewise, the *inverse-image* of a subset T of W will be

$$f^{-1}(T) := \pi_- \left((\pi_+|_f)^{-1}(T)\right) = \pi_- \left(\pi_+^{-1}(T) \cap f\right) \ .$$

Note that, for $T \subset W$, the set $f^{-1}(T)$ in our notation is the same as McGehee's $f^*(T)$ [20] and is different from his "$f^{-1}(T)$" which denotes only the set of points *all* of whose images under f lie in T. This latter set we shall write $f^{-McG}(T)$. As McGehee observes [20], there is a connection:

$$f^{-McG}(T) = (f^{-1}(T^c))^c \ .$$

For subsets T of W, we can generally only say

$$f\left(f^{-McG}(T)\right) \subset T \ .$$

However, we have equality precisely when T is an *image* set (of f), that is a set of the form $f(S)$ for some $S \subset Z$. Likewise, for subsets S of Z, we can generally only say

$$S \subset f^{-McG}(f(S)) \ .$$

This time we have equality precisely when S is a *co-image* set (of f^{-1}), that is a set of the form $(f^{-1}(T))^c$ for some $T \subset W$. Every set $S \subset Z$ naturally enlarges to a co-image - namely $f^{-McG}(f(S))$.

For the record we note that f is *symmetric* if $Z = W$ and $f^{-1} = f$.

We shall say that f is *symmetrizable* if there exists an identification of Z and W whereby f becomes symmetric, in other words, a bijection $M : Z \to W$ satisfying $f^{-1} \circ M = M^{-1} \circ f$. We shall say that f is *reversible* if $W = Z$, f is symmetrizable, and the automorphism $M : Z \to Z$ symmetrizing f is an *involution*.

2.1 Topological considerations

Given a correspondence (multi-valued map) f between topological spaces Z and W (defined by its graph $f \subset Z \times W$) we list some possible properties of f at a point $z \in Z$ which are needed to replace the now defunct concept of "continuity" which was the general working assumption in the case of f single-valued.

DEFINITIONS: We say f is *lower-semi-continuous* at z if, for each $w \in f\{z\}$ and each neighbourhood V of w there exists a neighbourhood U of z such that for all $z' \in U$ there exists a $w' \in V$ such that $w' \in f\{z'\}$. We say f is *upper-semi-continuous* at z if, for each neighbourhood V of $f\{z\}$, there is a neighbourhood U of z such that $f(U) \subset V$.

Note that

(i) f is lower-semi-continuous $\Leftrightarrow$ f^{-1} maps open sets to open sets;

(ii) f is upper-semi-continuous $\Leftrightarrow$ f^{-McG} maps open sets to open sets, in other words $\Leftrightarrow$ f^{-1} maps closed sets to closed sets.

DEFINITION: We say that f is a *closed relation* if the graph f is closed as a subset of $Z \times W$ (equipped with the product topology).

It is easily proved that when f is a relation between Hausdorff spaces Z and W then

(iii) f closed (as a relation) $\Rightarrow$ $f(K)$ closed for any compact subset K of Z (and in particular for K a single point);

(iv) f closed (as a relation) and W compact $\Rightarrow$ $\pi_-|_f$ is both proper and closed as a map - that is the pre-image of a compact set is compact and the image of a closed set is closed.

It follows from (ii) and (iii) that

(v) a closed relation f between Hausdorff Z and compact Hausdorff W is upper-semi-continuous.

DEFINITION: We say that $f(\subset Z \times W)$ is an *open relation* if the projections $\pi_-|_f$ and $\pi_+|_f$ are open maps, that is they project open subsets of f to open sets (in Z, W respectively).

From (i) we note that

(vi) if f is an open relation then both f and f^{-1} are lower-semi-continuous.

The time will come when we shall have to consider relations with graphs which are not closed, for example the relation obtained from a given holomorphic correspondence by removing some or all of the singular points of the graph, or that obtained from a holomorphic correspondence by taking the union of all its iterates. A useful construction then will be to consider the relation $\bar{f}$ which has graph the closure, in $Z \times W$, of the graph of f. The proposition and corollary below provide useful information about the relationship between $\bar{f}(S)$ and $\overline{f(S)}$ for a subset S of Z. The preliminary lemma is elementary.

Lemma 1 *If R is an open subset of $Z \times W$ containing the product of compact sets $K \times L$ then there exist open sets U in Z and V in W such that $K \times L \subset U \times V \subset R$.* □

Proposition 1 *If f is a relation between spaces Z and W and $S \subset Z$ then $\bar{f}(S) \subseteq \bigcap_{U^\circ \supset S} \overline{f(U)}$ and equality holds whenever S is compact.*

Proof. We show the contra-positive, namely that given $w \in W$, if there exists U, a neighbourhood of S, such that $\overline{f(U)}$ does not contain w then $w \notin \bar{f}(S)$ (and vice-versa in the case S is compact).

$$(\exists U^\circ \supset S)(w \notin \overline{f(U)}) \;\Leftrightarrow\; (\exists U^\circ \supset S, \exists V^\circ \ni w)((U \times V) \cap f = \emptyset)$$

$$(\Leftarrow) \Rightarrow\; (S \times \{w\}) \cap \bar{f} = \emptyset \;\Leftrightarrow\; w \notin \bar{f}(S)\ ,$$

(using the above Lemma for the reverse implication when S is compact). □

Corollary 1 *If U is open then $\bar{f}(U) \subset \overline{f(U)}$. If S is compact then $\bar{f}(S) \supset \overline{f(S)}$. In particular if $z \in Z$ then $\bar{f}\{z\} \supset \overline{f\{z\}}$.* □

2.2 Singular and non-singular points: branched-covering correspondences

DEFINITION: We say that a point $(z, w) \in f$ is *forward non-singular* (or that the *arrow* $f : z \to w$ is *non-singular*) if there exists a neighbourhood U of (z, w) in f such that $\pi_-|_U$ is a homeomorphism onto its image (a subset of Z). The (single-valued) composite $\pi_+ \circ (\pi_-|_U)^{-1}$ is then called a *branch* of f at z.

Analogously $(z, w) \in f$ is called *backward non-singular* if π_+ is locally a homeomorphism onto its image (a subset of W).

We shall also refer to a point $z \in Z$ as *forward non-singular*, meaning that for all w in $f\{z\}$ the point (z, w) of the graph is forward non-singular, or refer to a point $w \in W$ as *backward non-singular*, with the obvious analogous meaning.

EXAMPLE: When $f \subset \mathbf{C} \times \mathbf{C}$ is an algebraic curve, defined by a polynomial equation $P(z,w) = 0$, the arrow $f : z \to w$ is singular if and only if (z,w) is a point where $\partial P/\partial w = 0$, and the arrow $f^{-1} : w \to z$ is singular if and only if (z,w) is a point where $\partial P/\partial z = 0$. The 'singular points' in the sense of algebraic curves ([15]) are those points (z,w) which are singular for both f and f^{-1}.

We can generalize the notion of a 'branched-covering' map to the context of relations:

DEFINITION: We say that a relation f between locally-compact spaces Z and W is a *branched-covering correspondence* if it satisfies the following conditions:
(i) f is open and closed (as a relation);
(ii) $f\{z\}$ and $f^{-1}\{w\}$ are discrete for all $z \in Z$ and $w \in W$;
(iii) the forward singular points z of f are isolated (in Z), and the backward singular points w of f are isolated (in W).

In particular, if f is a branched-covering correspondence between compact Hausdorff Z and W then f is both lower and upper semi-continuous, f must have at most finitely many forward and backward singularities, and each $f\{z\}$ and $f^{-1}\{w\}$ must be finite. Moreover it follows at once from the local continuity of branches at non-singular points that $z \to \#(f\{z\})$ is locally constant at points z which are non-singular for f, and $w \to \#(f^{-1}\{w\})$ is locally constant at points w which are non-singular for f^{-1}. We say that f is an $m : n$ correspondence if all non-singular z have n images and all non-singular w have m inverse images.

2.3 Holomorphic correspondences, algebraic varieties and de-singularization

We are mainly interested in relations $f \subset Z \times W$ where Z and W are geometric manifolds (e.g. Riemann surfaces such as the Riemann sphere $\bar{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$) and f is a relation preserving some geometric (e.g. conformal) structure. Any polynomial equation of the form $P(z,w) = 0$ determines such a relation on the Riemann sphere. Here z and w are initially complex numbers, and $P(z,w) = 0$ initially defines an algebraic curve in $\mathbf{C} \times \mathbf{C}$ but by completing using homogeneous co-ordinates we obtain a subvariety of $\bar{\mathbf{C}} \times \bar{\mathbf{C}}$. Note that there are two different ways to 'complete' the curve $P(z,w) = 0$. The first (which is the standard construction in the study of algebraic curves) is to transform P into a homogeneous polynomial $P(z,w,t) = 0$ which can then be regarded as defining a curve in the complex projective plane $\mathbf{C}P^2$. The second is to introduce four variables z_0, z_1, w_0, w_1 homogenizing z and w separately via $z = z_0/z_1$ and $w = w_0/w_1$ and hence obtain a polynomial $P(z_0, z_1, w_0, w_1) = 0$ which we may regard as a subvariety of $\mathbf{C}P^1 \times \mathbf{C}P^1 = \bar{\mathbf{C}} \times \bar{\mathbf{C}}$. It is this second completion which we use to view the equation $P(z,w) = 0$ as the graph of a relation f on the Riemann sphere.

If $P(z,w)$ is an irreducible polynomial of degree $m \geq 1$ in z and degree $n \geq 1$ in w, then $P(z,w) = 0$ determines an $m : n$ branched-covering correspondence f

on the Riemann sphere in the sense of the previous subsection. The projection maps π_- and π_+ from the graph of f are branched coverings in the usual sense for maps, except at points $(z,w) \in f$ which correspond to arrows $f : z \to w$ and $f^{-1} : w \to z$ which are both singular. To cope with such points in f we appeal to a theory of *de-singularization* which constructs for $f \subset Z \times W$ a manifold cover, which we shall write $\Gamma(f)$, and branched covering maps

$$\tilde{Q}_-^{(f)} : \Gamma(f) \to Z \ ,$$

$$\tilde{Q}_+^{(f)} : \Gamma(f) \to W$$

such that the product map $\tilde{Q}_\times^{(f)} = \tilde{Q}_- \times \tilde{Q}_+ : \ \Gamma(f) \to Z \times W$ (given by $\tilde{Q}_\times(x) = (\tilde{Q}_-(x), \tilde{Q}_+(x))$ for $x \in \Gamma(f)$) has image-set the branched covering correspondence f and is one-to-one except over certain of the points (z,w) which are singular for both f and f^{-1}. See for example [14] and [15] for elementary accounts of the theory of Riemann surfaces (in particular in Chapter 7 of [15] there is an explanation of how to de-singularize an algebraic curve $P(z,w) = 0$).

More generally, f is a *singular Riemann surface* if it has a de-singularization whereby $\Gamma(f)$ is a Riemann surface and the above maps $\tilde{Q}_-$ and $\tilde{Q}_+$ are holomorphic branched coverings.

DEFINITION: A *holomorphic correspondence* is a multivalued map $f : Z \to W$ between compact Riemann surfaces Z and W, which has a factorisation $f = \tilde{Q}_+\tilde{Q}_-^{-1}$, where $\tilde{Q}_+$ and $\tilde{Q}_-$ are (single-valued) holomorphic maps from some Riemann surface X onto Z and W respectively.

Theorem 1 *If $f : Z \to W$ is a holomorphic correspondence of compact Riemann surfaces, determined by holomorphic maps $\tilde{Q}_- : X \to Z$ and $\tilde{Q}_+ : X \to W$ then the graph $f \subset Z \times W$ is a (singular) Riemann surface.*

Proof. Let $(z,w) \in f \subset Z \times W$ and suppose firstly that z is not a critical value of $\tilde{Q}_-$. Let U be a disc neighbourhood of z over which the fibre bundle $\tilde{Q}_-^{-1}(U) \to U$ is trivial. Then $f \cap (U \times W)$ is the union of the graphs of $deg(\tilde{Q}_-)$ functions from U to W. Some of these graphs may be identical, and each of them is the bijective image of a disc in X. By choosing U sufficiently small we may arrange that any two graphs in the collection which pass through (z,w) are either identical or intersect only at (z,w), since if two of the graphs intersect at points arbitrarily close to (z,w) the corresponding functions $U \to W$ agree at a sequence of points $\{z_n\}$ converging to z and being analytic are therefore identical (by a standard result of complex analysis, Theorem 1.35 of [24]).

The case that w is not a critical value of $\tilde{Q}_+$ is analogous, so we just have to deal with the (finite) set of points $(z,w) \in f$ for which both z and w are critical values. Given one of these points (z,w), let $U' = U \setminus \{z\}$ be a punctured disc neighbourhood of z in Z, small enough to avoid all other critical values of $\tilde{Q}_-$. De-singularize $f \cap (U' \times W)$ at any multiple points, that is to say points where two sheets intersect (as described in the previous paragraph), by replacing each with a separate point on each sheet. Each component of the de-singularized graph is a punctured disc, since it evenly covers the punctured disc U' (by the first part of the proof). Filling in the puncture point in each component makes it into a disc, which is a quotient of a disc in X by a (finite cyclic) covering group and is also the graph of a multivalued function $U \to W$ defined by a 'Puiseux expansion' (see Proposition 4 in Section 2.4). Finally observe that by choosing U to be sufficiently small we may again arrange that the sheets of the graph which pass through (z, w) intersect only at (z, w), since Puiseux expansions agreeing on a sequence of points converging to z define identical multivalued functions. $\square$

Note that the de-singularization of f is X except when $\tilde{Q}_-$ and $\tilde{Q}_+$ have a common right factor, in which case it is the quotient of X by the highest such factor. An extreme example is when $Z = W$ and $\tilde{Q}_- = \tilde{Q}_+$, when the graph f is the diagonal in $Z \times W$ whatever the Riemann surface X.

It follows from Theorem 1 that if f is a holomorphic correspondence between compact Riemann surfaces Z and W then without loss of generality we may take its X to be the de-singularized graph $\Gamma(f)$ of f. This Riemann surface is a disjoint union of finitely many compact connected Riemann surfaces and the 'projection' maps $\tilde{Q}_-$ and $\tilde{Q}_+$ are holomorphic branched coverings. We deduce that f is a branched-covering correspondence in the sense of the previous subsection. Moreover if Z and W are connected the projection maps will have well-defined degrees m and n respectively, and f will be an $m : n$ correspondence. We say that $P(z, w)$ is a *non-degenerate* polynomial if every irreducible factor is of degree at least one in both z and w. If P is a non-degenerate and square-free polynomial, then $P(z, w) = 0$ defines an $m : n$ holomorphic correspondence on the Riemann sphere, where m and n are the degrees of P in z and w respectively. Such a P is represented by a $(m + 1) \times (n + 1)$ matrix of coefficients and the correspondence is determined by the projective matrix $[P]$:

$$\underline{z}\,[P]\,\underline{w}^t = 0$$

where $\underline{z} = [z_0^m, z_0^{m-1}z_1, \ldots, z_0 z_1^{m-1}, z_1^m]$ and $\underline{w} = [w_0^n, w_0^{n-1}w_1, \ldots, w_0 w_1^{n-1}, w_1^n]$ in homogeneous coordinates. Conversely:

Theorem 2 *Every holomorphic correspondence on the Riemann sphere is algebraic, that is to say can be expressed in the form $P(z, w) = 0$ for some polynomial P in two variables.*

Proof. This follows from Chow's Theorem [10], [30] that every compact complex analytic projective variety is algebraic. Chow's Theorem concerns subvarieties of complex projective space $\mathbf{C}P^n$. However the product of two Riemann spheres $\mathbf{C}P^1 \times \mathbf{C}P^1$ embeds in $\mathbf{C}P^3$ (in homogeneous co-ordinates the map being $([z_0, z_1], [w_0, w_1]) \to [z_0 w_1, z_1 w_0, z_1 w_1, z_0 w_0]$) and so a codimension one subvariety of $\mathbf{C}P^1 \times \mathbf{C}P^1$ can be regarded as a codimension two subvariety of $\mathbf{C}P^3$. $\square$

For a holomorphic correspondence on the Riemann sphere it is a straightforward algebraic exercise to compute the singular points.

EXAMPLE: Consider the $2:2$ holomorphic correspondence $f : z \to w$ defined on the Riemann sphere by $P(z,w) = 0$ where

$$P(z,w) = zw^2 - (z-w)^2$$

The forward singular points of this correspondence are the values of z which have a single image w. To compute these we regard $P(z,w)$ as a polynomial in w and write down the condition on its coefficients that it have a repeated root:

$$z^2 = (1-z)z^2$$

Thus the forward singular points are $z = 0$ and $z = \infty$. All points z other than these have two distinct images under f. To find the backward singular points we regard $P(z,w)$ as a polynomial in z and observe that the condition for a repeated root yields $w = 0$ and $w = -4$. All points w other than these have two distinct inverse images. Thus the 'singular arrows' (z,w) are: (∞,∞), which is singular for f; $(4,-4)$, which is singular for f^{-1}; and $(0,0)$, which is singular for both f and f^{-1}.

We compute the maximal number of singular points for a general $m : n$ holomorphic correspondence f on the Riemann sphere:

Proposition 2 *An $m:n$ holomorphic correspondence on the Riemann sphere has at most $2(m-1)n$ backward singular points w and at most $2(n-1)m$ forward singular points z.*

Proof. Let the correspondence be defined by $P(z,w) = 0$. Write this equation as a polynomial in w:

$$(a_{00} + a_{10}z + \ldots + a_{m0}z^m) + \ldots + (a_{0n} + a_{1n}z + \ldots + a_{mn}z^m)w^n = 0$$

Partial differentiation with respect to z yields:

$$(a_{10} + \ldots + ma_{m0}z^{m-1}) + \ldots + (a_{1n} + \ldots + ma_{mn}z^{m-1})w^n = 0$$

A point w is backward singular if and only if for this value of w the first equation above has a repeated root z, and hence if and only if the two equations above have a common solution z. But we can eliminate the terms in z^m in the first equation (by multiplying the first equation by m and then subtracting z times the second equation) and now express the condition that the two equations have a common root z as the vanishing of the determinant of a $2(m-1)$ by $2(m-1)$ matrix of polynomials of degree n in w (the *resultant* of the two equations [15]). This determinant is a polynomial of degree $2(m-1)n$ in w. The same reasoning, but for f^{-1} in place of f, counts the forward singular points. □

Corollary 2 *If $P(z,w)$ is an irreducible polynomial of degree m in z and n in w, then the graph f of the holomorphic correspondence defined on the Riemann sphere by $P(z,w) = 0$ has genus at most $(m-1)(n-1)$.*

Proof. The graph is an m-fold branched cover of the sphere, branched over the (at most $2(m-1)n$) backward singular points. Hence f has Euler characteristic at least $2m - 2(m-1)n$. Resolving any multiple points only further increases Euler characteristic, and so the de-singularized graph $\Gamma(f)$ also has Euler characteristic at least $2m - 2(m-1)n = 2 - 2(m-1)(n-1)$. Hence, by the Riemann-Hurwitz formula, the genus of f is at most $(m-1)(n-1)$. $\square$

Remarks

1. By taking into account the exact number of backward critical points, and the number of points added in passing from the graph f to its de-singularization $\Gamma(f)$ (both of which will depend on the types of the doubly singular arrows $z \to w$) one can compute the exact value of the genus of f rather than simply an upper bound.

2. As already mentioned earlier in this subsection, the Riemann surface $f \subset \mathbf{C}P^1 \times \mathbf{C}P^1$ should not be confused with the subvariety of $\mathbf{C}P^2$ defined by the equation $P(z, w) = 0$ (where P is now regarded as a homogeneous polynomial in three variables in the obvious way). This latter variety is also a Riemann surface, and is a completion of the same algebraic curve in $\mathbf{C} \times \mathbf{C}$, but has genus $\leq (d-1)(d-2)/2$ where d is the degree of P as a homogeneous polynomial, the exact value being given by *Noether's formula* [15].

2.4 Local normal forms for holomorphic correspondences: simultaneous normalization, Puiseux expansions

We first consider the question of local normal forms for holomorphic maps between Riemann surfaces.

Given a Riemann surface X and a point $x \in X$, a chart map *at* x is a conformal homeomorphism δ_x from a neighbourhood of x to a neighbourhood of the origin in the complex plane $\mathbf{C}$, satisfying $\delta_x(x) = 0$. Every chart map (defined on a neighbourhood of x) can be normalised - e.g. by translation - to a chart map *at* x. Thus, fixing such a chart, by abuse of notation, we may write nearby points x' uniquely as "$x + \delta x$" where $\delta x = \delta_x(x')$. Thus

$$\text{“ } x + \delta x \text{ ”} \;=\; {\delta_x}^{-1}(\delta x)\ .$$

In the case $X = \mathbf{C}$ (and δ_x is purely the translation $x' \mapsto x' - x$) this agrees with the standard notation.

Recall that if Q is a holomorphic map between Riemann surfaces X and Y then given any $x \in X$ and $y \in Y$ satisfying $Q(x) = y$ and given chart maps δ_x and δ_y (at x and at y respectively) there exists a Taylor expansion

$$\delta y = \sum_{k \geq 1} \alpha_k \, (\delta x)^k$$

with positive radius of convergence and coefficients $\alpha_k \in \mathbf{C}$ such that for all $x' \in X$ with $|\delta_x(x')|$ sufficiently small, the value $\delta_y(y')$ for the point $y' = Q(x')$

satisfies

$$\delta_y(y') = \sum_{k\geq 1} \alpha_k \, (\delta_x(x'))^k \ .$$

The smallest k for which $\alpha_k \neq 0$ is independent of the chart maps δ_y and δ_x and is called the *local degree* of Q at x, written $\deg_x Q$.

Proposition 3 *If $Q : X \to Y$ is a holomorphic map of Riemann surfaces and has local degree d at a point $x \in X$ then for any chart map δ_y at $y = Q(x)$ there exists a chart map δ_x at x such that the Taylor expansion of Q about x becomes*

$$\delta y = (\delta x)^d \ .$$

Proof. We simply lift the chart map δ_y to the branched coverings corresponding to Q and to $x \to x^d$ on its domain and range respectively.

The reverse proposition may be false for $d > 1$: given a chart δ_x at x there may be *no* chart δ_y at y which gives the Taylor series for Q as $\delta y = (\delta x)^d$. For instance if $d = 2$ then we can modify a chart δ_y so that the expansion for Q becomes $\delta y = (\delta x)^2$ *if and only if*, for the original chart δ_y, the Taylor series of Q:

$$\delta y = \sum_{k\geq 2} \alpha_k \, (\delta x)^k$$

is an *even function* of δx, that is $\alpha_k = 0$ for all odd k. The criterion is that $Q(x + \delta x) = Q(x - \delta x)$ for all small δx.

We now move on to the question of local normal forms for holomorphic correspondences. Suppose $f : Z \to W$ is a holomorphic correspondence defined by holomorphic maps of Riemann surfaces $\tilde{Q}_- : X \to Z$ and $\tilde{Q}_+ : X \to W$. Without loss of generality we may take X to be the de-singularization $\Gamma(f)$ of the graph $f \subset Z \times W$ and $\tilde{Q}_-$ and $\tilde{Q}_+$ to be its projections onto Z and W. Suppose we are given a point $x \in X$ at which $\tilde{Q}_-$ has local degree n and at which $\tilde{Q}_+$ has local degree m. Given any chart δ_z at $z = \tilde{Q}_-(x)$ we can 'normalize' $\tilde{Q}_-$ by appropriate choice of chart δ_x (as in the above proposition). Likewise given any chart δ_w at $w = \tilde{Q}_+(x)$ we can 'normalize' $\tilde{Q}_+$ by an appropriate choice of chart δ_x.

DEFINITION: We say that $\tilde{Q}_-$ and $\tilde{Q}_+$ can be *simultaneously normalized* if there exist charts δ_x, δ_z, δ_w such that

$$\delta z \ = \ (\delta x)^n, \ \delta w \ = \ (\delta x)^m \ .$$

Clearly 'simultaneous normalization' is always possible if either $m = 1$ or $n = 1$. More generally, if the local degrees m and n of $\tilde{Q}_-$ and $\tilde{Q}_+$ at $x \in X$ are coprime, $\tilde{Q}_-$ and $\tilde{Q}_+$ can be simultaneously normalized at x if and only if the (local) covering transformations of $\tilde{Q}_-$ commute with those of $\tilde{Q}_+$.

COMMENTS:
1. 'Simultaneous normalization' is impossible if m and n have a common factor since otherwise δz and δw would both be powers of some non-trivial power of δx whence the supposed de-singularizing map

$$\tilde{Q}_\times(x+\delta x) = (z+(\delta x)^n, w+(\delta x)^m)$$

would not be one-to-one at any point in a neighbourhood of x (except possibly at the point x itself) - contrary to the fact that multiple points are isolated for a holomorphic correspondence.
2. 'Simultaneous normalizability' is equivalent to 'local separability': if $\tilde{Q}_- : U_x \to V_z$ and $\tilde{Q}_+ : U_x \to V_w$ are simultaneously normalizable on the neighbourhood U_x of x we can construct pushout maps $R : V_z \to W$ and $S : V_w \to W$ such that $\tilde{Q}_+\tilde{Q}_-^{-1}$ maps $z' \in V_z$ maps to $w' \in V_w$ if and only if $R(z') = S(w')$. Conversely if there exist such R and S then $\tilde{Q}_-$ and $\tilde{Q}_+$ are simultaneously normalizable in U_x. We shall have much more to say about separability later.

'Simultaneous normalizability' gives us a canonical local form for a correspondence. If $(z,w) \in f$ lifts to a unique point $x \in \Gamma(f)$, and if x is a point at which the projection maps $\tilde{Q}_-$ and $\tilde{Q}_+$ (to Z and W respectively) are 'simultaneously normalizable', then we can find charts $\delta z, \delta w$ around $z \in Z$ and $w \in W$ in which f has the local form

$$\delta w = (\delta z)^{m/n}$$

More generally, if $(z,w) \in f$ lifts to several points $x_1, .., x_l$ of $\Gamma(f)$ (that is, if (z,w) is a multiple point of the graph f), then if the projection maps $\tilde{Q}_-$ and $\tilde{Q}_+$ are simultaneously normalizable at each x_j we have a local form

$$\delta_j w = (\delta_j z)^{m_j/n_j}$$

for each sheet of the graph. In general we cannot expect to find a single pair of charts $\delta w, \delta z$ that will give this form simultaneously for all sheets.

When we do not have 'simultaneous normalization' the most useful general tool available to us is that of 'Puiseux expansions'. Consider the correspondence $P(z,w) = 0$ and suppose (without loss of generality) that the singular point we wish to analyse has coordinates $z = 0, w = 0$. A proof of the following proposition can be found (for example) in [15].

Proposition 4 *There exist holomorphic functions ('Puiseux expansions')*

$$g_j(t) = \sum_{r>0} a_r^{(j)} t^r$$

for $1 \le j \le l$, defined near 0, and positive integers $m_1, ..., m_l$, such that the curve $P(z,w) = 0$ is given in a neighbourhood of the point $(0,0) \in \bar{\mathbf{C}} \times \bar{\mathbf{C}}$ by

$$\bigcup_{1\le j\le l} \bigcup_{1\le s\le m_j} \{(z,w) : w = g_j(e^{2\pi i s/m_j} z^{1/m_j})\} \qquad \square$$

This is exactly what we would expect in view of our discussion earlier in this subsection: the chart δ_z lifts to a chart δ_z^{1/m_j} on the jth of the l intersecting sheets of the graph at $(z,w) \in f$, and the projection down to δ_w is a holomorphic function. Thus restricted to any sheet of f we can write our correspondence locally in the form

$$w = \sum_{r>0} a_r^{(j)} z^{r/m_j}$$

without even altering the charts at either end.

2.5 Separable correspondences

We say that a relation f, from Z to W, is *separable* or that the graph f is 'rectangle-complete', if

$$(z,w),(z',w),(z,w') \in f \;\Rightarrow\; (z',w') \in f \;.$$

When f is separable we get induced equivalence relations $\sim_+$ on Z and $\sim_-$ on W defined by

$$z \sim_+ z' \;\Leftrightarrow\; \exists w \in W, (z,w),(z',w) \in f \text{ (or } z = z') \;,$$

$$w \sim_- w' \;\Leftrightarrow\; \exists z \in Z, (z,w),(z,w') \in f \text{ (or } w = w') \;.$$

There is a natural identification of the quotient spaces $\pi_-(f)/\sim_+$ and $\pi_+(f)/\sim_-$, and so f can be written as $\{(z,w) \in Z \times W : Q_+(z) = Q_-(w)\}$ where $Q_+ : Z \to Z/\sim_+$, $Q_- : W \to W/\sim_-$ are the relevant quotient maps (figure 1a). Here the images $Z/\sim_+$ and $W/\sim_-$ are embedded in a common space Y such that $Y = Z/\sim_+ \cup W/\sim_-$ and $Z/\sim_+ \cap W/\sim_- = \pi_-(f)/\sim_+ = \pi_+(f)/\sim_-$. The space Y is thus the *push-out* of the maps $\pi_-|_f$ and $\pi_+|_f$ and will be called the *separating space* of f.

Conversely, given a space Y and maps $Q_+ : Z \to Y$ and $Q_- : W \to Y$, we can obtain a separable relation f between Z and W whose graph is $\{(z,w) \in Z \times W : Q_+(z) = Q_-(w)\}$. Assuming that $Y = image(Q_+) \cup image(Q_-)$ we see that Y is the separating space of f if and only if Q_+ is one-to-one outside $Q_+^{-1}(image(Q_-))$ and Q_- is one-to-one outside $Q_-^{-1}(image(Q_+))$ and otherwise Y is a quotient of the separating space of f.

If Z,W are compact Hausdorff and f is closed then separability of f implies that Q_+, Q_- are closed maps and that the separating space Y is Hausdorff. Conversely, if $Q_+ : Z \to Y$ and $Q_- : W \to Y$ are continuous maps to a Hausdorff space Y then the resulting separable correspondence between Z and W is closed.

In the case when Z and W are Riemann spheres, separability of a holomorphic correspondence f is equivalent to 'separation of variables' i.e. the correspondence can be written in the form $Q_+(z) = Q_-(w)$ where Q_+ and Q_- are rational maps of degrees d_+ and d_- respectively (here of course the separating space Y is also a Riemann sphere) and hence the polynomial relation $P(z,w)=0$ defining the correspondence can be expressed in the form $p_+(z)q_-(w)=q_+(z)p_-(w)$. Examples of separable correspondences are $d:1$ correspondences (rational maps) and $1:d$ correspondences (their inverses).

Given holomorphic maps Q_- and Q_+ of Riemann surfaces and an identification of their *domains* (figure 1b) the composite $Q_+\circ Q_-^{-1}$ is a holomorphic correspondence (by definition). We now check that the (separable) composite $Q_-^{-1}\circ Q_+$ obtained by *co-domain* identification (figure 1a) is also a holomorphic correspondence.

Proposition 5 *If Z, W and Y are Riemann surfaces and $Q_+:Z\to Y$, $Q_-:W\to Y$ are holomorphic maps then the (singular) Riemann surface*

$$Q_-^{-1}\circ Q_+ \ := \ \{(z,w)\in Z\times W : Q_+(z)=Q_-(w)\}$$

has a de-singularization $\tilde{Q}_\times:\Gamma(Q_-^{-1}\circ Q_+)\to Q_-^{-1}\circ Q_+$ such that $\tilde{Q}_-=\pi_-\circ\tilde{Q}_\times$ and $\tilde{Q}_+=\pi_+\circ\tilde{Q}_\times$ are holomorphic. Thus $Q_-^{-1}\circ Q_+$ is a holomorphic correspondence.

Proof. Given a point $(z,w)\in Q_-^{-1}\circ Q_+$, where $\deg_z Q_+=m$, $\deg_w Q_-=n$, and given a chart map δ_y at $y=Q_+(z)=Q_-(w)$ we obtain chart maps δ_z at z and δ_w at w satisfying

$$(\delta z)^m=\delta y=(\delta w)^n\ .$$

Thus $\delta w=\zeta^k(\delta z)^{\frac{m}{n}}$ where ζ is a primitive nth root of unity and k an integer. As $z+\delta z$ winds once around z, the image $w+\delta w$ performs m/n turns about w, so that the net effect is $k\mapsto k+m \pmod n$. Thus $\tilde{n}=\frac{\mathrm{LCM}(m,n)}{m}$ turns of δz are required to return δw to its original value. In so doing k sweeps through a $\frac{n}{\tilde{n}}\mathbf{Z}$-congruence class (mod $n\mathbf{Z}$). Hence, for each k in the range $0\le k<\mathrm{HCF}(m,n)=n/\tilde{n}$, we obtain a distinct de-singularization point x_k of (z,w) whence

$$\#\ \tilde{Q}_\times^{-1}(z,w) \ = \ \mathrm{HCF}(m,n)\ .$$

The local behaviour of $\tilde{Q}_-$ and $\tilde{Q}_+$ about each point x_k is given respectively by $\delta z=(\delta x_k)^{\tilde{n}}$ and $\delta w=\zeta^k(\delta x_k)^{\tilde{m}}$ (where $\tilde{m}=\frac{\mathrm{LCM}(m,n)}{n}$). Clearly, at each x_k, the maps $\tilde{Q}_-$ and $\tilde{Q}_+$ are simultaneously normalizable. $\square$

EXAMPLE. In the case of a separable 2:2 correspondence f of the Riemann sphere - i.e. a 'map of pairs' - given by $Q_+(z)=Q_-(w)$ where Q_+ and Q_- are degree two rational maps, there are three possibilities for the topology of the desingularized graph $\Gamma(f)$ according to the number of coincidences of critical values (in Y) of Q_+ and Q_-.

#(crit values(Q_+) ∩ crit values(Q_-))	topology of $\Gamma(f)$
0	torus
1	sphere
2	two spheres

To see this, observe firstly that by Corollary 2 (subsection 2.3) these are the only possible topological types for the graph of a $2:2$ correspondence, and secondly that each coincidence of a critical value of Q_- with one of Q_+ gives rise to a *double point* (z,w) of the graph f, since we can find coordinates in which the correspondence (being separable) takes the local form $w^2 = z^2$.

Finally we remark that given *any* holomorphic correspondence f from a Riemann surface X to itself the corresponding *graph correspondence* $\Gamma(f) \to \Gamma(f)$ defined by $x \mapsto y \Leftrightarrow \tilde{Q}_+(x) = \tilde{Q}_-(y)$ (which has the same dynamical behaviour as f) is separable, by its very definition. We shall have more to say about graph correspondences in subsection 2.7.

2.6 Diagram conditions on correspondences

We shall be interested in classes of correpondences which satisfy particular *digram conditions* relating images of points and the inverse images of these images. These conditions should be thought of as the analogue for correspondences of *relations* among the *generators* of a finitely generated Kleinian group.

The first such condition is that of *separability*, discussed in the previous subsection. One way to express this condition is to require that whenever f maps z to w every image of every inverse image of w is also an image of z and every inverse image of every image of z is also an inverse image of w. In other words the images of z and the inverse images of these images are the vertices of a complete bi-partite graph (with edges 'arrows' (z,w)).

With this example in mind, we define a *diagram template* G to be a (connected) bi-partite graph on a set of vertices. This means that G can be written as a subset of $U \times V$ where the set of vertices is the disjoint union of U and V. Certain rules must be satisfied:

(i) For every $u \in U$ there exists $v \in V$ such that $(u,v) \in G$,

(ii) For every $v \in V$ there exists $u \in U$ such that $(u,v) \in G$,

In addition we will usually insist that G be *connected*, that is:

(iii) Any function F on the vertices $U \cup V$ satisfying $(u,v) \in G \Rightarrow F(u) = F(v)$ is constant.

An isomorphism of diagram templates $G \subset U \times V$ and $G' \subset U' \times V'$ is a pair of bijections $U \to U'$ and $V \to V'$ transporting G to G'.

We say G is a *diagram condition* for a correspondence f if whenever $z_0 \in Z$ or $w_0 \in W$ there is an assignment $i : U \to Z$ and an assignment $j : V \to W$ satisfying

(i) $z_0 \in i(U)$ (or $w_0 \in j(V)$),

(ii) If $z = i(u)$ then $(z,w) \in f \;\Leftrightarrow\; (\exists v \in V)\,((u,v) \in G \text{ and } w = j(v))$,

(iii) If $w = j(v)$ then $(z,w) \in f \;\Leftrightarrow\; (\exists u \in U)\,((u,v) \in G \text{ and } z = i(u))$,

In addition we would normally insist that the diagram condition be *faithfully satisfied*:

(iv) Generically (in z_0 and in w_0) the assignments i and j are one-to-one.

For example when f is an m to n holomorphic correspondence the condition that f be separable is equivalent to requiring that f faithfully satisfies the diagram condition which has as template the complete bi-partite graph on sets of cardinality m and n.

It is clear that two connected diagram templates which are diagram conditions faithfully satisfied by the same correspondence are isomorphic and thus we can refer to *the* diagram condition of a correspondence.

Two examples of possible diagram conditions for a 2:2 correspondence are

$$\begin{tikzcd}
z_1 \arrow[r] \arrow[dr] & w_1 \\
z_2 \arrow[r] \arrow[ur] & w_2
\end{tikzcd}$$

for a separable correspondence (referred to as a "map of pairs" in [4]), and

$$\begin{tikzcd}
z_1 \arrow[r] \arrow[dr] & w_3 \\
z_2 \arrow[ur] \arrow[dr] & w_2 \\
z_3 \arrow[r] \arrow[ur] & w_1
\end{tikzcd}$$

for a "map of triples" [6].

2.7 Joins, splittings and compositions of correspondences

If f_1 and f_2 are correspondences from Z to W then we can form the "union" $f_1 \cup f_2$ which is also a correspondence from Z to W whose graph is the union of f_1 and f_2. For instance if f_1 is defined by $P_1(z, w) = 0$ and f_2 by $P_2(z, w) = 0$ then $f_1 \cup f_2$ is defined by the vanishing of the product polynomial $P_1(z, w)P_2(z, w)$. Clearly if S is a subset of Z then $(f_1 \cup f_2)(S) = f_1(S) \cup f_2(S)$.

If $f_1 \cap f_2$ is empty or contains at most finitely many points then we call the union $f_1 \cup f_2$ the "join" of f_1 and f_2 and we may write $f_1 + f_2$. Such will be the case if f_1 and f_2 are defined by polynomials $P_1(z, w)$ and $P_2(z, w)$ with no factor in common.

Conversely we say a (geometric) correspondence f admits a *splitting* $f = f_1 + f_2$ if f_1 and f_2 are (geometric) correspondences whose graphs intersect only at isolated points. This will mean that the de-singularized graph $\Gamma(f)$ is the *disjoint* union of $\Gamma(f_1)$ and $\Gamma(f_2)$. In the case f is defined by a squarefree polynomial $P(z, w)$ then splitting of f is equivalent to factorization of $P(z, w)$ as $P_1(z, w)P_2(z, w)$. Of particular interest is the case f is an $n : n$ correspondence of $\bar{\mathbf{C}}$ to itself whose defining polynomial P factorises completely into n distinct nondegenerate linear factors $P_1, \ldots, P_n$ (i.e. $P_i(z,w) = c_i zw + d_i w - a_i z - b_i$ with $a_i d_i \neq b_i c_i$). Thus f is the "join" of n Möbius transformations

$$f_i : z \mapsto \frac{a_i z + b_i}{c_i z + d_i} \ .$$

Properties of the group of Möbius transformations $< f_1, \ldots f_n >$ generated by composition of the f_i (and their inverses) can be obtained as properties of appropriate "iteration" of the correspondence f.

Suppose f is a correspondence from Z to W and suppose g is a correspondence from W to another space X (whose graph is a subset g of $W \times X$) then we can form the composed correspondence $g \circ f$ between Z and X where

$$g \circ f = \{(z,x) \in Z \times X : \exists w \in W, (z,w) \in f, (w,x) \in g\ \}.$$

Thus, for subsets S of Z, we have $(g \circ f)(S) = g(f(S))$. As for maps, composition "$\circ$" of correspondences is associative. Moreover it is both left and right distributive over "$\cup$".

Notice that there is also a separable 'graph correspondence' between the graphs (of) f and (of) g which is defined formally by its own graph

$$\{((z,w),(w',x)) \in f \times g : w = w'\}$$

but which we will embed in $Z \times W \times X$ as

$$f \wedge g \ = \ \{(z,w,x) \in Z \times W \times X : \ (z,w) \in f, \ (w,x) \in g\}$$

noting the projections $\pi_-^{(Z,W,X)} : (z,w,x) \mapsto (z,w)$ and $\pi_+^{(Z,W,X)} : (z,w,x) \mapsto (w,x)$ (whose restrictions to $f \wedge g$ map respectively to f and to g). The object $f \wedge g$ can also be thought of as the space of paths of length two. Its image under the projection $(z,w,x) \mapsto (z,x)$ is $g \circ f$.

An important observation is that if f and g are closed relations then $g \circ f$ will also be a closed relation provided that W is compact (see McGehee).

We are normally interested only in the case f and g are holomorphic correspondences, with de-singularizations:

$$\tilde{Q}_\triangle^{(f)} = \tilde{Q}_\times^{(f)} = \tilde{Q}_-^{(f)} \times \tilde{Q}_+^{(f)} : \Gamma(f) \ \to \ f(\subset Z \times W) \ ,$$

$$\tilde{Q}_\triangle^{(g)} = \tilde{Q}_\times^{(g)} = \tilde{Q}_-^{(g)} \times \tilde{Q}_+^{(g)} : \Gamma(g) \ \to \ g(\subset W \times X) \ .$$

Thus the geometric version of the "graph correspondence" is the separable correspondence between manifolds $\Gamma(f)$ and $\Gamma(g)$ given by

$$(\tilde{Q}_-^{(g)})^{-1} \circ \tilde{Q}_+^{(f)} \ = \ \{(u,v) \in \Gamma(f) \times \Gamma(g) : \tilde{Q}_+^{(f)}(u) = \tilde{Q}_-^{(g)}(v)\} \ .$$

Noting that the *de-singularized graph* of the graph correspondence is the same as the de-singularization $\Gamma(f \wedge g)$ of the path-space variety $f \wedge g$, we denote the graph projections (arising from the de-singularization of the separable correspondence $(\tilde{Q}^{(g)}_-)^{-1}\circ\tilde{Q}^{(f)}_+$ in $\Gamma(f)\times\Gamma(g)$ as described in the proof of Proposition 5, in subsection 2.5) by $\tilde{Q}^{(f\wedge g)}_-$ and $\tilde{Q}^{(f\wedge g)}_+$. It follows that the de-singularizing map $\tilde{Q}^{(f\wedge g)}_\triangle : \Gamma(f\wedge g)\to f\wedge g$ is a composition of the de-singularizing map

$$\tilde{Q}^{(f\wedge g)}_\times = \tilde{Q}^{(f\wedge g)}_- \times \tilde{Q}^{(f\wedge g)}_+ :\ \Gamma(f\wedge g)\ \to\ (\tilde{Q}^{(g)}_-)^{-1}\circ\tilde{Q}^{(f)}_+\ (\subset\Gamma(f)\times\Gamma(g))\ ,$$

followed by a map

$$\tilde{Q}_{\triangle\triangle} :\ (\tilde{Q}^{(g)}_-)^{-1}\circ\tilde{Q}^{(f)}_+\ \to\ f\wedge g\ (\subset Z\times W\times X).$$

Here we define $\tilde{Q}_{\triangle\triangle}(u,v)$ to be the concatenation of $\tilde{Q}^{(f)}_\triangle(u)\in f\ (\subset Z\times W)$ and $\tilde{Q}^{(g)}_\triangle(v)\in g\ (\subset W\times X)$ (where $u\in\Gamma(f)$, $v\in\Gamma(g)$); this is well-defined since $\pi_+\tilde{Q}^{(f)}_\triangle(u)=\tilde{Q}^{(f)}_+(u)=\tilde{Q}^{(g)}_-(v)=\pi_-\tilde{Q}^{(g)}_\triangle(v)$. We thus "de-singularize" $\pi^{(Z,W,X)}_-$ and $\pi^{(Z,W,X)}_+$ with the equations

$$\pi_-\circ\tilde{Q}^{(f\wedge g)}_\triangle=\tilde{Q}^{(f)}_\triangle\circ\tilde{Q}^{(f\wedge g)}_- \text{ and}$$

$$\pi_+\circ\tilde{Q}^{(f\wedge g)}_\triangle=\tilde{Q}^{(g)}_\triangle\circ\tilde{Q}^{(f\wedge g)}_+\ .$$

By an inductive process we can similarly de-singularize spaces of paths of greater length, noting the associative law $\Gamma((f\wedge g)\wedge h)=\Gamma(f\wedge(g\wedge h))$ and defining $\tilde{Q}^{(f\wedge g\wedge h)}_\triangle : \Gamma(f\wedge g\wedge h)\to f\wedge g\wedge h$, etc.

Returning to the composition $g\circ f$, in the case of f and g holomorphic correspondences, it is possible that $\Gamma(f\wedge g)$ may not be the de-singularization of this composition. Certainly we have holomorphic "projections":

$$\tilde{Q}^{(f\wedge g)}_{--}=\tilde{Q}^{(f)}_-\circ\tilde{Q}^{(f\wedge g)}_- :\ \Gamma(f\wedge g)\to Z\ ,$$

$$\tilde{Q}^{(f\wedge g)}_{++}=\tilde{Q}^{(g)}_+\circ\tilde{Q}^{(f\wedge g)}_+ :\ \Gamma(f\wedge g)\to X\ .$$

However, the image set of $\tilde{Q}_{--}\times\tilde{Q}_{++}$ in $Z\times X$ may have de-singularization *a quotient of* $\Gamma(f\wedge g)$ - in other words the maps $\tilde{Q}_{--}$ and $\tilde{Q}_{++}$ have a *holomorphic* common factor. The image of the "highest common factor" map we write as $\Gamma(g\circ f)$. From Theorem 1 (subsection 2.3) we see that the portrait $g\circ f\ (\subset Z\times X)$ is a branched-covering correspondence with $\Gamma(g\circ f)$ a genuine de-singularization. In the case that $Z=W=X=\bar{\mathbf{C}}$ and f and g are determined by the vanishing of polynomials in $\mathbf{C}[z,w]$ and $\mathbf{C}[w,x]$ respectively we may alternatively obtain $g\circ f$ as the vanishing set of the resultant of the polynomials for f and g with respect to the polynomial ring $\mathbf{C}[z,x](w)$.

A common situation where $\Gamma(g\circ f)\neq\Gamma(f\wedge g)$ is when g is the inverse correspondence to f.

2.8 Galois correspondences

Recall that $f^{-1} \circ f$ includes $id|_{\pi_-(f)}$ and $f \circ f^{-1}$ includes $id|_{\pi_+(f)}$ and, in general, these are strict inclusions. Indeed, at least in the case that f is a holomorphic correspondence of compact Riemann surfaces, we see that $f^{-1} \circ f$ is a splitting of id_Z and another symmetric correspondence. Likewise $f \circ f^{-1}$ is a splitting of id_W and a symmetric correspondence. We define:

$$\mathrm{Gal}_+^{(f)} = (f^{-1} \circ f) \setminus id_Z \ ,$$

$$\mathrm{Gal}_-^{(f)} = (f \circ f^{-1}) \setminus id_W \ .$$

In the case of a single-valued map $Q : X \to Y$ we simplify notation and write:

$$\mathrm{Gal}^Q = (Q^{-1} \circ Q) \setminus id_X \ .$$

In general $\mathrm{Gal}_+^{(f)}$ and $\mathrm{Gal}_-^{(f)}$ are rather complicated objects so we first examine the case of separable $f = Q_-^{-1} \circ Q_+$ (where Q_- and Q_+ have the same range). In this case $f^{-1} \circ f = Q_+^{-1} \circ Q_+$ and $f \circ f^{-1} = Q_-^{-1} \circ Q_-$. These resulting compositions are then separable symmetric correspondences. Striking out the graph-diagonals gives us $\mathrm{Gal}_+^{(f)} = \mathrm{Gal}^{Q_+}$ and $\mathrm{Gal}_-^{(f)} = \mathrm{Gal}^{Q_-}$. In the case Q ($= Q_+$ or Q_-) is continuous with Hausdorff quotient, putting back in any "fixed points" of Gal^Q gives us (via the closure of the graph) the *Galois correspondences* $\overline{\mathrm{Gal}}^{Q_+}$ and $\overline{\mathrm{Gal}}^{Q_-}$ which are the non-identity covering "transformations" of Q_+ and Q_- respectively. In the case that f is a separable holomorphic 2:2 correspondence $\bar{\mathbf{C}} \to \bar{\mathbf{C}}$ these are just Möbius involutions which we call the *forward involution* I_+ and the *backward involution* I_- respectively.

EXAMPLE: If $d_+ = 3$ and Q_+ is a cubic rational map of the form $z \mapsto z^3$ then $\overline{\mathrm{Gal}}^{Q_+} : z \mapsto w$ is represented by the polynomial equation

$$0 = \frac{Q_+(w) - Q_+(z)}{w - z} = \frac{w^3 - z^3}{w - z} = w^2 + zw + z^2 = (w - e^{2\pi i/3}z)(w - e^{-2\pi i/3}z)$$

and so splits into two Möbius rotations $z \mapsto e^{\pm 2\pi i/3}z$. If $d_+ = 3$ but Q_+ is of the form $z \mapsto z^3 - 3z$ then $\overline{\mathrm{Gal}}^{Q_+} : z \mapsto w$ is represented by the equation

$$0 = \frac{Q_+(w) - Q_+(z)}{w - z} = \frac{w^3 - z^3 - 3w + 3z}{w - z} = w^2 + zw + z^2 - 3$$

and does not split. However this correspondence can usefully be thought of as the "projection" of the Möbius transformations $u \mapsto v = e^{\pm 2\pi i/3}u^{\pm 1}$ under the degree 2 branched covering $z = u + 1/u$ (with $w = v + 1/v$).

More generally, if f is a $m : n$ separable correspondence then $\overline{\mathrm{Gal}}_+^{(f)}$ is a $(m-1) : (m-1)$ symmetric correspondence and $\overline{\mathrm{Gal}}_-^{(f)}$ is a $(n-1) : (n-1)$ symmetric correspondence.

The situation is more complicated for non-separable f and so we first look to the graph. For simplicity we take f holomorphic and work using the desingularization $\Gamma(f)$ of the graph. In place of $\overline{\mathrm{Gal}}^{Q_+}$ and $\overline{\mathrm{Gal}}^{Q_-}$ we must examine

$$\widetilde{\mathrm{Gal}}_+ := \overline{\mathrm{Gal}}^{\tilde{Q}_+^{(f)}} \text{ and } \widetilde{\mathrm{Gal}}_- := \overline{\mathrm{Gal}}^{\tilde{Q}_-^{(f)}}$$

(which act on $\Gamma(f)$). In the case f is 2:2 these are the covering *involutions* $\tilde{I_+}$ and $\tilde{I_-}$ (of $\tilde{Q}_+$ and $\tilde{Q}_-$ respectively). Separability of f is equivalent to the commuting of $\tilde{I_+}$ and $\tilde{I_-}$. If f is $m:n$ then instead of $\tilde{I_+}$ and $\tilde{I_-}$ we have $(m-1):(m-1)$ and $(n-1):(n-1))$ *correspondences* $\widetilde{\mathrm{Gal}}_+$ and $\widetilde{\mathrm{Gal}}_-$ on $\Gamma(f)$, and f is separable if and only if these correspondences "commute" in the obvious sense.

Our original $\overline{\mathrm{Gal}}_+^{(f)}$ and $\overline{\mathrm{Gal}}_-^{(f)}$ are thus "projections" of their upstairs counterparts. When f is not separable, they may be as bad as $n(m-1):n(m-1)$ and $m(n-1):m(n-1)$ correspondences, and should not in general be thought of as Galois correspondences, but as 'non-linear' Galois correspondences.

EXAMPLE: Consider the 2:2 holomorphic correspondence $f_t : z \mapsto w$, $(t \neq \pm 2)$ given by

$$z^2 - tzw + w^2 = 4 - t^2.$$

This has graph a Riemann sphere which can be parametrized by the complex variable u via the projection maps $\tilde{Q}_- : u \mapsto u + 1/u$ and $\tilde{Q}_+ : u \mapsto u/\zeta + \zeta/u$ where ζ is a solution of $t = \zeta + 1/\zeta$. Note that the points $u = 0$ and $u = \infty$ are identified (corresponding to $(z, w) = (\infty, \infty)$). The graph involutions are thus $\tilde{I_-} : u \mapsto 1/u$ and $\tilde{I_+} : u \mapsto \zeta^2/u$. Thus $\tilde{I_-}$ and $\tilde{I_+}$ generate a dihedral group with cyclic subgroup generator $\tilde{I_+} \circ \tilde{I_-} : u \mapsto \zeta^2 u$ (which has trace t). One sees that f_t is separable only in the case $\zeta = \pm i$ which is the case $t = 0$. Since f_t is symmetric the correspondence $\overline{\mathrm{Gal}}_+^{(f_t)}$ equals $\overline{\mathrm{Gal}}_-^{(f_t)}$ and is given by

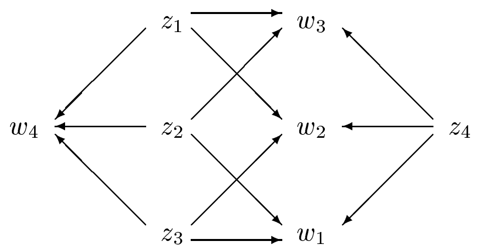
$$w^2 - (t^2 - 2)zw + z^2 = t^2(4 - t^2)$$

(which only in the separable case $t = 0$ simplifies to a $1:1$ map $(z \mapsto -z)$).

2.9 Off-separable correspondences

We say a correspondence f from a set X to itself is *off-separable* if there is a bijection $M : X \to X$ whose graph intersects that of f (if at all) in a finite set of points, and such that the correspondence $f + M$ is separable. If we wish to specify M then we say f is *M-off-separable*. For example, a 2:2 correspondence which has the diagram condition of a "map of triples" (subsection 2.6) is off-separable, provided that the there are at most finitely many z_i where the diagram condition is not faithfully satisfied. Here M is the map which sends z_i to w_i, and when joined to f gives a separable 3:3 correspondence. It is only at the finitely many points z_i where the diagram 'collapses' that the arrow $z_i \to w_i$ may already be in f. A particular example of an off-separable f is the correspondence $f_{-1} \circ M$ (with equation $(Mz)^2 + (Mz)w + w^2 = 3$) involving the symmetric correspondence f_t from the previous subsection on Galois correspondences, in the case that the graph parameter ζ is a cube root of unity.

A 3:3 off-separable correspondence has diagram condition:



Again M is the map sending z_i to the 'antipodal' vertex w_i. An $m:n$ off-separable correspondence is one which satisfies the diagram condition having as template G a bipartite graph with vertex sets U and V of cardinalities m and n respectively, and where each vertex of U has valency $n-1$ and each vertex of V has valency $m-1$. (The 'missing' arrows, which when added to the graph make it separable, define a bijection $M: Z \to W$.)

There is a close connection between off-separable correspondences and Galois correspondences (implying, in particular, that $m = n$):

Lemma 2 *If f is M-off-separable then $M^{-1} \circ f + id$ is an equivalence relation. Hence $M \circ \mathrm{Gal}^Q \subset f \subset M \circ (Q^{-1}Q)$ for a (unique) quotient map Q.*

Proof. Suppose that $(z, z') \in M^{-1} \circ f$, that is $M(z') \in f\{z\}$. Separability of $f + M$ implies that $f\{z\} \cup M(z) = f\{z'\} \cup M(z')$. It follows that, in all cases, $M(z) \subset f\{z'\} \subset f\{z\} \cup M(z)$ (since the case $M(z) = M(z')$ yields $z = z'$ and so $(z', z) \in M^{-1} \circ f$). Applying M^{-1} gives $\{z\} \subset M^{-1}f\{z'\} \subset M^{-1}f\{z\} \cup \{z\}$. The first inclusion thus tells us that the relation $M^{-1} \circ f$ (and so its union with id) is symmetric. The second inclusion implies that $M^{-1} \circ f \cup \mathrm{id}$ is transitive and so an equivalence relation. $\square$

The graphs of Gal^Q and $M \circ \mathrm{Gal}^Q$ are isomorphic (it is just that the 'axes' Z and W are identified via the identity in one case and via M in the other). Thus the diagram condition satisfied by an off-separable correspondence is precisely that associated to Gal^Q for the corresponding quotient map Q. Indeed an alternative way to characterize off-separable correspondences is as those which satisfy the diagram condition of a Galois correspondence.

We now move from the set-theoretic level to the topological level.

DEFINITION: A continuous surjection $Q: X \to Y$ of locally compact Hausdorff spaces X and Y is said to be a *branched covering map* if

(i) Q is an open mapping;

(ii) $Q^{-1}(y)$ is discrete for each $y \in Y$;

(iii) Q is a local homeomorphism except at a (possibly empty) set $S \subset X$ having image $Q(S)$ a discrete subset of Y.

Note that since any continuous surjection $f : X \to Y$ with Y Hausdorff has closed graph, a branched covering map is in particular a branched covering correspondence in the sense of our earlier definition. We now restrict our attention to correspondences on *compact* Hausdorff spaces X, so the *discrete* sets in both definitions become *finite* sets. For a branched covering map Q on such a space X, the difference in graphs between Gal^Q and its closure is:

$$\overline{\mathrm{Gal}}^Q - \mathrm{Gal}^Q = \{(x,x) : x \text{ critical for } Q\}$$

where a *critical point* of Q is any point $x \in X$ at which Q fails to be locally injective. These points form the set S of (iii) above.

Theorem 3 *Let X be a compact Hausdorff space and $M : X \to X$ be a homeomorphism.*

(i) If Q is a branched covering map defined on X, then both $M \circ \mathrm{Gal}^Q$ and its closure the branched covering correspondence $M \circ \overline{\mathrm{Gal}}^Q$ are $M-$off-separable. If Q has degree $d+1$ (almost everywhere) then $M \circ \overline{\mathrm{Gal}}^Q$ is a $d : d$ correspondence.

(ii) Conversely, if f is an M-off-separable branched covering correspondence on X then $f = M \circ \overline{\mathrm{Gal}}^Q$ for a (unique) quotient map Q defined on X. Moreover X/Q is compact Hausdorff and Q is a branched covering map.

Proof. (i) follows at once from the definition of M-off-separability and the fact that $Q^{-1}Q$ is closed in $X \times X$ (since X/Q is Hausdorff). For (ii), we first note that by Lemma 2 we have a quotient map Q such that $M \circ \mathrm{Gal}^Q \subset f \subset M \circ (Q^{-1}Q)$. Now, by the definition of M-off-separability, the graph of f contains at most finitely many points in addition to those of $M \circ \mathrm{Gal}^Q$, but if f is a branched covering correspondence its graph contains $M \circ \overline{\mathrm{Gal}}^Q$ since f is closed, and has no isolated points since f is open. Thus $f = M \circ \overline{\mathrm{Gal}}^Q$. Moreover $M \circ (Q^{-1}Q)$ is closed in $X \times X$, being the union of closed graphs f and M. Hence $Q^{-1}Q$ is also closed in $X \times X$. Thus the diagonal is closed in $X/Q \times X/Q$, whence X/Q is Hausdorff. To show that Q is an open map we first note that $M \circ (Q^{-1}Q)$, being the union of f and M, is a branched covering correspondence itself, and thus so is $Q^{-1}Q$. In particular the projections from the graph $Q^{-1}Q \subset X \times X$ to X are open maps. Thus for any open subset U of X, the set $Q^{-1}Q(U)$ is open in X, and so $Q(U)$ is open in X/Q (by the definition of the quotient topology). That Q satisfies the remaining conditions ((ii) and (iii)) of the definition of a branched covering map also follows easily from the fact that $Q^{-1}Q$ is a branched covering correspondence. □

Corollary 3 *A holomorphic $n : n$ correspondence on a compact Riemann surface X, with $n \geq 2$, is off-separable if and only if it has the form $M \circ \overline{\mathrm{Gal}}^Q$ for some conformal automorphism M of X and holomorphic branched covering map of Riemann surfaces $Q : X \to X/Q$.*

Proof. If f is holomorphic and M-off-separable, and $n \geq 2$, then M is necessarily conformal since $M \subset f \circ f^{-1} \circ f$. The rest is easy. □

3 Regular and limit sets

If f is a correspondence between X and itself we can consider the iterates f^n, $n \geq 0$ with graph $f^n \subset X \times X$. For ease of notation we shall write f^{-n} for $(f^{-1})^n$. One must be aware however that f^{m+n} will only equal $f^m \circ f^n$ if m and n have the same sign. We shall write f^* for the *orbit correspondence*, the union of arbitrarily long compositions of mixtures of f and f^{-1}.

For an iterated rational map f the dynamics naturally partitions the Riemann sphere into two completely invariant subsets, the *Fatou set* where behaviour is 'regular' and the *Julia set* where behaviour is 'chaotic'. For a Kleinian group G there is a similar partition of the Riemann sphere into a *regular set* and *limit set*. In both cases there are a number of different, but equivalent, ways to define these sets. So when it comes to generalising to correspondences we have a number of choices to make. Our difficulty is that it is no longer clear which versions of definitions are equivalent, and, where they are not, which versions will be the most useful. Our main concern in this chapter is with the analogue for correspondences of the *regular* (or *proper discontinuity*) set of a Kleinian group, an analogue we shall make use of in the remaining chapters of this article. In the final subsection of the chapter we also discuss how one might generalise the *Fatou* (or *equicontinuity*) set of a rational map to an appropriate notion for holomorphic correspondences.

3.1 The regular set $\Omega(f)$

Given a holomorphic correspondence $f : z \mapsto w$ on a compact Riemann surface X, we construct a canonical *regular set* $\Omega(f)$ and its complement - the *(non-regular) limit set* - both fully-invariant. We shall see later (Corollary 4 in subsection 3.2) that $\Omega(f)$ has Hausdorff orbit-quotient under f. In the case that f is a union of Möbius transformations which define a discrete (Kleinian) group G, the set $\Omega(f)$ is the regular set of G in the usual sense of Kleinian groups, and its complement is the limit set of the group.

Both f^{-1} and f map open sets to open sets. As a consequence the orbit correspondence f^* also maps open sets to open sets. Note that f^*, being an equivalence relation, is the covering relation of an *orbit-quotient* map $Q_*(= Q_*^{(f)})$, that is $f^* = Q_*^{-1} \circ Q_*$. Thus Q_* is an open mapping (where the orbit-quotient space has the standard quotient topology - the maximal making Q_* continuous). Note however that in general f^* will not be a closed relation on the whole of X, the quotient by Q_* not being Hausdorff.

Since the motivation for regular sets and limit sets comes from Kleinian groups, we will first examine the case f is an $n : n$ correspondence which splits into n Möbius transformations $f_1, \ldots, f_n$ which generate a Kleinian group G. Recall that the regular set $\Omega(G)$ of G consists of those points z where G acts *discontinuously*, that is, having a neighbourhood U such that $g(U) \cap U \neq \emptyset$ for only finitely many $g \in G$. In the case $f_1, \ldots, f_n$ generate G *freely*, this is equivalent to saying $g_w(U) \cap U \neq \emptyset$ for only finitely many words w in the free group $< f_1, \ldots, f_n >$ (where $g_w = w_k \circ \ldots \circ w_1$ for $w = w_1 \ldots w_k$, $w_i \in \{f_1, f_1^{-1}, \ldots, f_n, f_n^{-1}\}$). If, on the other hand, the f_i do not generate G freely, then arbitrarily long words w give $g_w(U) \cap U \neq \emptyset$. However we can say that

$$\{(z', g_w(z')) \in U \times U : \ w \in \{f_1^{\pm 1}, \ldots, f_n^{\pm 1}\}^*\}$$

is accounted for as the union of $\{(z', g_w(z')) : z' \in X\} \cap (U \times U)$ over *finitely many* words $w \in \{f_1^{\pm 1}, \ldots, f_n^{\pm 1}\}^*$. (This uses the observation that the global action of a Möbius transformation g is determined by its local action near z.)

As a holomorphic correspondence f (or at least component(s) thereof) is likewise determined by its local behaviour near a point z we can formulate a similar notion of "discontinuous action" for f:

We first define f^e, for $e \in \{-1, 1\}^*$, to be the correspondence-composition $f^{e_k} \circ \ldots \circ f^{e_1}$ where $e = e_1 \ldots e_k$, $e_i \in \{-1, 1\}$. In the case f splits into Möbius generators $f_1, \ldots, f_n$ one can think of this as the join (or union) of g_w over all words $w \in \{f_1^{\pm 1}, \ldots, f_n^{\pm 1}\}^k$ with *signature* e.

Thus f^* is the union over all $e \in \{-1, 1\}^*$ of f^e. We say a correspondence f acts *discontinuously* at a point z if there exists a neighbourhood U and a number $N \geq 1$ such that

$$f^* \cap (U \times U) \subset \bigcup_{|e| < N} f^e$$

where $|e|$ denotes the length of e. In other words U has only finitely many distinct returns under f^*.

DEFINITION: The *regular set* of f is the set $\Omega(f)$ of points z where f acts discontinuously.

This set is clearly open. Our main aim in this subsection is to prove that it is completely invariant under f and that its orbit quotient $\Omega(f)/f^*$ is Hausdorff.

To prove these results it is necessary to reduce U to a "nice neighbourhood" in a sense similar to that for Kleinian groups: Recall that a *nice neighbourhood* of a point z for a Kleinian group G is one satisfying $g(U) = U$ for all g in the stabiliser G_z of z and where $g(U) \cap U = \emptyset$ for all $g \in G \setminus G_z$. (Note that the hypothesis $z \in \Omega(G)$ implies that G_z is finite.) (See Maskit [19])

DEFINITION: A connected open subset U of X is called a *nice neighbourhood* of a point z in X if the bi-restriction of f^* to U splits into a finite union of self-homeomorphisms f_z^λ of U, each fixing z. Thus

$$f^* \cap (U \times U) = \bigcup_\lambda f_z^\lambda$$

When z has a nice neighbourhood we write $\mathrm{Stab}_{f^*}(z)$, or just $\mathrm{Stab}(z)$, for $\bigcup_\lambda f_z^\lambda$, the *stabilizer group* of f^* at z. Defining $\deg_z Q_*$ to be the degree of $Q_*|_U$ at z, we obtain $\deg_z Q_* = \#\mathrm{Stab}_{f^*}(z)$.

EXAMPLE: Let f be the map $z \to z^2 + 1/4$, regarded as a $2:1$ correspondence, let $z_0 = i/2$ and let $D = \{z : |z| < 1/5\}$. Then the component of $f^{-1}(D)$ containing z_0 is a nice neighbourhood of z_0 and $\mathrm{Stab}(z_0)$ is a cyclic group of of order 2, the pull back of the group $\{z \mapsto \pm z\}$ which acts on D.

Lemma 3 *If z has neighbourhood U in X satisfying $f^* \cap (U \times U) \subset \bigcup_{|e|<N} f^e$, that is $z \in \Omega(f)$, then U contains a nice neighbourhood of z.*

Proof. For each word $e = e_1 \ldots e_k \in \{-1, 1\}^*$ with f^e fixing z (and $k < N$) we claim that z is both forwards and backwards non-singular, or de-singularizable, for f^e. In other words we claim that each $u \in \left(\tilde{Q}_\times^{(f^e)}\right)^{-1}(z, z)$ (in $\Gamma(f^e)$) is neither a critical point of $\tilde{Q}_-^{(f^e)}$ nor a critical point of $\tilde{Q}_+^{(f^e)}$:

Suppose $\tilde{Q}_-^{(f^e)}$ and $\tilde{Q}_+^{(f^e)}$ have local degrees, respectively, n and m at u. Given a chart δ_z at z we can choose a chart δ_u at u so that at least $\tilde{Q}_-^{(f^e)}$ is normalized. Writing the Taylor expansion of $\tilde{Q}_+^{(f^e)}$, with respect to the same charts $\delta_w = \delta_z$ (at $w = z$) and δ_u (at u), as

$$\delta w = \sum_{k \geq m} \alpha_k (\delta u)^k$$

(where δw may be different from δz) we obtain an expression for the local behaviour the (u-)branch of f^e, through (z, w), as

$$\delta w = \sum_{k \geq m} \alpha_k (\delta z)^{k/n} \ .$$

If we had $n \neq m$ then for δz small, this branch of f^e would behave like

$$z + \delta z \ \mapsto \ z + \alpha_m \, (\delta z)^{m/n}$$

(where α_m is non-zero). In the case $n < m$, forward iterating this branch produces an orbit accumulating on z. In the case $m < n$, backward iterating this branch produces an orbit accumulating on z. Neither is possible for $z \in \Omega(f)$. The case $m = n$ is more tricky.

Write λ for $\alpha_m (= \alpha_n)$ so that the branch of f^e becomes

$$z + \delta z \ \mapsto \ z + \lambda \, \delta z + \sum_{k > n} \alpha_k \, (\delta z)^{k/n} \ .$$

In the case $|\lambda| < 1$, forward iterating this branch produces an orbit accumulating on z. Likewise in the case $|\lambda| > 1$, backward iterating this branch produces an orbit accumulating on z. Thus $|\lambda| = 1$. Furthermore, the fact that U has only finitely many distinct returns under f^* is sufficient to prove that λ must be a root of unity.

Now observe that the highest common factor of $\{k : \alpha_k \neq 0\}$ must be one since if d is a common factor then both $\tilde{Q}_-^{(f^e)}$ and $\tilde{Q}_+^{(f^e)}$ locally factor through a map of the form

$$u + \delta u \ \mapsto \ v + (\delta u)^d$$

whence so will their product $\tilde{Q}_\times^{(f^e)}$. Since de-singularization is almost everywhere one-to-one we must have $d = 1$. It follows that if $n > 1$ there must be a least k (with $\alpha_k \neq 0$) which is not a multiple of n. Write k/n in lowest terms as p/q (with $1 < q < p$ and $q|n$.) Thus although this branch is n-valued, the first multi-valued approximation is q-valued:

$$\delta w \ = \ \lambda\, \delta z + \sum_{j=2}^{[p/q]} \alpha_{nj}\, (\delta z)^j + \mu\, (\delta z)^{p/q} + ...$$

where $\mu = \alpha_{pn/q} \neq 0$. Given δw small but non-zero the first approximation to the solutions for δz is $\lambda^{-1}\, \delta w$. We need the first multi-valued approximation, which is given by

$$h_\lambda(\delta z) \ = \ \delta w - \mu\, \lambda^{-p/q}\, (\delta w)^{p/q}$$

where h_λ is the (locally) bi-holomorphic transformation

$$h_\lambda(\delta z) \ = \ \sum_{j=1}^{[p/q]} \alpha_{nj}\, (\delta z)^j \ .$$

Thus the images of $w + \delta w$ under the inverse-branch followed by the forward branch are $w + \delta w'$ (as in a local "non-linear Galois" approximation) are (to the first multi-valued approximation) given by

$$\delta w' \ = \ \delta w - (1 - \zeta^r)\, \mu\, \lambda^{-p/q}\, (\delta w)^{p/q}$$

where ζ is a primitive q-th root of unity and r ranges over integers between 0 and q. Since $\delta w'/\delta w$ tends to one as δw tends to zero, an iteration can be set up (say fixing $r \neq 0$) to produce arbitrarily large local orbits.

Thus the only possibility is $m = n = 1$ as claimed, whence u defines a locally invertible single-valued branch of f^e at z. Since the de-singularization of (z, z) for each f^e is finite and there are less than 2^N words e to consider, there exists a common neighbourhood U' ($\subset U$) on which is defined a single-valued and invertible map f_u for each u, each representing the branch of the appropriate f^e through u. A similar line of argument to the above now shows that the single-valued map in fact only depends on the value of λ. Thus we write it as f_z^λ. The set of possible λs is a finite, hence cyclic, group of roots of unity. We now create a further sub-neighbourhood U'' defined by

$$U'' := \{z' \in U' : \ f_z^\lambda(z') \in U' \ \forall f_z^\lambda\} \ .$$

Observe that U'' is our desired *nice neighbourhood.* $\square$

Lemma 4 *A nice neighbourhood U of a point $z \in \Omega(f)$ is a component of its image under global iteration $f^*(U)$.*

Proof. Suppose, on the contrary, that a point x of ∂U is contained in $f^*(U)$. This means that there exists $w \in U$ and some word $e \in \{-1,1\}^*$ such that $w \in f^e(x)$. Since f^e is lower-semi-continuous at x we can find a sequence x_n, *in* U, tending to x with a sequence w_n (also in U) tending to w, satisfying $(x_n, w_n) \in f^e$. By hypothesis, we have $(x_n, w_n) \in f_z^{\lambda_n}$, i.e. $f_z^{\lambda_n}(x_n) = (w_n)$, for some sequence λ_n among the finite collection of λs. Thus some (infinite) sub-subsequence (x_{n_r}, w_{n_r}) is contained in *one* f_z^λ, (λ fixed). Since f_z^λ is a homeomorphism of U and since $w_{n_r} \to w$, we must have x_{n_r} tending to a limit *in* U, namely $(f_z^\lambda)^{-1}(w)$. This contradicts the assumption that x was in the boundary of U. $\square$

Lemma 5 *The image of any component of a fully-invariant open set W under either f or f^{-1} is a finite union of components of W.*

Proof. Clearly if U is open in W then so is $f(U)$ (by the lower-semi-continuity of f^{-1}). Likewise $f^{-1}(U)$ is open. On the other hand if U is closed in W (say U is a component of W) then for any $w \in W \setminus f(U)$ we have that $f^{-1}(w)$ is disjoint not only from U but also from the closure of U in X. It follows (by the upper-semi-continuity of f^{-1}) that the same is true for $f^{-1}(w')$ for all w' sufficiently close to w. We have therefore proved that $f(U)$ is closed in W. Similarly $f^{-1}(U)$ is closed in W. $\square$

REMARK: Likewise, the inverse-image $f^{-1}(V)$ of a component V of $f(U)$ is closed in W and hence contains U. As a consequence the components U and V of W are components of $f^*(V)$.

Theorem 4 *The set $\Omega(f)$ is completely invariant under f.*

Proof. Given an open set U which is a component of $f^*(U)$ and a component V of $f(U)$, we have seen that U and V are components of $f^*(V)$ and (hence) that $f^*(V) = f^*(U)$. It is an easy matter to verify the equalities

$$(f \cap (U \times V)) \circ (f^* \cap (U \times U)) = f^* \cap (U \times V) = (f^* \cap (V \times V)) \circ (f \cap (U \times V))$$

It follows that if $f^* \cap (U \times U) \subset \bigcup_{|e|<N} f^e$ then

$$f^* \cap (V \times V) \subset \bigcup_{|e|<N} f \circ f^e \circ f^{-1} \subset \bigcup_{|e|<N+2} f^e .$$

Similarly if $f^* \cap (V \times V) \subset \bigcup_{|e|<N} f^e$ then

$$f^* \cap (U \times U) \subset \bigcup_{|e|<N} f^{-1} \circ f^e \circ f \subset \bigcup_{|e|<N+2} f^e .$$

$\square$

Corollary 4 *The correspondence $f \cap (U \times V)$ induces an isomorphism between the local orbifolds U/f^* and V/f^*.* $\square$

As a prelude to proving equicontinuity of the regular set in the following subsection, we ask even more of a *nice neighbourhood*:

DEFINITION: A *very nice neighbourhood* of z is a neighbourhood U in $\Omega(f)$ of z which is a connected component of $f^*(U)$, all of whose components are simply-connected and intersect $f^*\{z\}$ in exactly one point. The following Lemma shows that this is equivalent to saying that U is a simply-connected nice neighbourhood of z such that the singularities of f intersect $f^*(U)$ only in $f^*\{z\}$.

Lemma 6 *If U is a neighbourhood in $\Omega(f)$ of z which is a component of $f^*(U)$, is simply-connected and intersects $f^*(z)$ only in z, then*
(i) U is a nice neighbourhood of z;
(ii) a component V of $f(U)$ is simply-connected and intersects $f^*(z)$ in a unique point if and only if $f \cap (U \times V)$ has no forward singular points in $U \setminus \{z\}$;
(iii) a component V of $f^{-1}(U)$ is simply-connected and intersects $f^*(z)$ in a unique point if and only if $f \cap (V \times U)$ has no backward singular points in $U \setminus \{z\}$.

Proof. (i) Since z is the only element of $f^*(z)$ in U we have

$$\deg(Q_*|_U) = \deg_z Q_* = \#\mathrm{Stab}_{f^*}(z) \ .$$

Since U is simply-connected, an Euler characteristic computation (counting critical points with mulitiplicities) yields:

$$\#\{\text{critical points of } Q_*|_U\} = \deg(Q_*|_U) \cdot \chi(U/f^*) - \chi(U) = \deg(Q_*|_U) - 1$$

whence U contains no critical point of $Q_*|_U$ other than z.

(ii) Recall from Lemma 5 that V is a component of $f^*(U)$ and note the identification of quotients U/f^* and V/f^* as in Corollary 4.

Since U is a nice neighbourhood of z, the assumption for $f \cap (U \times V)$ to have no forward singular points in $U \setminus \{z\}$ also applies to the separable correspondence

$$f^* \cap (U \times V) = (f \cap (U \times V)) \circ (f^* \cap (U \times U)) \ .$$

Hence, by the proof of Proposition 5 (subsection 2.5), the only critical points of $Q_*|_V$ lie on $f^*(z)$. Thus a similar Euler characteristic computation yields:

$$\sum_{w \in V \cap f^*(z)} (\deg_w Q_* - 1) = \#\{\text{critical points of } Q_*|_V\} = \deg(Q_*|_V) \cdot \chi(V/f^*) - \chi(V) \ .$$

Since $\chi(V/f^*) = \chi(U/f^*) = 1$ and $\deg(Q_*|_V) = \sum_{w \in V \cap f^*(z)} \deg_w Q_*$ we deduce that $\chi(V)$ equals $\#(V \cap f^*(z))$. Since V is connected and $V \cap f^*(z) \neq \emptyset$ we deduce both numbers are equal to 1.

Conversely, supposing that V is simply-connected and containing a unique element w of $f^*(z)$, the argument in (i) applied to V shows that w is the only possible critical point of $Q_*|_V$. It follows, again by the proof of Proposition 5 (subsection 2.5), that the separable correspondence $f^* \cap (U \times V)$ has singularities only on the orbit $f^*(z)$, whence the same is true of $f \cap (U \times V) \subset f^* \cap (U \times V)$.

(iii) The proof is completely analogous. □

Lemma 7 *If $z \in \Omega(f)$ then z has a very nice neighbourhood.*

Proof. Let U'' be the nice neighbourhood constructed in Lemma 3. For some k sufficiently large the set $\bigcup_{|e|<k} f^e(U'')$ covers all (forward and backward) singularities of f which are contained in $f^*(U'')$. Now as we shrink U'' down to $\{z\}$ (among simply-connected nice neighbourhoods) we see that $\bigcup_{|e|<k} f^e(U'')$ shrinks down towards the finite set $\bigcup_{|e|<k} f^e\{z\}$ and so eventually evades any singularities of f which are not in this finite set. It follows, by the equivalent definition of very nice neighbourhood, that U'' is now very nice. □

EXAMPLES
1. If a finitely generated Kleinian group G is regarded as the correspondence f having graph the union of the graphs of a finite set of generators of G, then $\Omega(f)$ is just the usual *regular set* $\Omega(G)$ in the sense of Kleinian groups.
2. If a rational map f is regarded as a correspondence, then $\Omega(f)$ consists of a union of components and punctured components of the Fatou set of f. To be precise, $\Omega(f)$ is made up of the basins of parabolic cycles of f and the basins of attractive (but not superattractive) periodic cycles, the latter punctured by the removal of the grand orbits of the attracting cycles themselves.

DEFINITION: The *(non-regular) limit set* $\Lambda(f)$, for a holomorphic correspondence f, is the complement of the regular set $\Omega(f)$.

3.2 Equicontinuity sets

Defining the equicontinuity set for a general correspondence is quite tricky. First we must define the notion of a branch of iteration along a path:

DEFINITION: A *path of global iteration* of f is a sequence $\underline{z}$ of points $z_0, z_1, \ldots, z_n$ in X and a sequence e of signs $e_1, \ldots, e_n$ in $\{-1, 1\}$ satisfying $(z_{i-1}, z_i) \in f^{e_i}$ for each $i = 1, \ldots, n$.

A primitive notion of a branch of iteration of f along a path ascribes to each connected (open) neighbouhood U_0 of z_0 a sequence of connected (open) neighbourhoods U_i of z_i given by: U_i is the component of $f^{e_i}(U_{i-1})$ which contains z_i.

This notion is good enough for the definition of the *regular* or *proper discontinuity* set (in the previous subsection) but tends to be too coarse in some other situations, with the resulting connected neighbourhood U_n of z_n too large. An example to bear in mind is behaviour in the basin of a superattractive periodic point. Consider the map $f : z \mapsto z^2$ and let z_0 be in the basin of 0. However small a neighbourhood U of z_0 we take, for sufficiently large n the image $f^n(U)$ will be an annulus surrounding 0 and hence $f^{-n}(f^n(U))$ will also be an annulus around 0, so that the path which takes $z_0 \to z_1 \to \ldots \to z_n$ and then retraces its steps back to z_0 will map U to this large annulus, rather than simply back to U as we would wish. To overcome this problem we revise our definition of a 'branch':

DEFINITION: Given a path $\underline{z} = z_0, z_1, \ldots, z_n$ where $(z_{i-1}, z_i) \in f^{e_i}$ and $e_i \in \{-1, 1\}$, the *branch of iteration* along the path, denoted $f^e_{\underline{z}}$, assigns to each (open) connected neighbourhood U of z a multi-valued map from U onto a connected neighbourhood of z_n, written $f^e_{\underline{z}}(U)$. The graph of this multi-valued map is obtained from the connected component $\hat{U}_n$ of $(f^{e_1} \wedge \ldots \wedge f^{e_n}) \cap (U \times X^n)$ which contains $\underline{z} = (z_0, z_1, \ldots, z_n)$, by projecting, via $\pi_0 \times \pi_n$, into $U \times f^e_{\underline{z}}(U)$ ($\subset U \times X$). In effect the connected component $\hat{U}_n$ of $(f^{e_1} \wedge \ldots \wedge f^{e_n}) \cap (U \times X^n)$ is a "de-singularized component" of the graph of the composition

$$f^{e_n}|_{U_{n-1}} \circ \ldots \circ f^{e_1}|_{U_0}$$

where $U_0 = U$ and each U_i is the component of $f^{e_i}(U_{i-1})$ which contains z_i, as in the primitive version. Assuming that f is an open and closed relation we have $\pi_0(\hat{U}_n) = U_0$ but note that in general $\pi_i(\hat{U}_n) \subset U_i$ without equality.

In the case that f is a holomorphic correspondence we may further desingularize (using the procedure outlined in Section 2.7) and in effect split $\hat{U}_n$ into algebraically irreducible components. Let $\tilde{U}_{n,\alpha}$ be the (desingularized) irreducible components which contain $\underline{z}$. Each $\tilde{U}_{n,\alpha}$ is an open (connected) Riemann surface which is a ramified cover over U, the component of $(\tilde{Q}_{-n}^{(f^{e_1} \wedge \ldots \wedge f^{e_n})})^{-1}(U)$ which contains $(\underline{z}, \alpha)$ in $(\tilde{Q}_{\triangle}^{(f^{e_1} \wedge \ldots \wedge f^{e_n})})^{-1}(\underline{z})$. Here π_0 induces the holomorphic projection $\tilde{Q}_{-n} = \pi_0 \circ \tilde{Q}_{\triangle}^{(f^{e_1} \wedge \ldots \wedge f^{e_n})}$. Likewise π_n induces the holomorphic projection $\tilde{Q}_n = \pi_n \circ \tilde{Q}_{\triangle}^{(f^{e_1} \wedge \ldots \wedge f^{e_n})}$ which maps $\tilde{U}_{n,\alpha}$ onto a neighbourhood $f^e_{\underline{z},\alpha}(U) = \tilde{Q}_n(\tilde{U}_{n,\alpha})$ of z_n. We call each $f^e_{\underline{z},\alpha}$ an *(irreducible) branch* of the iterated holomorphic correspondence f. Usually only one $\tilde{Q}_{\triangle}(\tilde{U}_{n,\alpha})$ will contain $\underline{z}$ but an example to bear in mind is that of a correspondence which consists of a finite union of Möbius transformations: where two sheets of the graph cross at (z_{n-1}, z_n) we regard them as separate branches.

To summarise, we have $U_n \supset \pi_n(\hat{U}_n) \supset \tilde{Q}_n(\tilde{U}_{n,\alpha}) \ni z_n$.

DEFINITION: The iterates of the holomorphic correspondence f at z_0 are said to be *equicontinuous* if $\forall \varepsilon > 0$ there exists $\delta > 0$ such that for all branches $f^e_{\underline{z},\alpha}$ along paths $\underline{z}$ starting at z_0

$$f^e_{\underline{z},\alpha}(B_\delta(z_0)) \subset B_\varepsilon(z_n)$$

The above definition of equicontinuity is not well-adapted to the task of developing a theory analogous to the classical Fatou-Julia theory for rational maps. The difficulty is that it is by no means clear what one should mean by saying that the (possibly multi-valued) branches of an iterated correspondence f form a *normal family* at z. However if the point z is such that only a finite degree of branching occurs on the grand orbit $f^*(z)$ we can adapt the definition for maps without too much difficulty, following an idea developed by Münzner and Rasch in their analysis of (bi-directional) iteration of algebraic functions [25]. Let U be a neighbourhood of z_0. As described above, associated with any (irreducible) branch of iteration $f^e_{\underline{z},\alpha}$ along a path $\underline{z}$ of length n emanating from z_0 is a ramified cover $\tilde{U}_{n,\alpha}$ of U equipped with a covering projection $\tilde{Q}^{\underline{z},\alpha}_{-n}$ ($= \pi_0 \circ \tilde{Q}_\triangle|_{\tilde{U}_{n,\alpha}}$) to U, and a forward projection $\tilde{Q}^{\underline{z},\alpha}_{n}$ ($= \pi_n \circ \tilde{Q}_\triangle|_{\tilde{U}_{n,\alpha}}$) to the Riemann sphere. Removing the final step from any path of iteration induces a factor map $\tilde{Q}_- : \tilde{U}_{n,\alpha} \to \tilde{U}_{n-1,\alpha}$ satisfying $\tilde{Q}^{(z_0,\ldots,z_{n-1}),\alpha}_{-(n-1)} \circ \tilde{Q}_- = \tilde{Q}^{(z_0,\ldots,z_n),\alpha}_{-n}$. Let $\tilde{U}$ be a finite ramified cover of U and z_* be a marked point in $\tilde{U}$. A collection of covering projections $\tilde{\pi}^{\underline{z},\alpha} : \tilde{U} \to \tilde{U}_{n,\alpha}$, one for each branch of iteration starting at z_0, is said to be *compatible* if each sends the marked point z_* to the canonical point $(\underline{z},\alpha) \in \tilde{U}_{n,\alpha}$ and

$$\tilde{\pi}^{(z_0,\ldots,z_{n-1}),\alpha} = \tilde{Q}_- \circ \tilde{\pi}^{(z_0,\ldots,z_n),\alpha}.$$

DEFINITION: A point z_0 is in the *normality set* $N(f)$ of f if there exists a neighbourhood U of z_0, a finite ramified cover $\tilde{U}$ of U, and a compatible set of covering projections $\tilde{\pi}^{\underline{z},\alpha}$ for the branches emanating from z_0, such that when lifted to $\tilde{U}$, where they become single-valued maps $\tilde{Q}^{\underline{z},\alpha}_n \circ \tilde{\pi}^{\underline{z},\alpha}$, these branches form a normal family.

From the classical result of complex analysis that a normal family of maps on a subset of the Riemann sphere is equicontinuous it follows at once that the normality set $N(f)$ of a holomorphic correspondence f is contained in the equicontinuity set. For a counterexample to the converse consider the critical value c of a quadratic map $z \mapsto z^2 + c$ where c is outside the Mandelbrot set (i.e. the critical value c is in the basin of the superattracting fixed point at infinity). There are points arbitrarily close to c with grand orbits passing through c and hence there is no way to construct a single ramified cover $\tilde{U}$ on which to lift all branches to single-valued maps. In fact in this example $N(f)$ omits a whole circle of points (those on the 'equipotential' of c) and its pre- and post-images: however under any reasonable definition of 'equicontinuity of branches' all these missing points are in the equicontinuity set.

The following definition should be regarded as provisional until such time as the questions above concerning the definitions of 'equicontinuity' and 'normality' are resolved.

DEFINITION: The *Julia set* $\mathcal{J}(f)$ of a holomorphic correspondence f is the complement of the equicontinuity set.

COMMENTS:

1. If f is a rational map then with this definition $\mathcal{J}(f)$ will contain the grand orbits of attractive and superattractive periodic points as well as those points in the conventional Julia set.
2. The *forwards equicontinuity set* of a correspondence f is defined in the obvious way by restricting attention to *forwards* branches of iteration. Its complement $\mathcal{J}_-(f)$ is the conventional Julia set when f is a rational map.
3. The regular set $\Omega(f)$ of a correspondence f is a subset of the equicontinuity set. In fact:

Theorem 5 *The regular set $\Omega(f)$ is a subset of the normality set $N(f)$.*

Proof. Take U_z to be a very nice neighbourhood of the point $z \in \Omega(f)$ (as in Lemma 7). Having defined $\deg_w^* = \#\mathrm{Stab}_{f^*}(w)$ for $w \in \Omega(f)$, define $\mathrm{LCM} = \mathrm{LCM}_{w\in f^*\{z\}} \deg_w^*$ (which will be finite since $\deg^*$ is bounded over $f^*\{z\}$) and write $\tilde{U}_z$ for a degree $\mathrm{LCM}/\deg_z^*$ cover of U_z ramified totally over z.

Now writing U_w as the components of $f^*(U_z)$ containing the respective $w \in f^*\{z\}$, the action of f induces a commuting system of isomorphisms between the local orbifolds $\{U_w/f^* : w \in f^*\{z\}\}$. Since U_w is a nice neighbourhood for each $w \in f^*\{z\}$ (by Lemma 6), it follows that pre-composing the isomorphism $U_z/f^* \to U_w/f^*$ by the projection $\tilde{U}_z \to U_z/f^*$ induces $\deg_w^*$ single-valued maps (lifts of the action of f) from $\tilde{U}_z$ to U_w.

It remains to observe that the union of all these single-valued maps taken over all $w \in f^*\{z\}$ is a normal family, since they are uniformly bounded. □

Corollary 5 *The orbit-quotient $\Omega(f)/f^*$ of the regular set is Hausdorff.*

Proof. This is equivalent to saying that the equivalence relation $f^* \cap (\Omega(f) \times \Omega(f))$ is closed (as a subset of $\Omega(f) \times \Omega(f)$). To prove this, consider any (z, w) in $(\Omega(f) \times \Omega(f)) \setminus f^*$. The point $\{w\}$ is disjoint from the closed (even discrete) subset $f^*\{z\}$ of $\Omega(f)$. By equicontinuity it follows that $f^*(U) \cap V = \emptyset$ for some neighbourhoods U of z and V of w. □

EXAMPLES:
1. For a rational map f, the equicontinuity set of f (regarded as a correspondence) consists of the Fatou set of f (regarded as a rational map) but with the grand orbit of any attractive or superattractive periodic orbit of f excised from it. In particular grand orbits of Siegel discs or Herman rings are in the equicontinuity set.
2. For a finitely generated Kleinian group G (regarded as a correspondence f) the equicontinuity set is equal to the regular set $\Omega(f)$, (which of course is itself the same as the regular set $\Omega(G)$ defined in the usual way for a Kleinian group).
3. For a critically finite correspondence (i.e. where the grand orbits of all critical points, forwards or backwards, are finite) [4],[5], the equicontinuity set contains the complement of the union of the critical orbits, provided this union contains at least three points.

Finally in this Section we propose the following conjecture, which we shall verify later in certain special cases (see the Comment at the end of Section 5).

CONJECTURE: For holomorphic correspondences $\partial\Lambda(f) \subset \mathcal{J}(f)$.

4 Directionalities, and fundamental sets for forward and bi-directional iteration

4.1 Directionalities

A set S defines a *directionality* for f $(\subset Z \times Z)$ if

$$f(\bar{S}) \subset S^{\circ}$$

or, in other words, if

$$(\overline{S} \times \overline{S^c}) \cap f = \emptyset.$$

This is thus equivalent to saying that S^c is a directionality for f^{-1}. In practice we shall exclude the trivial directionalities given by $S = \emptyset$ and $S = Z$.

Associated to a directionality is an "attractor-repeller pair" where the attractor and repeller are respectively given by

$$\omega_+(S) = \omega_+(f, S) := \bigcap_{n \to +\infty} f^n(S),$$

$$\omega_-(S) = \omega_-(f, S) := \omega_+(f^{-1}, S^c) = \bigcap_{n \to +\infty} (f^{-1})^n(S^c).$$

Note that we can replace S by any S' such that $S \subset S' \subset f(S)$ and obtain exactly the same attractor-repellor pair, so there is a notion of "equivalence" for directionalities. There is a duality between attractors and repellers (of directionalities) when Z is a compact Hausdorff space and f is a closed relation. The general theory is covered by McGehee's paper [20]. However we record here for later use some properties of the boundary $\partial\omega_+(f, S)$ of the forward attractor. First some terminology:

DEFINITION: We say that a subset S of Z is *backward complete* if $f^{-1}\{z\} \cap S \neq \emptyset$ for all $z \in S$, and that it is *forward invariant* if $f(S) = S$.

The notions of *forward complete* and *backward invariant* are defined analogously by exchanging the roles of f and f^{-1} in the definition above.

Note that as well as satisfying $f(S) \subset S$, forward invariant sets are necessarily backward complete, for the statement that $f^{-1}\{z\} \cap S \neq \emptyset$ for all $z \in S$ is just the inclusion $f(S) \supset S$.

Proposition 6 *The boundary of the attractor of a directionality for an open and closed relation f is backward-complete.*

Proof. Since the attractor $\omega_+(f,S)$ for a directionality S for a closed relation f is always backward-complete [20], it suffices, for the analogous property for $\partial\omega_+(f,S)$, to prove that all images w under f of a point $z \in (\omega_+(f,S))^\circ$ lie in $(\omega_+(f,S))^\circ$. By lower-semi-continuity of f^{-1} at such a w there exists a neighbourhood V such that for all $w' \in V$ there exists $z' \in (\omega_+(f,S))^\circ$ such that $w' \in f(z')$. It follows that $V \subset \omega_+(f,S)$ and w lies in the interior. □

Proposition 7 *The boundary of the attractor of a directionality S for an open and closed relation f is forward-invariant if f is injective on S (that is if $f\{z\} \cap f\{z'\} = \emptyset$ for $z, z' \in S$, with $z \neq z'$).*

Proof. Clearly, $f(\partial\omega_+(f,S)) \subset \omega_+(f,S)$. On the other hand, if $w \in (\omega_+(f,S))^\circ$ then, by injectivity and backward-completeness, $f^{-1}(w) \cap S = \{z\}$ for some $z \in \omega_+(f,S)$. We claim that, always, $z \in (\omega_+(f,S))^\circ$. By lower-semi-continuity of f at z, there exists a neighbourhood U of z such that for all $z' \in U$ there exists $w' \in f(z') \cap \omega_+(f,S)$. Choosing $U \subset S$, we have by injectivity, $f^{-1}(w') \cap S = \{z'\}$, whence by backward-completeness, $z' \in \omega_+(f,S)$. Thus $U \subset \omega_+(f,S)$ and z is in the interior. It follows therefore, that $f(\partial\omega_+(f,S)) \subset \partial\omega_+(f,S)$. Since we know the reverse inclusion, the result follows. □

Proposition 8 *The boundary of the attractor of a directionality for an open and closed relation f on a compact metric space is contained in $\mathcal{J}_+(f)$ (the backward non-equicontinuity set).*

Proof. Let the distance between $S \setminus f(S)$ and $\omega_+(f,S)$ be ϵ. Given any point $z_0 \in \partial\omega_+(f,S)$ and any δ there exists some point $z \in f^n(S) \setminus f^{n+1}(S)$ for some n, with the distance from z_0 to z less than δ. Now by the backwards completeness of $\partial\omega_+(f,S)$ the backwards orbit of z_0 has a branch remaining in $\partial\omega_+(f,S)$, but the corresponding branch of the backwards orbit of z has its nth point in $f(S)^c$ and therefore distance at least ϵ away from $\partial\omega_+(f,S)$. □

4.2 Fundamental sets for forward and bi-directional iteration

For the remainder of Section 4 we shall be working at the set-theoretic level, returning once again to the topological level in Section 5. A *set-directionality* for a correspondence f is a set S which merely satisfies

$$f(S) \subset S$$

DEFINITION: We say that S is *injective* as a set-directionality if $f\{z\} \cap f\{z'\} = \emptyset$ whenever $z, z' \in S$ with $z \neq z'$.

Note that while $\omega_+(f,S) = \bigcap f^n(S)$ is no longer an attractor in the topological sense, the following proposition is an immediate corollary of these definitions:

Proposition 9 *If S is an injective set-directionality for f then $F = S \setminus f(S)$ is a fundamental set for the forward iteration of f on $S \setminus \omega_+(f,S)$ in the sense that*

(F1): $S \setminus \bigcap_{n \geq 0} f^n(S) = \bigcup_{n \geq 0} f^n(F)$,

(F2): $F \cap f^n(F) = \emptyset$ for all $n > 0$,

(F3): for each $z \in \bigcup_{n>0} f^n(F)$ exactly one element of $f^{-1}\{z\}$ lies in $\bigcup_{n \geq 0} f^n(F)$. $\Box$

We now move on to the question of fundamental sets for *bi-directional* iteration, where from a given starting point we allow purely *forwards* and purely *backwards* but not *mixed* iteration. If T is a *transversal* to the single-valued map $Q : X \to Y$, that is a subset of X mapped bijectively by Q onto Y, then T is a *maximal* injective set for Q. Furthermore the image $\mathrm{Gal}^Q(T)$ equals $X \setminus T$. We shall see that a transversal is a special case of a more general concept, which is the key to the construction of bi-directional fundamental sets.

DEFINITION: We say that S is a *co-injective set* for a correspondence f if:
(CO1) S is a co-image - i.e. $f^{-1}\left((f(S))^c\right) = S^c$,
(CO2) S is injective for f,
(CO3) $(f(S))^c$ is injective for f^{-1}.

A co-injective set S for a correspondence f partitions the graph f into three sets:

$$CO_+ = \{(z,w) \in f : z \in S\} \ ,$$

$$CO_- = \{(z,w) \in f : w \in (f(S))^c\} \ ,$$

$$AMB = \{(z,w) \in f : z \in S^c \text{ and } w \in f(S)\} \ .$$

Thus π_+ maps CO_+ injectively onto $f(S)$ and π_- maps CO_- injectively onto S^c. Furthermore $\mathrm{Gal}^{(\pi_+|_f)}(CO_+) = AMB = \mathrm{Gal}^{(\pi_-|_f)}(CO_-)$. Observe also that $\pi_-(CO_+) = \pi_-(f) \cap S$ and that $\pi_+(CO_-) = \pi_+(f) \cap f(S)^c$.

Lemma 8 *Let f be of the form $M \circ \mathrm{Gal}^Q$ where M is a bijection. Then any set T which is transversal to Q is a co-injective set for f.*

Proof. The correspondence f maps T injectively onto $(M(T))^c$ and f^{-1} maps $M(T)$ injectively onto T^c. $\Box$

Given a point $z \in X$ define:

$$O_+(z) = O_+^{(f)}(z) := \bigcup_{n>0} f^n\{z\} \ ,$$

$$O_-(z) = O_-^{(f)}(z) := \bigcup_{n>0} f^{-n}\{z\} \ ,$$

$$O_\pm(z) = O_\pm^{(f)}(z) := \bigcup_{n \in \mathbf{Z}} f^n\{z\} = O_-(z) \cup \{z\} \cup O_+(z) \ .$$

We say a set U is a *bi-directional fundamental set* for a subset Ω of X if:

(U1) $\Omega = \bigcup_{n\in\mathbf{Z}} f^n(U)$,
(U2) $U \cap f^n(U) = \emptyset$ for all $n \neq 0$,
(U3) If $z \in \bigcup_{n<0} f^n(U)$ then exactly one element of $f\{z\}$ lies in $\bigcup_{n\leq 0} f^n(U)$,
(U4) If $z \in \bigcup_{n>0} f^n(U)$ then exactly one element of $f^{-1}\{z\}$ lies in $\bigcup_{n\geq 0} f^n(U)$.

Observe that conditions (U2), (U3) and (U4) are equivalent to the conditions:
(U2') $f^m(U) \cap f^n(U) = \emptyset$ whenever $m \neq n$.
(U3') $f^{-1}|_{f^{-n}(U)}$ is injective for all $n \geq 0$.
(U4') $f|_{f^n(U)}$ is injective for all $n \geq 0$.
The main point about bi-directional fundamental sets is the following:

Proposition 10 *For every $z \in \Omega$ the set $O_\pm(z)$ intersects U in exactly one point. If $z \notin \Omega$ then $O_\pm(z) \cap U = \emptyset$.* $\square$

One should note that, a priori, Ω is not fully-invariant under f and in particular that $z \in \Omega$ does not imply $O_\pm(z) \subset \Omega$.

DEFINITION: S is a *co-injective* set-directionality for f if S is a co-injective set and $f(S) \subset S$.

Proposition 11 *If S is a co-injective set-directionality for f then the set $U = S \setminus f(S)$ is a bi-directional fundamental set for f on the complement $\Omega(f,S)$ of $\bigcap_{n\to+\infty} f^n(S) \cup \bigcap_{n\to+\infty} f^{-n}(S^c)$.*

Proof. First observe that $U = S \setminus f(S) = S \cap f(S)^c = f(S)^c \setminus S^c = f(S)^c \setminus f^{-1}(f(S)^c)$. Now since S is an injective set-directionality for f and $f(S)^c$ is an injective set-directionality for f^{-1} the fact that U satisfies conditions (U1) to (U4) follows at once from Proposition 9. $\square$

If we call S the region of "forward convergence", condition (U3) says that for $z_0 \in S^c$ there is a unique forward "diverging" path (z_i) (where $z_{i+1} \in f\{z_i\}$) which terminates (i.e. enters S) (if at all) at a point of U. Thus the points z_0 which have un-ending forward "diverging" paths constitute precisely the "repeller" $\bigcap_{n\to+\infty} f^{-n}(S^c)$. Likewise calling $f(S)^c$ the region of "backward convergence", condition (U4) says that for $z_0 \in f(S)$ there is a unique backward "diverging" path (z_{-i}) (where $z_{-i-1} \in f^{-1}\{z_{-i}\}$) which terminates (i.e. enters $f(S)^c$) (if at all) at a point of U. The points z_0 which have un-ending backward "diverging" paths constitute precisely the "attractor" $\bigcap_{n\to+\infty} f^n(S)$.

$$\begin{tikzcd} & U \arrow[dr, "CO_+"] & \\ S^c \arrow[ur, "CO_-"] \arrow[rr, "AMB"'] \arrow[loop left, "CO_-"] & & f(S) \arrow[loop right, "CO_+"] \end{tikzcd}$$

For our applications the key example of a co-injective set-directionality and associated bi-directional fundamental set is given by the following (which follows at once from Lemma 8 and Proposition 11):

Lemma 9 *If $f = M \circ \mathrm{Gal}^Q$ and T is a transversal to Q with the additional property that $T \cup M(T) = X$, then T is a co-injective set-directionality for f and $T \cap M(T)$ is the associated bi-directional-fundamental set U for $\Omega(f,T)$.* $\square$

From Lemma 8 it is clear that co-injective sets are plentiful when f is of the form $M \circ \mathrm{Gal}^Q$. In general for holomorphic correspondences the existence or non-existence of co-injective sets is purely a feature of the *diagram condition*. Let f be a holomorphic correspondence satisfying a diagram condition G. A necessary condition for the existence of co-injective sets for f is the existence of a co-injective set for the *template correspondence* G. For example in the diagram template for an off-separable $3:3$ correspondence (subsection 2.9) the set S consisting of the singleton $\{z_1\}$ is a co-injective set and $(f(S))^c$ is the singleton $\{w_1\}$. This condition on templates is also a sufficient condition for the existence of a co-injective set for f provided there are no points where there is "collapse of the diagram template". Given a subset S of the left-vertices U which is co-injective under G we can form a "transversal" T which is a subset of X satisfying $T \cap i(U) = i(S)$ for a "left-assignment" $i : U \to X$ corresponding to each "generic" $z_0 \in X$. If there are instances of collapse of the diagram template we may be lucky and find that all of them also admit co-injective sets so that we can complete the "transversal" T as a co-injective set for f. However it is not hard to construct examples of collapses to diagrams not admitting co-injective sets - e.g. in the diagram for an off-separable $3:3$ correspondence if we identify z_1 with z_2, z_3 with z_4, w_1 with w_2 and w_3 with w_4 we obtain

$$\begin{tikzcd} z_1 = z_2 \arrow[r] \arrow[dr] & w_1 = w_2 \\ z_3 = z_4 \arrow[r] \arrow[ur] & w_3 = w_4 \end{tikzcd}$$

which no longer admits a co-injective set. To deal with this problem we resort to deleting from the graph of f the relevant points (z_0, w_0) where degeneracy of the diagram condition occurs, and to considering co-injective sets for the resulting correspondence. In particular although $f = M \circ \mathrm{Gal}^Q$ admits co-injective sets, its closure $\bar{f} = M \circ \overline{\mathrm{Gal}}^Q$ (with graph including (z, Mz) for multiple points of Q) will not do so in general. This technical difficulty will oblige us to carefully distinguish between f and $\bar{f}$ in a number of proofs below.

4.3 Relations between bi-directional orbits

Recall (Lemma 2, in subsection 2.9) that if f is M-off-separable then $M \circ \mathrm{Gal}^Q \subset f \subset M \circ (Q^{-1}Q)$ for some quotient map Q. One reason for studying off-separable correspondences is that there is some inter-relation of "bi-directional orbits" $O_\pm$:

Proposition 12 *If f is an M-off-separable correspondence then*

$$(z,w) \in f \;\Rightarrow\; O_\pm(z) \cup O_\pm(M(z)) = O_\pm(w) \cup O_\pm(M^{-1}(w))$$

Proof. Firstly see that

$$\{z\} \cup f^{-1}\{M(z)\} = \{z\} \cup \mathrm{Gal}^Q\{z\} = Q^{-1}Q(z) = Q^{-1}Q(M^{-1}(w))$$
$$= \mathrm{Gal}^Q\{M^{-1}(w)\} \cup \{M^{-1}(w)\} = f^{-1}\{w\} \cup \{M^{-1}(w)\}$$

and so taking the union over all backward images gives

$$O_-(z) \ \cup \{z\} \cup O_-(Mz) \ = O_-(w) \ \cup \{M^{-1}w\} \cup \ O_-(M^{-1}w) \ .$$

Secondly we see

$$f\{z\} \cup \{M(z)\} = M(\mathrm{Gal}^Q\{z\}) \cup \{M(z)\} = MQ^{-1}Q(z) = MQ^{-1}Q(M^{-1}(w))$$
$$= \{MM^{-1}(w)\} \ \cup \ M(\mathrm{Gal}^Q\{M^{-1}(w)\}) = \{w\} \cup f\{M^{-1}(w)\}$$

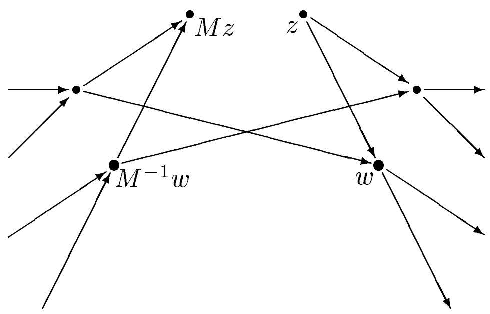
and so taking the union over all forward images gives

$$O_+(z) \ \cup \ \{Mz\} \cup O_+(Mz) \ = \ O_+(w) \ \cup \{w\} \cup O_+(M^{-1}w) \ .$$

Finally, observing that the union of the left-hand-sides equals the union of the right-hand-sides gives the result. □

Proposition 12 is valid for any M-off-separable correspondence, in particular for $f = M \circ \mathrm{Gal}^Q$ and its closure $\bar{f} = M \circ \overline{\mathrm{Gal}}^Q$. We now specialise to the situation of Lemma 9 and consider the possible paths of forward iteration from $M^{-1}w$ to w. Proposition 13 (below) is true for f but not for $\bar{f}$.

Proposition 13 *Let $f = M \circ \mathrm{Gal}^Q$, and T be a transversal to Q such that $T \cup M(T) = X$. Given $w \in f(T) \setminus \omega_+(f,T)$ such that $M^{-1}w \in T^c \setminus \omega_-(f,T)$, let z be the unique pre-image of w under f which is contained in T, so Mz is the unique image of $M^{-1}w$ which is contained in $M(T)$. Then any path of forward iteration from $M^{-1}w$ to w must go via Mz and z.*



Proof. Forward images of $M^{-1}w$ and of z, and backward images of w and of Mz are arranged as in the diagram above. Since the hypotheses of Lemma 9 are satisfied, and $z \in T \setminus \omega_+(f,T)$, we have $z \in f^{n-1}(T) \setminus f^n(T)$ for a unique $n \geq 1$.

Now $f\{M^{-1}w\} \cup \{w\} = \{Mz\} \cup f\{z\}$, from the proof of Proposition 12, and since $Mz \notin f\{z\}$ and $w \notin f\{M^{-1}w\}$ (as $z \notin \mathrm{Gal}^Q\{z\}$ and $M^{-1}w \notin \mathrm{Gal}^Q\{M^{-1}w\}$, by definition) we deduce that

$$f\{M^{-1}w\} \setminus \{Mz\} = f\{z\} \setminus \{w\} \subset f^n(T) \setminus \{w\}.$$

Thus any path of forward iteration from $M^{-1}w$ which does not pass through Mz goes to a point in $f^n(T) \setminus \{w\}$ at its first step, and then at subsequent steps it goes to points in $f^m(T), m > n$. So it cannot reach w. Hence a path of forward iteration from $M^{-1}w$ to w must pass through Mz on its first step. Similarly, since by hypothesis $Mz \in f^{-(n-1)}(MT) \setminus f^{-n}(MT)$ for some unique $n \geq 1$ (not necessarily the same n as before), and since $\{z\} \cup f^{-1}\{Mz\} = f^{-1}\{w\} \cup \{M^{-1}w\}$ (also from the proof of Proposition 12), we deduce

$$f^{-1}\{w\} \setminus \{z\} = f^{-1}\{Mz\} \setminus \{M^{-1}w\} \subset f^{-n}(MT) \setminus \{M^{-1}w\}$$

and so any path from $M^{-1}w$ to w must pass through z on its last step. □

Corollary 6 *Let J be an involution, $f = J \circ \mathrm{Gal}^Q$, and T be a transversal to Q such that $T \cup J(T) = X$. Then any path of forward iteration from $Jw \in T^c \setminus \omega_-(f,T)$ to $w \in f(T) \setminus \omega_+(f,T)$ must pass through a fixed point of J and this fixed point must lie in $T \cap J(T)$.*

Proof. Since $J = J^{-1}$ we can simply apply the Proposition repeatedly until we reach the level where $z \in T \setminus f(T) = T \cap J(T)$. □

5 'Fundamental domains' for global iteration of reversible off-separable correspondences

5.1 Reversible-off-separable correspondences

Suppose now that f is a *reversible-off-separable* correspondence, that is to say f is J-off-separable where J is an involution. This implies in particular that $f^{-1} = JfJ$. By Lemma 2 (subsection 2.9) there exists a quotient map Q such that $J \circ \mathrm{Gal}^Q \subset f \subset J \circ (Q^{-1}Q)$.

Given a point $x \in X$, we define

$$O(x) := O_\pm(x) \cup O_\pm(J(x)) \ .$$

Proposition 12 (in subsection 4.3) now becomes

$$(z, w) \in f \Rightarrow O(z) = O(w)$$

and this has the following consequence:

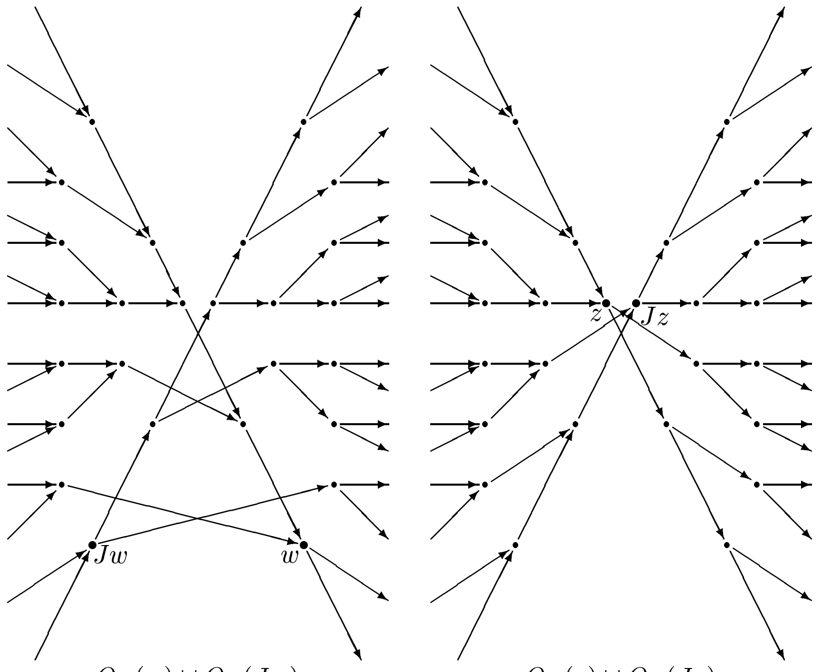
Theorem 6 *Let f be a reversible off-separable correspondence on a set X. Then (i) the grand orbit $f^*(x)$ of any $x \in X$ is contained in $O(x)$, and (ii) if the quotient map Q associated to f is of degree at least 3 at all but finitely many points then any infinite grand orbit $f^*(x)$ is equal to $O(x)$.*

Proof. By Proposition 12, if $(x, y) \in f$ then $O(x) = O(y)$. It follows at once that $f^*(x)$ is contained in $O(x)$ for any $x \in X$. Next, if $f^*(x)$ is infinite then the condition on Q in (ii) ensures that there is a point $y \in f^*(x)$ such that Jy is also in $f^*(x)$, since if $Q(y) = Q(y') = Q(y'')$ for y, y', y'' distinct then

$$J(y) \in f\{y'\} \subset (f \circ f^{-1})\{J(y'')\} \subset (f \circ f^{-1} \circ f)\{y\}.$$

Thus $O(y) = O_\pm(y) \cup O_\pm(Jy)$ is contained in $f^*(y)$ and therefore $f^*(y) = O(y)$. But $f^*(y) = f^*(x)$ and $O(y) = O(x)$ (by Proposition 12). □

Different choices of point x on a single grand orbit of f yield different partitions of that orbit into bi-directional orbits $O_\pm(x)$ and $O_\pm(J(x))$. The diagrams below illustrate a single grand orbit split into a bi-directional orbit and its J-twin in two different ways. Suppose that T is a transversal to Q such that $T \cup J(T) = X$, our grand orbit passes through $T \cap J(T)$, and in each diagram we use the convention that moving to the right moves one to a 'deeper' level $f^n(T)$ and moving to the left moves one to a 'deeper' level $f^{-n}(J(T))$. Then the points z and Jz (in the right-hand diagram) are the only two points on the grand orbit to lie in $T \cap J(T)$. As we shall see in Corollary 7, every grand orbit passing through $T \cap J(T)$ contains a unique such pair, or a point fixed by J.



$O_\pm(w) \cup O_\pm(Jw)$ $\qquad$ $O_\pm(z) \cup O_\pm(Jz)$

Corollary 7 *If f is as in Theorem 6(ii) and T is a transversal to Q such that $T \cup J(T) = X$, then any transversal Δ for the action of J on $T \cap J(T)$ is a transversal to the grand orbits of f passing through $T \cap J(T)$.*

Proof. We know from Lemma 9 that $T \cap J(T)$ is a fundamental set for the bi-directional orbits $\{O_\pm(z) : z \in T \cap J(T)\}$ of $J \circ \mathrm{Gal}^Q$ passing through it, and that these orbits are all infinite. In fact Corollary 6 of subsection 4.3 implies that any point z in $T \cap J(T)$ which is not fixed by J has $O_\pm(z)$ disjoint from $O_\pm(Jz)$. The statement that Δ is a transversal to the grand orbits of $J \circ \mathrm{Gal}^Q$ passing through Δ then follows from Theorem 6(i). The statement that Δ is a transversal to the grand orbits of $J \circ \mathrm{Gal}^Q$, and so of $J \circ \overline{\mathrm{Gal}}^Q$ passing through $T \cap J(T)$ follows from Theorem 6(ii). □

5.2 Transversal directionality for off-separable correspondences

We return to the topological level and consider correspondences of the form $M \circ \overline{\mathrm{Gal}}^Q$, where Q is a branched covering map on a compact Hausdorff space X. Any off-separable branched covering correspondence f on a compact Hausdorff space is of this type (Theorem 3, in subsection 2.9) and in particular any off-separable holomorphic correspondence on a compact Riemann surface is of this type, with Q holomorphic and M conformal (Corollary 3, same subsection).

DEFINITION: A *transversal directionality* for an off-separable branched covering correspondence $f = M \circ \overline{\mathrm{Gal}}^Q$ on a compact Hausdorff space X is a (topological) directionality $D \subset X$ for f such that D is a (set-theoretic) transversal to Q.

Proposition 14 *If f is an M-off-separable branched covering correspondence on a compact Hausdorff space X, then a transversal D for the associated map Q is a transversal directionality for f if and only if $D^\circ \cup M(D^\circ) = X$.*

Proof. By definition D is a transversal directionality if and only if $\overline{\mathrm{Gal}}^Q(\bar{D}) \subset M^{-1}(D^\circ)$, since M is a homeomorphism. Thus what we must show is that if D is a transversal to Q then $\overline{\mathrm{Gal}}^Q(\bar{D}) = X - D^\circ$. Since D is transversal to Q we know that $\mathrm{Gal}^Q(D) = X - D$, and we are reduced to showing that

$$\overline{\mathrm{Gal}}^Q(\bar{D}) - \mathrm{Gal}^Q(D) = D \cap \partial D$$

where ∂D is the boundary $\bar{D} - D^\circ$ of D. It will suffice to prove that

$$\overline{\mathrm{Gal}}^Q(\bar{D}) - \mathrm{Gal}^Q(\bar{D}) = \{x \in D \cap \partial D : \mathrm{Gal}^Q(x) \cap \partial D = \emptyset\}$$

and

$$\mathrm{Gal}^Q(\bar{D}) - \mathrm{Gal}^Q(D) = \{x \in D \cap \partial D : \mathrm{Gal}^Q(x) \cap \partial D \neq \emptyset\}.$$

But both these equalities follow from the following observations, all of which are quite straightforward to prove:

(i) $y \in \mathrm{Gal}^Q(x) \Leftrightarrow x \in \mathrm{Gal}^Q(y)$;
(ii) $\mathrm{Gal}^Q(D) = X - D$;
(iii) x is non-critical $\Leftrightarrow \overline{\mathrm{Gal}}^Q(x) = \mathrm{Gal}^Q(x)$;
(iv) x is critical $\Leftrightarrow \overline{\mathrm{Gal}}^Q(x) = \mathrm{Gal}^Q(x) \cup \{x\}$;
(v) $x \in D$ is critical $\Rightarrow x \in \partial D$;
(vi) $x \in \partial D$ is non-critical $\Rightarrow \mathrm{Gal}^Q(x) \cap \partial D \neq \emptyset$;
(vii) $x \in \partial D \Rightarrow \mathrm{Gal}^Q(x) \cap D^\circ = \emptyset$.

□

REMARKS

1. If f is any M-off-separable correspondence, then any (topological) directionality S for f such that $f(\bar{S}) \subset M(\bar{S})^c \subseteq S^\circ$ can be enlarged to a transveral D to Q such that $D^\circ \cup M(D^\circ) = X$.

2. The term *equivariant* directionality was used in [6],[7],[9] for what we now call a *transversal* directionality.

The notion of a transversal directionality for $f = M \circ \overline{\mathrm{Gal}}^Q$ can be defined in alternative ways at different 'levels'. Definitions II and III below are equivalent to the definition above, which we shall now refer to as *Definition I.*

DEFINITION II: A *transversal directionality* for f is a directionality D such that X is the disjoint union of $M(D)$ and $M \circ \mathrm{Gal}^Q(D)$.

DEFINITION III: A *transversal directionality* for f, satisfying Q has degree almost everywhere greater than 2, is a directionality which is the projection of a transversal for the (finite) correspondence generated by the covering correspondences $\widetilde{\mathrm{Gal}}_+$ and $\widetilde{\mathrm{Gal}}_-$ on the graph f.

For f holomorphic, Definition III excludes the case f is $1:1$. As a motivating example, consider a $2:2$ holomorphic M-off-separable correspondence f. The pair of covering involutions $\tilde{I}_-$, $\tilde{I}_+$ for the graph projections $\tilde{Q}_-$ and $\tilde{Q}_+$ generate an action of the symmetric group S_3 on the de-singularized graph $\Gamma(f)$ (it is easily checked that $\tilde{I}_+\tilde{I}_-\tilde{I}_+(z,w) = (M^{-1}w, Mz) = \tilde{I}_-\tilde{I}_+\tilde{I}_-(z,w)$). A directionality $D \subset X$ for f is a *transversal* directionality for f if and only if $D = \tilde{Q}_-(\Delta)$ for some transversal Δ for this S_3-action. This was the definition adopted in [6].

One may regard III as being the 'upstairs' definition (on the graph), II as on the 'ground floor' (the dynamical space X) and the original Definition I as in the 'basement' (since it arises from considering X/Q). It can be useful to switch levels to select the version of the definition most suitable to deal with a particular problem.

Figure 2a illustrates a transversal directionality for a $2:2$ correspondence on the real unit interval.

5.3 The main theorem

Theorem 7 *Let f be an $n : n$ J-off-separable branched covering correspondence on a compact Hausdorff space X, where $n \geq 2$ (almost everywhere) and J is an involution. If D is a transversal directionality for f then any transversal Δ for the action of J on $D \cap J(D)$ is a transversal for the (global) action of f on the complement $\Omega(f, D)$ of the global attractor $\omega(f, D) = \omega_+(f, D) \cup \omega_-(f, D)$ (where $\omega_+(f, D)$ denotes $\bigcap f^n(D)$ and $\omega_-(f, D)$ denotes $\bigcap f^{-n}(JD)$).*

If X is a Riemann surface and f is holomorphic then $\Omega(f, D)$ is contained in the regular set $\Omega(f)$ of f, and the various limit sets satisfy

$$\partial\omega(f, D) \subset \mathcal{J}(f) \subset \Lambda(f) \subset \omega(f, D) \ .$$

Proof. Write g for $J \circ \mathrm{Gal}^Q$ (so that $f = \bar{g}$). The graphs of $\overline{\mathrm{Gal}}^Q$ and Gal^Q only differ by the (finite) set of points $\{(x, x) : x \text{ critical for } Q\}$. As already noted in the proof of the proposition above, no critical point x of Q can lie in D°. Using the fact that $f(\bar{D}) \subset D^\circ$ we deduce that $\omega_+(f, D) = \omega_+(g, D)$. Similarly $\omega_-(f, D) = \omega_-(g, D)$. Hence $\Omega(f, D) = \Omega(g, D)$. But $\Omega(g, D)$ is the union of the bi-directional orbits of g passing through $D \cap J(D)$. Since Q has degree at least three, except at the (finitely many) branch points, we deduce that these orbits are identical (as sets) to grand orbits of f, and Δ is a transversal to them, by Theorem 6 and Corollary 7 (in subsection 5.1).

We next show that if f is a holomorphic correspondence then $\Omega(f, D)$ is contained in the regular set $\Omega(f)$ (as defined in Section 3). It is clear that if $U \subset D^\circ \cap J(D^\circ)$ then $f^* \cap (U \times U) \subset id \cup J$, whence $U \subset \Omega(f)$. Any boundary point p of $D \cap J(D)$ which is not a critical point of Q is easily dealt with by modifying D to put p inside D°. This just leaves the problem of critical points of Q on the boundary of D. The set $U = Q^{-1}Q(D \cap J(D))$ is a neighbourhood of each such critical point. But U is made up of a finite number of 'translates' of $D \cap J(D)$ and hence $f^* \cap (U \times U)$ is a union of finitely many $f^e \cap (U \times U)$, in other words U has only finitely many 'distinct returns'. This, together with complete invariance of $\Omega(f)$, establishes that $\Omega(f, D) \subset \Omega(f)$.

Finally the inequalities in the chain of limit sets are proved as follows:

$\Lambda(f) \subset \omega(f, D)$: from $\Omega(f, D) \subset \Omega(f)$;

$\mathcal{J}(f) \subset \Lambda(f)$: from Theorem 5 (subsection 3.2);

$\partial\omega(f, D) \subset \mathcal{J}(f)$: Proposition 8 (subsection 4.1).

□

5.4 Polynomial-like behaviour on limit sets

If $f = J \circ \overline{\mathrm{Gal}}^Q$ has a transversal directionality D which is a topological disc, then $f(D)(= J(D)^c)$ is also a disc. The single-valued branch of f^{-1} mapping $f(D)(\subset D)$ onto D will not be continuous in general, but its restriction to $f(D^\circ)$ will be. If f is holomorphic and $f(D^\circ)$ is a topological disc then this restriction is a *polynomial-like* mapping in the sense of [12]. It then follows from the Douady-Hubbard Straightening Theorem [12] that $\omega_+(f,D)$ is homeomorphic to the filled Julia set $\mathcal{K}_P$ of a polynomial map P, via a homeomorphism which is conformal on the interior of $\omega_+(f,D)$ and which extends to a quasiconformal conjugacy from f^{-1} on a neighbourhood of $\omega_+(f,D)$ to P on a neighbourhood of $\mathcal{K}_P$.

EXAMPLE: Let Q be any cubic rational map and J be an involution. Then $f = J \circ \overline{\mathrm{Gal}}^Q$ is given by $z \mapsto w$ where

$$\frac{Q(J(w)) - Q(z)}{J(w) - z} = 0 \; .$$

If Q has exactly one cubic critical point then (by conjugating f by an appropriate Möbius transformation) we can assume Q is of the form $z^3 - 3z$ so that f is given by the equation

$$(J(w))^2 + zJ(w) + z^2 = 3$$

and does not split. If there exists a tranversal directionality D then the Theorem above guarantees that f acts properly discontinuously on the complement of $\omega_+(f,D) \cup \omega_-(f,D)$. If D is a topological disc, the sets $\omega_+(f,D)$ and $\omega_-(f,D)$ are homeomorphic to filled Julia sets of quadratics (as explained above), and if $\omega_+(f,D)$ and $\omega_-(f,D)$ contain singular points of f they are copies of *connected* filled quadratic Julia sets (see [6]).

Figure 2b illustrates an example of the regular and limit set of a $2:2$ *reversible off-separable holomorphic correspondence* f *on the Riemann sphere, for which* $\omega_+(f,D)$ *and* $\omega_-(f,D)$ *are homeomorphic to filled quadratic Julia sets.*

In the limiting case when the directionality D degenerates into a 'contact directionality', so that instead of their being disjoint $\omega_+(f,D)$ meets $\omega_-(f,D)$ at a single point, the correspondence f becomes a 'mating of a quadratic map with the modular group $PSL(2,\mathbf{Z})$' [6], in that $\Omega(f,D)$ is homeomorphic to the upper half plane via a conformal bijection which conjugates the action of f to that of $PSL(2,\mathbf{Z})$.

Note that the $\Omega(f,D)$ need not be the whole of the *regular set* $\Omega(f)$ as defined in Section 3. If the corresponding quadratic maps $q_c : z \mapsto z^2 + c$ have parabolic or attracting (but not superattracting) periodic orbits, then the interior of $\omega(f,D)$ (less the grand orbit of the non-repelling periodic orbit itself) also forms part of the regular set $\Omega(f)$.

If Q has two cubic critical points then, by conjugating f by a suitable Möbius transformation, we can assume that Q is of the form $Q(z) = z^3$ so that f is given by the equation

$$(J(w))^2 + zJ(w) + z^2 = 0$$

and f splits into the pair of maps $z \mapsto J(e^{\pm 2\pi i/3}z)$. These maps generate a representation of $C_3 * C_2$. The condition that f has a transversal directionality D is the same as the condition that the cyclic subgroups $\langle z \mapsto e^{2\pi i/3}z\rangle$ and

$\langle z \mapsto J(z)\rangle$ have boundary-overlapping fundamental domains whose union is the whole Riemann sphere, and hence guarantees, by Klein's "Combination Theorem", that these subgroups freely generate a discrete faithful representation G of $C_3 * C_2$, and moreover that $\omega_+(f,D) \cup \omega_-(f,D)$ (which in this case is a Cantor set) is the *limit set* of the Kleinian group G.

REMARK: For examples like that in figure 2b, and for matings of non-parabolic quadratic maps with the modular group [6], or more generally, whenever there exists a D', not necessarily connected, satisfying the hypotheses of Theorem 7 and such that $\Omega(f,D') = \Omega(f)$, it follows from Theorem 7 that $\partial\Lambda(f) \subset J(f)$, confirming, in these cases, the conjecture at the end of Section 3.

6 'Fundamental domains' for global iteration of separable correspondences

For general correspondences, the global orbit of a point cannot be made out of finitely many "bi-directional" orbits $O_\pm$. However we may be able to recover the whole global orbit by a process of "global iteration" which amounts to iterating a new correspondence whose "bi-directional" orbits are multi-directional orbits of the original correspondence.

The global orbit of a point z_0 is made up of paths which are one-sided sequences $z_0, z_1, z_2, \ldots$ where, for each i, either $(z_i, z_{i+1}) \in f$ or $(z_{i+1}, z_i) \in f$. It is also convenient to consider paths which do not "back-track" so we exclude the possibility that $z_i = z_{i+2}$ where a change of direction of iteration has occurred at z_{i+1}.

In the case where f is separable we can further restrict to paths which do not admit two reversals of direction of iteration in immediate succession. This is because (for f separable):

$$(z_i, z_{i+1}), (z_{i+2}, z_{i+1}), (z_{i+2}, z_{i+3}) \in f \;\Rightarrow\; (z_i, z_{i+3}) \in f \;;$$

$$(z_{i+1}, z_i), (z_{i+1}, z_{i+2}), (z_{i+3}, z_{i+2}) \in f \;\Rightarrow\; (z_{i+3}, z_i) \in f \;.$$

In effect, we are considering orbits generated by applications of words in f and f^{-1} which do not contain either of the subwords f, f^{-1}, f or f^{-1}, f, f^{-1}. Such words can be uniquely "bracketed": that is, replaced by words in f, f^{-1}, (f^{-1}, f) and (f, f^{-1}) such that every occurrence of f^{-1}, f has been replaced by (f^{-1}, f) and every occurrence of f, f^{-1} has been replaced by (f, f^{-1}). We can now incorporate the original "no back-tracking" restriction by replacing (f^{-1}, f) and (f, f^{-1}) by the "Galois" correspondences:

$$\mathrm{Gal}_+^{(f)} : z \mapsto z' \;\Leftrightarrow\; \exists w, (z,w), (z',w) \in f, z \neq z' \;;$$

$$\mathrm{Gal}_-^{(f)} : w \mapsto w' \;\Leftrightarrow\; \exists z, (z,w), (z,w') \in f, w \neq w' \;,$$

Thus we are considering orbits which are generated by applications of certain "allowed" words in f, f^{-1}, $\mathrm{Gal}_+^{(f)}$, $\mathrm{Gal}_-^{(f)}$. The process of iteration by such words is best described by keeping track of which "mode" of iteration we are in. These are "forward mode" $\{+\}$ and "backward mode" $\{-\}$, and can be incorporated into the dynamics by means of a double cover $Z^{\pm} = Z^{+} \cup Z^{-}$ of the dynamic space Z.

Global iteration (for a separable correspondence f) is thus equivalent to iteration of a new correspondence $\mathrm{DIR} + \mathrm{GAL}$, acting on $Z^{\pm}$, defined by:

$$\mathrm{DIR} : (z, +) \mapsto (w, +) \;\Leftrightarrow\; f : z \mapsto w \;;$$

$$\mathrm{DIR} : (z, -) \mapsto (w, -) \;\Leftrightarrow\; f^{-1} : z \mapsto w \;;$$

$$\mathrm{GAL} : (z, +) \mapsto (w, -) \;\Leftrightarrow\; \mathrm{Gal}_{+}^{(f)} : z \mapsto w \;;$$

$$\mathrm{GAL} : (z, -) \mapsto (w, +) \;\Leftrightarrow\; \mathrm{Gal}_{-}^{(f)} : z \mapsto w \;.$$

The global orbit (for separable f) of a point z_0 will then be given by the union of the Z-projections of the forward images under $\mathrm{DIR} + \mathrm{GAL}$ of the two points $(z_0, +)$ and $(z_0, -)$. (For f not separable the above union will only give a subset of the global orbit of z_0.)

6.1 Bi-injective directionality

Here we assume that f is a closed relation on a compact Hausdorff space Z.

DEFINITION: S defines a *bi-injective directionality* for f (not necessarily separable) if $f(\overline{S}) \subset S^{\circ}$, f restricted to a neighbourhood of $\overline{S}$ is injective and f^{-1} restricted to a neighbourhood of $\overline{S^c}$ is injective.

Lemma 10 *If S is a bi-injective directionality for f then the image of $\overline{S}$ under the closure of $\mathrm{Gal}_{+}^{(f)}$ is disjoint from $\overline{S}$. Similarly the image of $\overline{S^c}$ under the closure of $\mathrm{Gal}_{-}^{(f)}$ is disjoint from $\overline{S^c}$.*

Proof. The graph of $\mathrm{Gal}_{+}^{(f)}$ is the set of pairs (z, z') with $z \neq z'$ such that there exists w with both (z, w) and (z', w) in the graph of f. Note that the closure of $\mathrm{Gal}_{+}^{(f)}$ may contain points on the diagonal (corresponding to "fixed points" of $\mathrm{Gal}_{+}^{(f)}$). Consider a point (z, z') with $z, z' \in \bar{S}$. This has a neighbourhood $U \times U'$ which, by the injectivity of f, intersects the graph of $f^{-1} \circ f$ only in the diagonal and hence intersects $\mathrm{Gal}_{+}^{(f)}$ nowhere. Similarly we obtain for any two points w,w' in $\overline{S^c}$ respective neighbourhoods V,V' satisfying $(V \times V') \cap f \circ f^{-1} \subset \mathrm{id}$. $\square$

Proposition 15 *If S defines a bi-injective directionality for f then $(S \times \{+\}) \cup (S^c \times \{-\})$ defines a directionality for $\mathrm{DIR} + \overline{\mathrm{GAL}}^{(f)}$.*

Proof. To say that $(S \times \{+\}) \cup (S^c \times \{-\})$ is a directionality is to say that $\mathrm{DIR} + \overline{\mathrm{GAL}}^{(f)}$ has no transitions from $(\overline{S} \times \{+\}) \cup (\overline{S^c} \times \{-\})$ to $(\overline{S^c} \times \{+\}) \cup (\overline{S} \times \{-\})$. This follows because:

$$(\overline{S} \times \overline{S^c}) \cap f = \emptyset \ ,$$

$$(\overline{S^c} \times \overline{S}) \cap f^{-1} = \emptyset \ ,$$

$$(\overline{S} \times \overline{S}) \cap \overline{\mathrm{Gal}}_+^{(f)} = \emptyset \ ,$$

$$(\overline{S^c} \times \overline{S^c}) \cap \overline{\mathrm{Gal}}_-^{(f)} = \emptyset \ .$$

□

6.2 Bi-injective directionality for separable correspondences

Proposition 16 *If f is a separable correspondence on a compact Hausdorff space Z, with graph given by $\{(z,w) : Q_+(z) = Q_-(w)\}$ where Q_+ and Q_- are continuous maps onto a Hausdorff "separating" space Y, then f has a bi-injective directionality if and only if there exist sets F and B with the properties that*

(i) F° and B° have union the whole dynamical space;

(ii) $Q_+|_F$, $Q_-|_B$ are injective, and

(iii) $Q_+(F)$ and $Q_-(B)$ form a partition of Y.

Proof. Given sets F and B satisfying the above we can endow f with a bi-injective directionality merely by choosing the set S to satisfy:

$$\overline{B^c} \subset S^\circ \subset \overline{S} \subset F^\circ$$

and observing that injectivity of f is equivalent to that of Q_+ (and likewise that injectivity of f^{-1} is equivalent to that of Q_-).

Conversely, given S, a bi-injective directionality for the correspondence $z \mapsto w$, $Q_+(z) = Q_-(w)$, we get $Q_+(\overline{S})$ disjoint from $Q_-(\overline{S^c})$. Choose a subset D of the "separating" space Y which satisfies:

(1) D is a neighbourhood (in $image(Q_+)$) of $Q_+(\overline{S})$;

(2) D^c is a neighbourhood (in $image(Q_-)$) of $Q_-(\overline{S^c})$.

It follows that $Q_+^{-1}(D)$ is a neighbourhood of $\bar{S}$ and that $Q_-^{-1}(D^c)$ is a neighbourhood of $\overline{S^c}$. Since Q_+ is injective wherever f is, it is possible to choose a branch of inverse of Q_+ which is defined on D and whose image F contains a neighbourhood of $\bar{S}$. Likewise, since Q_- is injective wherever f^{-1} is, one can choose a branch of inverse of Q_- which is defined on D^c and whose image B contains a neighbourhood of $\overline{S^c}$. □

Figure 3a illustrates the graph of a $2:2$ separable correspondence on the real unit interval, and sets F and B satisfying the hypotheses of Proposition 16.

Consider $Q^{(f)} : X = Z^+ \cup Z^- \to Y$ defined by:

$$Q^{(f)}(x) = \begin{cases} Q_+(z) & \text{if } x = (z,+) \\ Q_-(w) & \text{if } x = (w,-) \end{cases} .$$

The fundamental observation is that $\mathrm{DIR} + \mathrm{GAL}^{(f)}$ is none other than $\mathrm{id}_\pm \circ \mathrm{Gal}^{Q^{(f)}}$, where $\mathrm{id}_\pm$ is the involution of X which exchanges $(z,+)$ with $(z,-)$. Furthermore the above Proposition shows that a bi-injective directionality for f gives rise to a transversal for $Q^{(f)}$ which is also a (set-theoretic) directionality for $\mathrm{id}_\pm \circ \mathrm{Gal}^{Q^{(f)}}$.

Theorem 8 *If S is a bi-injective directionality for a separable correspondence f satisfying the hypotheses of the above Proposition then the directionality $(S \times \{+\}) \cup (S^c \times \{-\})$ for $\mathrm{DIR} + \overline{\mathrm{GAL}}^{(f)}$ is itself bi-injective. Furthermore the sets F and B of the Proposition intersect in a fundamental domain $F \cap B$ for the global action of f on the complement $\Omega(f,S)$ of the "global attractor" $\omega(f,S)$ which is the projection of the attractor (or repeller) associated with the directionality on $Z \times \{+,-\}$.*

If Z is a Riemann surface and f is holomorphic then $\Omega(f,S)$ is contained in the regular set $\Omega(f)$ of f, and the various limit sets satisfy

$$\partial\omega(f,S) \subset \mathcal{J}(f) \subset \Lambda(f) \subset \omega(f,S) .$$

Proof. By the preceding observation, the sets F and B constructed above form a set $T := F^+ \cup B^-$ (where F^+ denotes $F \times \{+\}$ and B^- denotes $B \times \{-\}$), transversal to $Q^{(f)}$, and since

$$T^\circ \cup \mathrm{id}_\pm(T^\circ) = (F^+ \cup B^-)^\circ \cup (B^+ \cup F^-)^\circ = (F^\circ \cup B^\circ)^\pm = Z^\pm$$

the set T satisfies the conditions of Lemma 9 (subsection 4.2) for the correspondence $\mathcal{F} = \mathrm{id}_\pm \circ \mathrm{Gal}^{Q^{(f)}}$. Hence the conclusion follows that $T \cap \mathrm{id}_\pm(T)$ is a bi-directional-fundamental set for the complement of $\omega_+(\mathcal{F},T) \cup \omega_-(\mathcal{F},T)$ where $\omega_+(\mathcal{F},T) = \bigcap_{n\to\infty} \mathcal{F}^n(T)$ and $\omega_-(\mathcal{F},T) = \bigcap_{n\to\infty} \mathcal{F}^{-n}(\mathrm{id}_\pm(T))$. The global orbit under f of a point z is given by the union of Z-projections of the orbits $O_+^{(\mathcal{F})}(z,+)$ and $O_+^{(\mathcal{F})}(z,-)$ (and hence by the Z-projection of the orbit $O_\pm^{(\mathcal{F})}(z,+)$) under $\mathcal{F} = \mathrm{DIR} + \mathrm{GAL}^{(f)}$. By Proposition 10 (subsection 4.2), $O_\pm^{(\mathcal{F})}(z,+)$ has a unique point in the "bi-directional fundamental set"

$$T \cap \mathrm{id}_\pm(T) = (F^+ \cup B^-) \cap (F^- \cup B^+) = (F \cap B)^\pm$$

provided that $(z,+)$ does not lie in either the attractor $\omega_+(\mathcal{F},T)$ or the repeller $\omega_-(\mathcal{F},T) (= \mathrm{id}_\pm(\omega_+(\mathcal{F},T)))$. Thus the global orbit of z under f has a unique point in $F \cap B$ unless z lies in the *global attractor* which is the Z-projection of $\omega_+(\mathcal{F},T)$ (or of $\omega_-(\mathcal{F},T)$).

Since $\mathcal{F}(T) = Z^{\pm} \setminus \mathrm{id}_{\pm}(T) = (B^c)^+ \cup (F^c)^-$ the construction in the preceding Proposition gives us

$$\overline{\mathcal{F}(T)} \subset \Sigma^\circ \subset \overline{\Sigma} \subset T^\circ$$

where $\Sigma = S^+ \cup (S^c)^-$ is the directionality of $\mathrm{DIR} + \overline{\mathrm{GAL}}^{(f)} = \overline{\mathcal{F}}$ provided by Proposition 15. This directionality is itself bi-injective because T is mapped injectively by $Q^{(f)}$ and hence by $\mathrm{id}_{\pm} \circ \overline{\mathrm{Gal}}^{Q^{(f)}} = \overline{\mathcal{F}}$ and we obtain that $\omega_+(\mathcal{F}, T)$ equals the attractor $\omega_+(\overline{\mathcal{F}}, \Sigma)$ and similarly for the repeller. When f is a branched covering correspondence then T is a *transversal directionality* for $\mathrm{id}_{\pm} \circ \overline{\mathrm{Gal}}^{Q^{(f)}}$ (in the sense of subsection 5.2, by Proposition 14). When f is holomorphic we obtain (by Theorem 7, subsection 5.3) that the global attractor denoted $\omega(f, S)$ satisfies the desired series of inclusions. □

Proposition 17 *If f is a separable correspondence with a bi-injective directionality S then contained in the global attractor $\omega(f, S)$ are two disjoint completely-invariant sets, one generated by the attractor $\omega_+(f, S)$, and the other generated by the repeller $\omega_-(f, S)$.*

Proof. It will suffice to show that forward iterating $\mathcal{F} = \mathrm{DIR} + \mathrm{GAL}^{(f)}$ from any point $z^+ = (z, +)$ or $z^- = (z, -)$, where $z \in \omega_-(f, S)$, yields an orbit which fails to intersect $(\omega_+(f, S))^+$ or $(\omega_+(f, S))^-$.

The forward completeness under f of $\omega_-(f, S)$ ($\subset Z \setminus F$) produces from z^+ a path of forward iteration under $\mathcal{F}$ entirely contained in $(\omega_-(f, S))^+ \subset Z^+ \setminus F^+ = Z^+ \setminus T$. Thus we have a non-terminating "diverging" path from z^+ under $\mathcal{F}$; in other words z^+ lies in the repeller $\omega_-(\mathcal{F}, S^+ \cup (S^c)^-)$ for $\mathcal{F}$. In our case the non-terminating "diverging" path from z^+ proceeds via a branch of DIR at every stage.

Consider a forward path under $\mathcal{F}$ starting from z^+ which enters $\mathcal{F}(T)$. Then, by uniqueness of "diverging" paths, we know that the last point of our path to lie outside $\mathcal{F}(T)$ must lie in $(\omega_-(f, S))^+ \subset Z^+ \setminus T$ and hence (without loss of generality) we may now write z^+ for this point. The next point of the path is either a point w'^+ with $w' \in \mathrm{Gal}_-^{(f)}(w)$ or a point z'^- with $z' \in f^{-1}(w)$, where in both cases $w \in f(z)$ is the unique point of $f(z)$ such that w^+ lies outside $\mathcal{F}(T)$ (in other words w^+ is the next point after z^+ on the "diverging" path described above).

Now $w^- \in B^- \subset T \setminus (\omega_+(f, S))^+$. But both w'^+ and z'^- lie in $\mathcal{F}(w^-)$, so they both lie in $\mathcal{F}(T \setminus (\omega_+(f, S))^+)$. If we can show that

$$(*) \qquad \mathcal{F}(T \setminus (\omega_+(f, S))^+) \subset T \setminus (\omega_+(f, S))^+$$

it will follow that the forward orbit of z^+ under $\mathcal{F}$ cannot reach $(\omega_+(f, S))^+$. But

$$\mathcal{F}(T \setminus (\omega_+(f, S))^+) = \mathcal{F}(T) \setminus (\omega_+(f, S))^+$$

because $\mathcal{F}$ restricted to T is injective (T being transversal to $Q^{(f)}$) and because $\omega_+(f, S)$ is backward complete under f. Since

$$\mathcal{F}(T) \setminus (\omega_+(f, S))^+ \subset T \setminus (\omega_+(f, S))^+$$

we are done.

Next observe that the forward orbit of z^+ under $\mathcal{F}$ is also disjoint from $(\omega_+(f,S))^-$, since the latter is disjoint from T and the unique forward path from z^+ which remains outside T stays in $(\omega_-(f,S))^+$, which is disjoint from $(\omega_+(f,S))^-$. Finally we turn our attention to the forward orbit under $\mathcal{F}$ of z^-, where z is still any point in $\omega_-(f,S)$. Such a point z^- lies in $B^- \subset T \setminus (\omega_+(f,S))^+$, so $(*)$ shows that the forward orbit of z^- is disjoint from $(\omega_+(f,S))^+$. It is also disjoint from $(\omega_+(f,S))^-$ since the latter does not meet T. $\Box$

Figure 3b illustrates the global orbits of $\omega_+(f,S)$ and of $\omega_-(f,S)$ for an example of a $2:2$ correspondence equipped with a bi-injective directionality. In this example $\omega_+(f,S)$ is a single 'heart', $\omega_-(f,S)$ is a single (Cantor) 'worm' and $\omega(f,S)$ is the closure of the union of all the hearts and worms. Similar examples can be constructed with any specified pair of 'quadratic-like' filled Julia sets for $\omega_\pm(f,S)$ (see [28]).

NOTE: The closures of the global orbits of $\omega_+(f,S)$ and of $\omega_-(f,S)$ need not be disjoint, nor need the union of their closures be the whole of $\omega(f,S)$, even for $2:2$ separable correspondences ("maps of pairs") of the Riemann sphere. However in the $2:2$ case, with the additional condition that S is a topological disc with boundary a Jordan curve, more can be said (see [28]). In the general case, with no such condition, the best we can hope for is:

CONJECTURE (see [28]): For separable holomorphic correspondences f with a bi-injective directionality S of any Markov-type, the closure of $f^*(\partial\omega_-(f,S)) \cup f^*(\partial\omega_+(f,S))$ equals $\partial\omega(f,S)$.

6.3 Bi-injective directionality for reversible separable correspondences

We conclude with an examination of a special class of separable correspondences f where the 'DIR+GAL' construction can be adapted to define a new reversible-off-separable correspondence $\mathcal{F}$ on the dynamical space Z itself, instead of requiring a double cover.

We say that a correspondence f is *J-reversible* if J is an involution on Z such that $f^{-1} = J \circ f \circ J$. If moreover f is separable, say $f = Q_-^{-1}Q_+$, then using the fact that J conjugates Gal^{Q_+} to Gal^{Q_-} it is a straightforward algebraic exercise to see that J passes to an involution $\underline{J}$ on the separating space Y, well-defined by $\underline{J}(y) = Q_-JQ_+^{-1}(y) = Q_+JQ_-^{-1}(y)$. Let $q_{\underline{J}}$ denote the quotient map $Y \to Y/\underline{J}$. The pair of sets $(D, J(D))$ is called a *transversal (bi-injective) directionality* for f if and only if

(i) $D^\circ \cup J(D^\circ) = Z$ (the dynamical space);

(ii) D is a transversal for $q_{\underline{J}} \circ Q_+$.

These conditions imply that $(D, J(D))$ is a bi-injective directionality in the sense of subsections 6.1 and 6.2, with $F = D$ and $B = J(D)$ satisfying (i)-(iii) of Proposition 16.

REMARK: As in the case of a transversal directionality, equivalent definitions can be given at different levels. The above is the 'sub-basement' definition, which we shall call 'Definition 0'. We give 'basement', 'ground floor' and 'upstairs' definitions for the special case of f a branched covering $2:2$ correspondence:

DEFINITION I: $(D, J(D))$ with $D^\circ \cup J(D^\circ) = Z$ (the dynamical space) is a transversal bi-injective directionality if and only if Q_+ is injective on D, and $Q_+(D)$ is the complement of $Q_-(J(D))$ in the separating space, modulo fixed points of $\underline{J}$.

DEFINITION II: $(D, J(D))$ with $D^\circ \cup J(D^\circ) = Z$ is a transversal bi-injective directionality if and only if $J(D)$ is a transversal for the action of I_- on the complement of $f(D)$, modulo points $x \in D$ such that $J(x) \in f(x)$. (In particular this implies that Z is the disjoint union of $f(D), J(D)$ and $I_-J(D)$, modulo a finite set of points).

DEFINITION III: $(D, J(D))$ with $D^\circ \cup J(D^\circ) = Z$ is a transversal bi-injective directionality if and only if $D = \tilde{Q}_-(\Delta)$ for a transversal Δ of the action of the dihedral group $< \tilde{I}_+, \tilde{I}_-, \tilde{J} >$ of order 8 on the graph of f, where $\tilde{J} : (z, w) \to (Jw, Jz)$.

Figure 4a illustrates a transversal (bi-injective) directionality for a $2:2$ correspondence on the real unit interval.

When f is a J-reversible separable correspondence on Z, say $f = Q_-^{-1} \circ Q_+$ (with $Q_+ = \underline{J}Q_-J$), the 'DIR+GAL' construction gives us a new correspondence $\mathcal{F}$ on X, defined by $\mathcal{F} = f \cup (J \circ \overline{\mathrm{Gal}}^{Q_+})$. If f is $n : n$, then $\mathcal{F}$ is $2n-1 : 2n-1$.

Figure 4b illustrates the $3:3$ correspondence $\mathcal{F}$ associated to the reversible separable correspondence f illustrated in figure 4a.

Proposition 18 *If f is a J-reversible and separable branched covering correspondence with $J \cap f$ finite, then $\mathcal{F}$ is a J-off-separable branched covering correspondence. If $(D, J(D))$ is a transversal bi-injective directionality (in the sense defined above) for f, then D is a transversal directionality (in the sense of Section 5) for the reversible-off-separable correspondence $\mathcal{F}$.*

Proof

$$\mathcal{F} = f \cup (J \circ \overline{\mathrm{Gal}}^{Q_+}) = (J \circ Q_+^{-1} \underline{J} Q_+) \cup (J \circ \overline{\mathrm{Gal}}^{Q_+})$$

$$= J \circ (Q_+^{-1} \underline{J} Q_+ \cup \overline{\mathrm{Gal}}^{Q_+}) = J \circ \overline{\mathrm{Gal}}^{q_{\underline{J}} Q_+}$$

The assertion that D is a transversal directionality for $\mathcal{F}$ is now just a reinterpretation of the definition of a transversal bi-injective directionality for f. $\square$.

Assuming that Q_+ is a branched covering map of degree at least 2 and that J is the covering involution of a branched covering map (both of which are the case when f is an $n : n$ holomorphic correspondence with $n \geq 2$), we can now apply Theorem 7 and deduce:

Corollary 8 *Modulo (the finite set of) points z such that $Jz \in f^{\pm 1}(z)$, the intersection $D \cap J(D)$ is a transversal for the (global) action of f on the complement $\Omega(f, D)$ of the "global attractor" $\omega(f, D)$.*

Proof
The 'DIR+GAL' construction of $\mathcal{F}$ gives us the following relationship between global orbits:

$$\mathcal{F}^*\{z\} = f^*\{z\} \cup f^*\{Jz\}$$

but any transversal for the action of J on $D \cap J(D)$ is a transversal for the global action of $\mathcal{F}$ (since D is a transversal directionality for the J-off-separable correspondence $\mathcal{F}$). Hence, modulo the finite set stated, $D \cap J(D)$ is a transversal for the global action of f on $\Omega(f, D)$. $\square$

Figure 4c illustrates a particular example. Here $\Omega(f, D)$ is the complement of the closure $\omega(f, D)$ of the union of the (filled-in) 'hearts'.

See [9] for the generalisation of the corollary above to "contact directionalities", in the context of correspondence perturbations of circle-packing Kleinian groups.

FINAL COMMENT: Since every correspondence $f = \tilde{Q}_+ \circ \tilde{Q}_-^{-1}$ is semi-conjugate to a separable correspondence, namely the lift of f which is the graph correspondence $\tilde{Q}_-^{-1} \circ \tilde{Q}_+$, in principle the methods of Section 6 are applicable to the analysis of correspondences in general. However in practice it is comparatively rare for a graph correspondence to have a bi-injective directionality, and the best way to proceed may be to apply a combination theorem suited to the particular class of correspondences under consideration.

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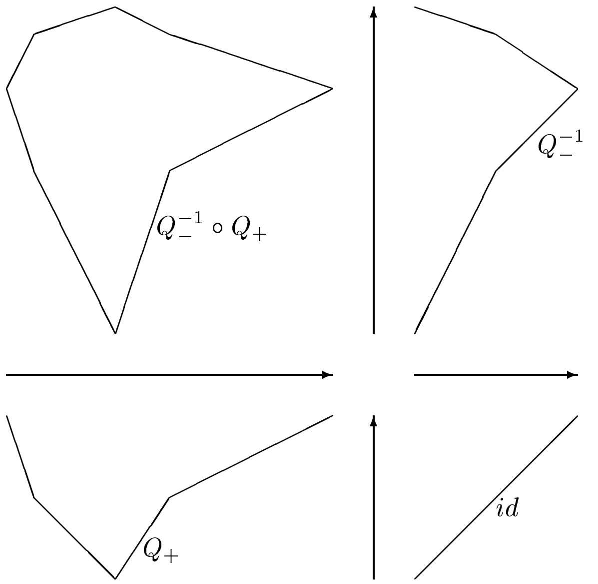


Figure 1a
The separable correspondence formed by co-domain identification of Q_+ and Q_-.

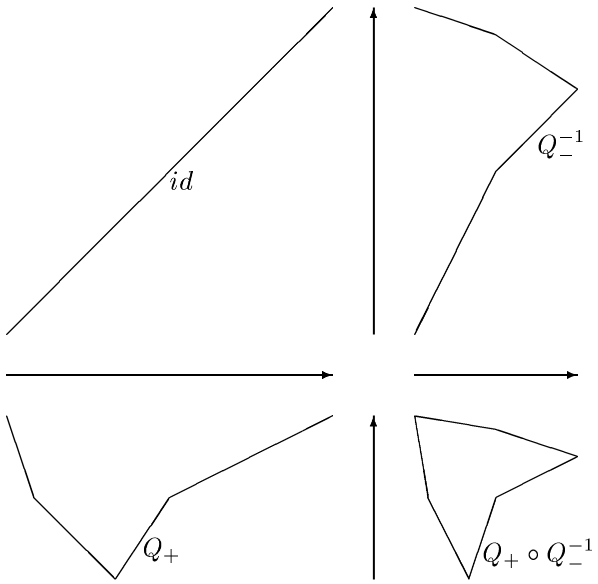


Figure 1b
The product correspondence formed by domain identification of Q_- and Q_+.

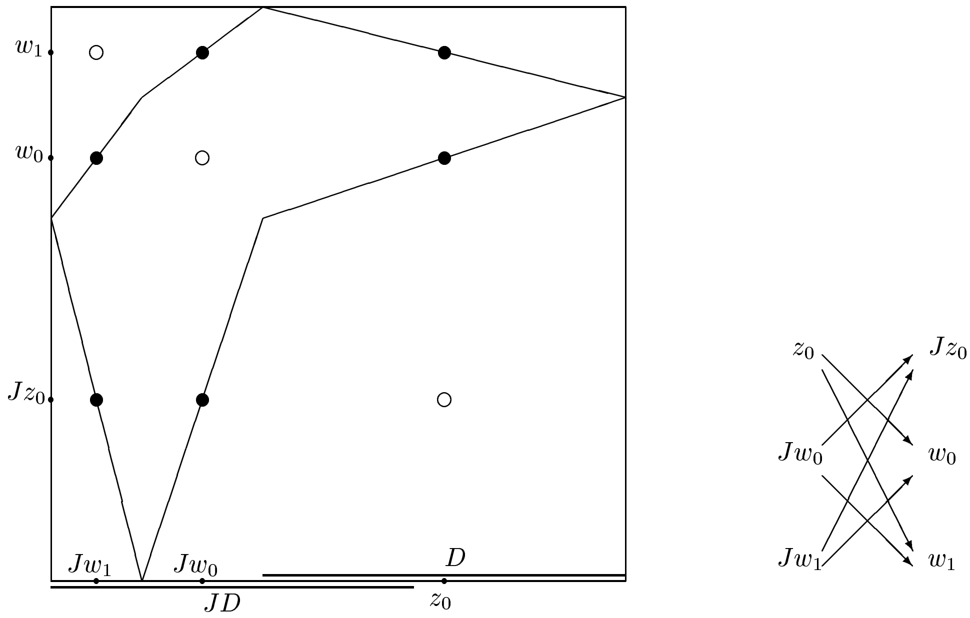
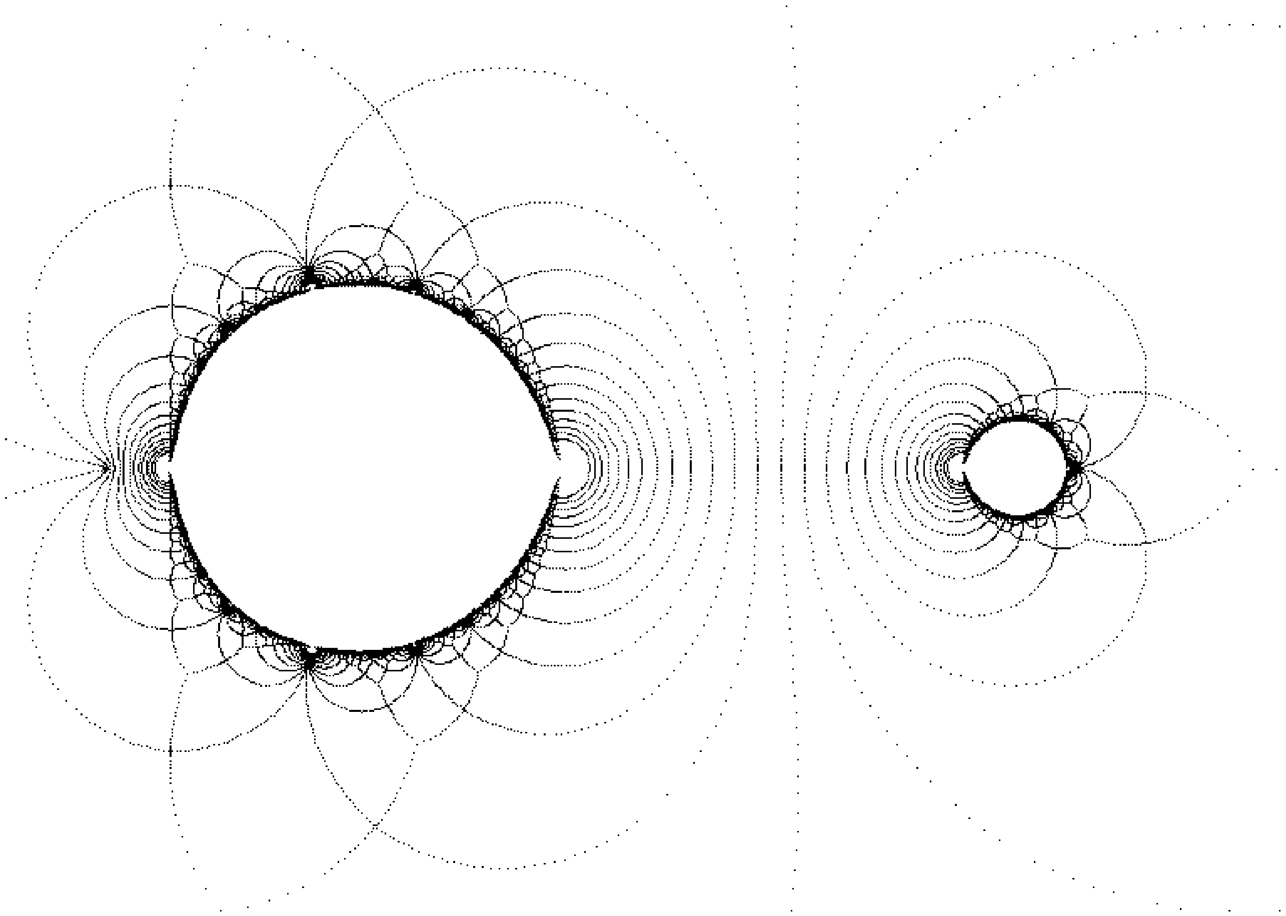


Figure 2a
Graph of a 2 : 2 reversible off-separable correspondence on the real interval $[0,1]$. Here J is the involution $z \mapsto 1-z$. The open circles represent 'missing arrows'. A transversal directionality D is shown. Note that it is the projection of a fundamental domain for the action of S_3 on the graph.

Figure 2b (overpage)
Computer plot of a 2 : 2 reversible off-separable correspondence f on the Riemman sphere, defined by $f : z \mapsto w$ where

$$z^2 + z\left(\frac{pw-a}{w-p}\right) + \left(\frac{pw-a}{w-p}\right)^2 = 3 \quad \text{with} \quad p = 3, a = 5.2$$

A transversal directionality D and associated 'tiling' are plotted. In this example $\omega_+(f,D)$ and $\omega_-(f,D)$ are each connected sets and $\Omega(f,D)$ is the complement of their union. The restriction of f to $\omega_+(f,D)$ is conjugate to the inverse $q_c^{-1} : z \mapsto \sqrt{z-c}$ of a quadratic map $q_c : z \mapsto z^2 + c$ acting on the filled Julia set $\mathcal{K}(q_c)$ of q_c, as is the restriction of f^{-1} to $\omega_-(f,D)$. In the example plotted c is small (though non-zero) so $\mathcal{K}(q_c)$ is a topological disc.



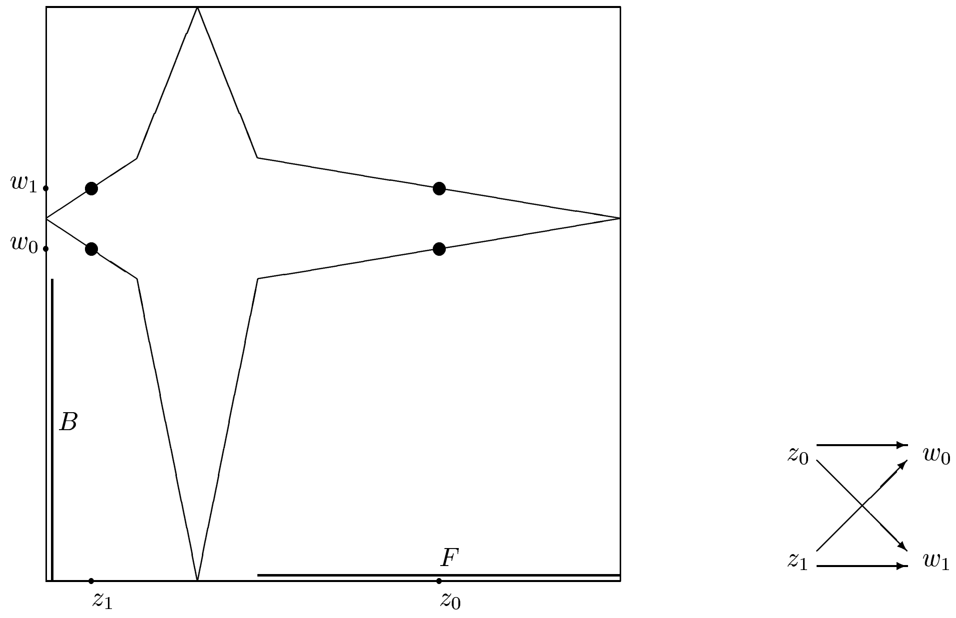
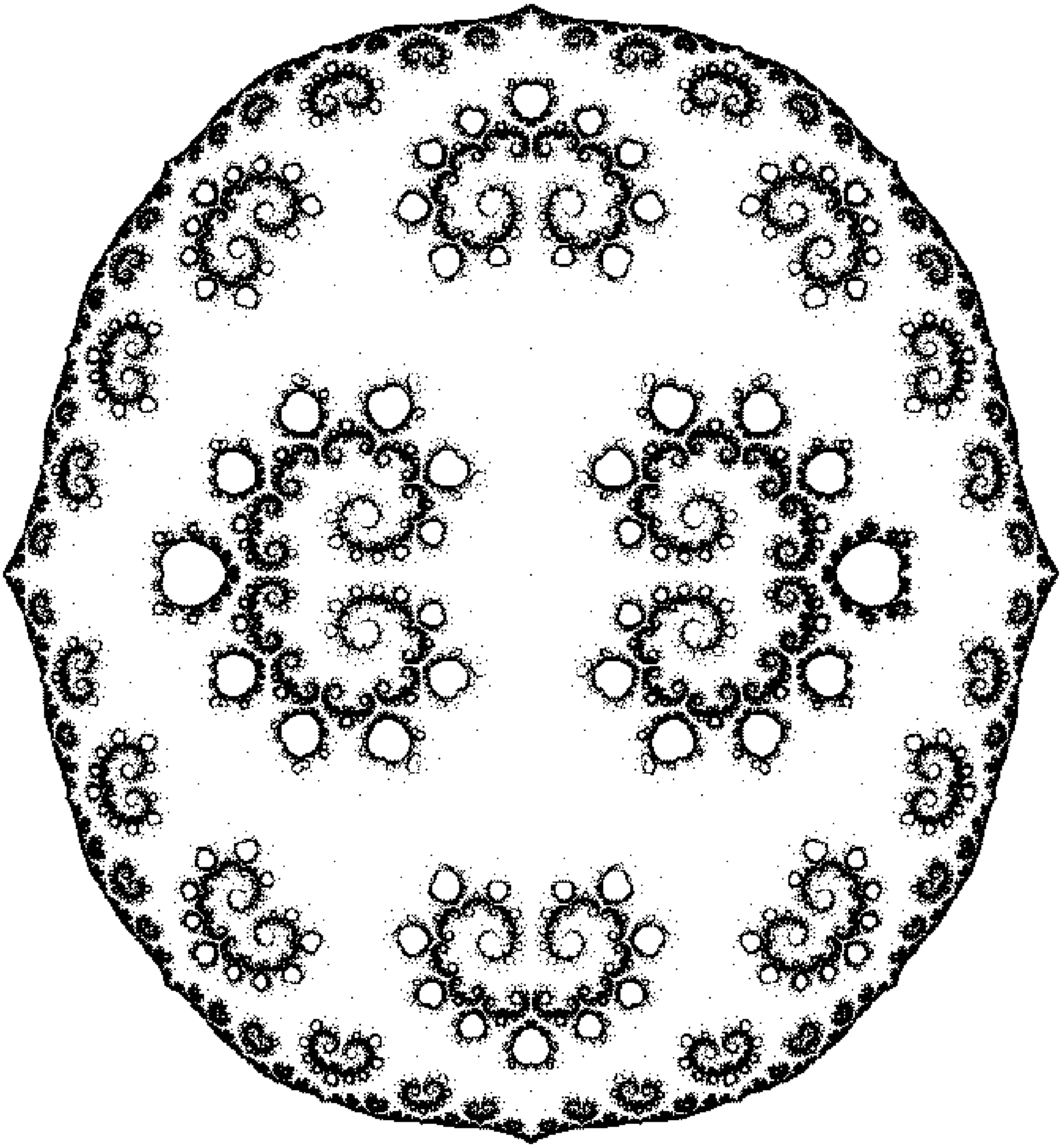


Figure 3a
Graph of a 2 : 2 separable correspondence f on the real interval $[0, 1]$. A bi-injective directionality (F, B) is shown.

Figure 3b (overpage)
Computer plot of a 2 : 2 separable (but not reversible) f on the Riemann sphere, defined by $f : z \mapsto w$ where

$$z(z+a) = \frac{w^2}{w + b/a} + abd \quad \text{with} \quad a = 0.8, b = -0.15, d = 0.5$$

Here F and B are discs (not shown), intersecting in an annulus. The forward limit set $\omega_+(f, F)$ is a (Cantor) 'worm', the backward limit set $\omega_-(f, B)$ is a 'heart', and the global limit set $\omega(f, D)$ is the closure of a union of copies of these (see [7] for further pictures).



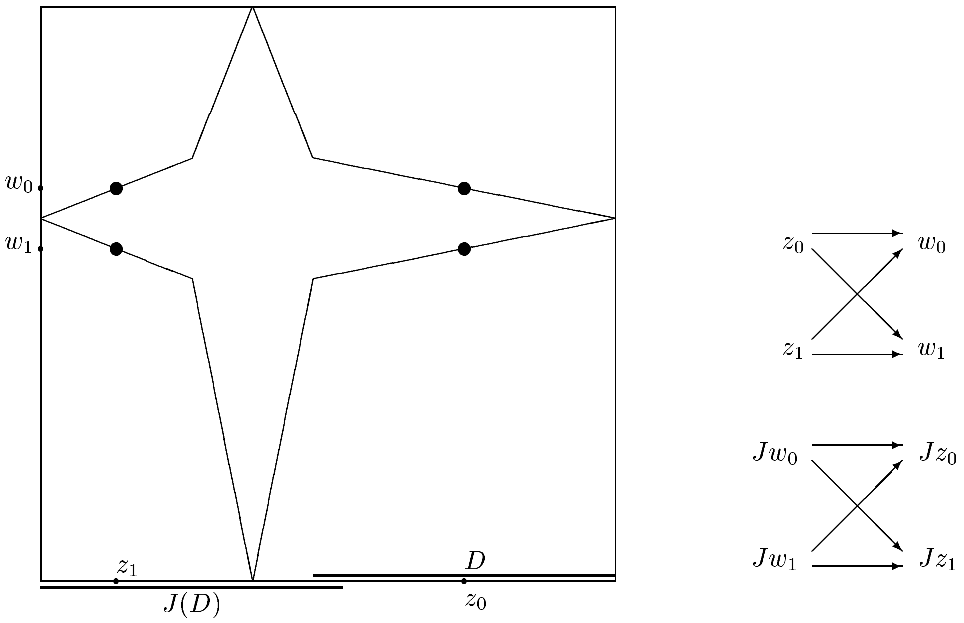


Figure 4a
Graph of a 2 : 2 reversible separable f on the real interval $[0, 1]$. The involution J maps z to $1 - z$. A (bi-injective) transversal directionality D is shown. Note that it is the projection of a fundamental domain for the action of the dihedral group of order 8 on the graph.

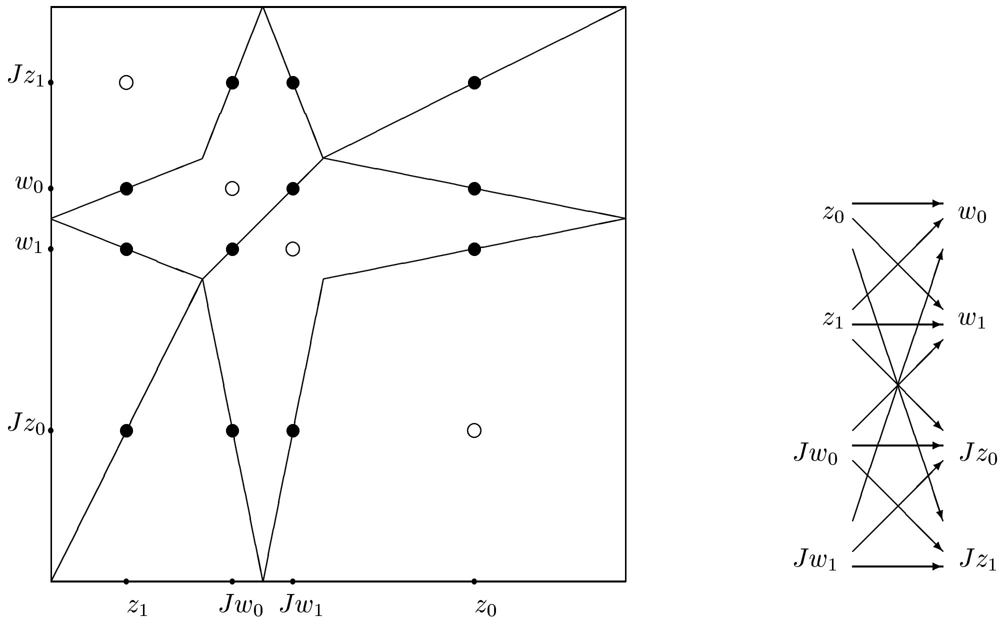


Figure 4b
Graph of the 3 : 3 reversible off-separable correspondence $\mathcal{F}$ associated to the 2 : 2 reversible separable correspondence f of figure 4a. Open circles represent 'missing arrows'.

Figure 4c (overpage)
Computer plot of a 2 : 2 reversible separable f on the Riemann sphere, defined by $f : z \mapsto w$ where

$$z(z+a) = \frac{w^2}{w + b/a} \quad \text{with} \quad a = 0.8, b = -0.15$$

Now $\omega_+(f, D)$ and $\omega_-(f, D)$ are both 'hearts', and $\omega(f, D) = \omega(\mathcal{F}, D)$ is the closure of a countable union of copies of them.