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Conformal Measures and Holomorphic Correspondences

Background

It is well known that there are strong parallels between the theories of Kleinian groups and rational maps (see [8] for a recent account of the state of the 'Sullivan dictionary' between the two fields). *Holomorphic correspondences* on the Riemann sphere are multi-valued maps $z \to w$ defined by polynomial equations $p(z, w) = 0$. They form a common generalisation of rational maps and finitely generated Kleinian groups. Striking examples of correspondences which are matings between $PSL(2, \mathbb{Z})$ and quadratic maps were exhibited in [1], leading to:

Conjecture 0.1 *For every quadratic polynomial $q_c : z \to z^2 + c$ having a connected Julia set, that is to say with $c \in \mathcal{M}$, the Mandelbrot set, there exists a polynomial relation $p(z, w) = 0$ defining a $(2:2)$ correspondence which is a mating between q_c and $PSL(2, \mathbb{Z})$.*

While this conjecture remains open (but see Corollary 1.1 below), the existence of 'unpinched matings' between quadratic maps and generic discrete representations of $C_2 * C_3$ (those with connected ordinary set or equivalently with limit set a Cantor set) was proved in [2]:

Theorem 0.1 *For every quadratic polynomial q_c with $c \in \mathcal{M}$ and every faithful discrete representation r of $C_2 * C_3$ in $PSL(2, \mathbb{C})$ having connected ordinary set, there exists a polynomial relation $p(z, w) = 0$ defining a $(2:2)$ correspondence which is a mating between q_c and r.*

The limit sets of such matings are well-understood combinatorially but before this project nothing was known about their metric/measure-theoretic properties. The main thrust of the project was to undertake a study of these by exploiting the sophisticated methods developed for rational maps and Kleinian groups, and also to investigate the construction of more general classes of matings and their associated parameter spaces.

Main Results

The advances resulting from the project are summarised under four sub-headings below, corresponding to the publications [BH],[F2],[BF2] and [BF3] in the bibliography. But first let us say what we mean by 'pinched' and 'unpinched' matings between Hecke groups and maps. The Hecke group $H_n \subset PSL(2, \mathbb{R})$ is a generalisation of $H_3 = PSL(2, \mathbb{Z})$, generated by the involution $\sigma(z) = -1/z$ and a rotation ρ of order n. As an abstract group it is the free product $C_2 * C_n$ of the subgroups generated by these elements. An $(n-1 : n-1)$ holomorphic correspondence $\mathcal{F}$ is a *(pinched) mating* between H_n and a polynomial map g of degree $n-1$ if the Riemann sphere $\hat{\mathbb{C}}$ is partitioned into an open domain Ω and a closed connected set Λ, each completely invariant under $\mathcal{F}$, and there is an involution J conjugating $\mathcal{F}$ to $\mathcal{F}^{-1}$, such that:

• there is a conformal bijection ϕ from Ω to the upper half-plane, carrying $\mathcal{F}|_\Omega$ to $\{\sigma\rho, \ldots, \sigma\rho^{n-1}\}$ and $J|_\Omega$ to σ;
• Λ is a union $\Lambda_+ \cup \Lambda_-$, where $\Lambda_+ \cap \Lambda_-$ is a single point, $\Lambda_+ = J(\Lambda_-)$, and $\mathcal{F}$ restricted to Λ_- as domain and codomain is a single-valued map, conjugate to $g|_{K(g)}$ and conformally so on interiors (where K(g) denotes the filled Julia set of g).

The reason for describing this mating as 'pinched' is that Λ_+ meets Λ_- (in a single point), and the representation of $C_2 * C_n$ involved has limit set a continuum (a 'pinched' Cantor set). We say that a correspondence $\mathcal{F}$ is an *(unpinched) mating*, between a generic discrete representation r of $C_2 * C_n$ and a polynomial map g of degree $n-1$, if the situation is as above except that now $\Lambda_+ \cap \Lambda_-$ is empty, Ω is an annular region, and there is a conformal bijection between the grand orbit space $\Omega/\mathcal{F}$ and the quotient of the ordinary set of r by the canonical C_2-extension of $C_2 * C_n$ which acts upon it (see [BF2]).

1. Pinching holomorphic correspondences

A major aim of the project was to establish measure-theoretic properties of 'pinched' and 'unpinched' matings for $n = 3$, and to examine how these vary as an unpinched mating converges to a pinched mating. A key step is to determine when the convergence is uniform. This is a technically difficult issue, as one has to deal with a family of quasiconformal deformations of increasing dilatation, and in the limit pinch arcs to points. However it proved possible to adapt techniques developed by Haïssinsky and co-workers [4, 5] for pinching rational maps.

Definition *A convergent pinching deformation for a union* Γ *of arcs which is invariant under the correspondence* $p_0(z, w) = 0$ *is a family of quasiconformal maps* $(\varphi_t)_{0\leq t<1}$ *of the Riemann sphere such that the conjugate correspondences* p_t *defined by* $p_t(z, w) = p_0(\varphi_t^{-1}(z), \varphi_t^{-1}(w))$ *are holomorphic and satisfy the following:*
- (p_t, φ_t) *are uniformly convergent to a pair* (p_1, φ_1) *as* t *tends to* 1*;*
- *the non-trivial fibres of* φ_1 *are exactly the closures of the connected components of* Γ*.*

There are two technical conditions that we require the quadratic map $q_c : z \to z^2 + c$ to satisfy in order to apply the techniques of [5]:

(i) if the critical point 0 of q_c is recurrent, the β-fixed point of q_c is not in the ω-limit set of 0;
(ii) q_c is weakly hyperbolic (see [BH] for the definition of this term).

Theorem 1.1 [BH] *Let* $p_0(z, w)$ *be a mating between a generic representation of* $C_2 * C_3$ *and* q_c*, where* q_c *satisfies conditions (i) and (ii) above. Then there exists a pinching deformation of* p_0 *such that* $(p_t)_{0\leq t<1}$ *converges uniformly to a mating* p_1 *between* $PSL(2, \mathbb{Z})$ *and* q_c*.*

Apart from its application to questions concerning Hausdorff dimensions (for example Theorem 2.4 below), Theorem 1.1 also resolves Conjecture 0.1 in a large class of cases:

Corollary 1.1 [BH] *Conjecture 0.1 is true for all* $q_c : z \to z^2 + c$ *satisfying (i) and (ii).*

The modular group $PSL(2, \mathbb{Z})$ is given by the simplest of Maskit's pinchings [6] of a loxodromic to a parabolic element in a generic representation of $C_2 * C_3$, but there is a circle-packing representation of $C_2 * C_3$ for each rational p/q. A variation on the techniques of [4] yields:

Theorem 1.2 [BH] *Let* $p_0(z, w)$ *be a mating between a generic representation* r *of* $C_2 * C_3$ *and* $q_0 : z \to z^2$*, and let* p/q *be any rational. Then there exists a pinching deformation of* p_0 *such that* $(p_t)_{0\leq t<1}$ *converges uniformly to a mating* p_1 *between the circle-packing representation* $r_{p/2q}$ *of* $C_2 * C_3$ *in* $PSL(2, \mathbb{C})$ *and* q_0*.*

2. Conformal measures for pinched and unpinched matings

To prove measure-theoretic results about the *limit sets* $\partial\Lambda$ of matings, we followed and adapted McMullen's work on rational maps in [7]. This is based on the Patterson/Sullivan approach to construct 'conformal measures' on the limit sets, which, under suitable conditions, are equivalent to Hausdorff measure and thus yield the Hausdorff dimension of these sets.

Let f be a pinched or unpinched mating between a representation of $C_2 * C_3$ in $PSL(2, \mathbb{C})$ and a holomorphic map g of degree 2. An α-conformal f-invariant measure is a positive probability measure μ supported on the Riemann sphere such that for any Borel set E and for any branch h of f or f^{-1} which is injective and single-valued on E we have

$$\mu(h(E)) = \int_E |h'(z)|^\alpha d\mu(z). \tag{1}$$

The *critical dimension* $\alpha(f)$ is defined as the infimum of the set of values of α for which there exists an α-conformal f-invariant measure supported on Λ_+.

By the definition of a mating, f restricted to Λ_- as both domain and codomain is a single-valued map of degree 2. In the unpinched case this map, which we shall denote g_f, is conformally conjugate to a quadratic polynomial acting on its filled Julia set (by [3]) and it is also conjectured to be conjugate to such a polynomial in the pinched case. We say that f is *geometrically finite* if g_f is geometrically finite, and define the radial limit set $L_{rad}(f)$ in the same way as one defines the radial limit set of a polynomial (see [7]). The main results proved in [F2] are:

Theorem 2.1 [F2] *Let f be a $(2:2)$ correspondence which is a mating (pinched or unpinched). Then there exists a real number $0 < \delta \leq 2$ such that $\partial\Lambda$ carries a δ-conformal f-invariant measure μ with no atoms on repelling or parabolic periodic points of g_f or any of their inverse images under g_f.*

Theorem 2.2 [F2] *If f is as above and geometrically finite, then*

$$\delta(f) = HD(L_{rad}(f)) = HD(\partial\Lambda_+) = \alpha(f).$$

Moreover, the measure μ constructed in Theorem 2.1 is the unique normalised $\delta(f)$-conformal measure with support in $\bar{\Omega} - \{p\}$.

Corollary 2.1 [F2] *If f is geometrically finite and μ is a conformal measure supported on $\partial\Lambda_+$ then either it is the canonical measure μ constructed in Theorem 2.1, or it is an atomic measure of dimension greater than $\alpha(f)$ supported on the orbit under f of parabolic periodic points and the critical point of g_f.*

Corollary 2.2 [F2] *If f is geometrically finite then $HD(\partial\Lambda_+) < 2$.*

Theorem 2.3 [F2] *If f_t is a pinching deformation with limit f then the limit sets $\partial\Lambda_+^t$ of f_t converge to the limit set $\partial\Lambda_+$ of f in the Hausdorff topology.*

Theorem 2.4 [F2] *Let $q_c : z \to z^2 + c$ be a geometrically finite quadratic polynomial with $c \neq 1/4$, connected Julia set and such that the orbit of the critical point of q_c does not land on the β-fixed point. Let f_0 be a mating between q_c and a representation of $C_2 * C_3$, and let $\{f_t\}_{0 \leq t < 1}$ be a pinching deformation with limit f. Let p_t denote the β-fixed points of g_{f_t}, so $p_t \to p$. If either*
(i) $p_t \to p$ radially, or (ii) $p_t \to p$ horocyclically and $\liminf(HD(\partial\Lambda_+^t)) > 1$,
then $HD(\partial\Lambda_+^t) \to HD(\partial\Lambda_+)$.

Here, radial and horocyclical convergence are defined analogously to the rational map case. When the parameters involved in the pinching deformation are real, radial convergence is automatic and a stronger result can be stated (see [F2] for details).

3. Structure theorems for matings between polynomials and Hecke groups

Another theme of our work was the construction and analysis of new classes of matings involving maps of higher degrees. Our first result is a structure theorem identifying the candidates for (pinched) matings between Hecke groups and maps. For Q a rational map of degree n and J an involution we shall use the notation Cov_0^Q for the *deleted covering correspondence* of Q, the $(n-1:n-1)$ correspondence $z \to w$ defined by $(Q(w) - Q(z))/(w - z) = 0$, and the notation $J \circ Cov_0^Q$ for the correspondence defined by $(Q(J(w)) - Q(z))/(J(w) - z) = 0$.

Theorem 3.1 [BF2] *Let $\mathcal{F}$ be a holomorphic correspondence which is a mating between a Hecke group and a map. Then $\mathcal{F} = J \circ Cov_0^Q$ for some polynomial Q and involution J.*

Holomorphic correspondences which are generated by $(1:1)$-subcorrespondences are groups of Möbius transformations, i.e. subgroups of $PSL(2,\mathbb{C})$. The most elementary type of correspondences which are not of this type are those correspondences $\mathcal{F}$ which are *generated by a single* $(2:2)$*-subcorrespondence* f, in other words the grand orbits of $\mathcal{F}$ (under mixed iteration, forward and back) are those of the $(2:2)$-subcorrespondence f.

Theorem 3.2 [BF2] *Let n be odd and $\mathcal{F} = J \circ Cov_0^Q$ be an $(n-1:n-1)$ correspondence which is a mating between a map g and the Hecke group H_n. If the $(2:2)$ subcorrespondence $\phi h_1 \phi^{-1}$ of $\mathcal{F}|_\Omega$ extends to the whole of $\hat{\mathbb{C}}$, where $h_1 = \{\sigma\rho, \sigma\rho^{-1}\}$, then:*
(i) $\mathcal{F}$ splits into $(2:2)$ correspondences $\{f_j\}_{1\le j\le (n-1)/2}$;
(ii) $\mathcal{F}$ is generated by any f_j with j coprime to n;
(iii) up to pre- and post- composition by Möbius transformations, $Q = T_n$ where T_n denotes the nth Chebyshev polynomial (the polynomial which expresses $\cos n\theta$ in terms of $\cos\theta$).

When n is even there is a similar, but slightly more complicated, statement (see [BF2]). Theorem 3.2 reduces us to the consideration of a one complex parameter family of correspondences, since having set Q to be T_n our only freedom is in the choice of the fixed points of J, and for a mating one of these has to be at the intersection $\Lambda_+ \cap \Lambda_-$. In our final structure theorem we find that the Chebyshev polynomial of one degree lower now becomes part of the picture in a surprising way. Let R_m denote the conjugate of the Chebyshev polynomial T_m defined by $R_m(z) = \frac{1}{2}(1 - T_m(1-2z))$. Thus R_m fixes 0 and maps the real interval $[0,1]$ to itself.

Theorem 3.3 [BF2] *If the $(n-1:n-1)$ correspondence $\mathcal{F} = J \circ Cov_0^{T_n}$ is a mating between the Hecke group H_n and a holomorphic map g of degree $n-1$, then g has the combinatorial behaviour of a scalar multiple of R_{n-1}.*

For $n=3$ we have $R_2(z) = 4z(1-z)$, and the maps $\{\lambda R_2 : \lambda \in \mathbb{C}\}$ are (up to conjugacy) the family of all quadratics. For $n>3$ the polynomials in the corresponding family $\{\lambda R_{n-1} : \lambda \in \mathbb{C}\}$ have considerably more complicated dynamics. However for any map λR_{n-1} for which the Julia set $J(\lambda R_{n-1})$ is connected it is shown in [BF2] that $J(\lambda R_{n-1})$ is the closure of the union of a countable set of copies of the Julia set of a single quadratic polynomial. The proof also yields a combinatorial model for the connectedness locus in the space of the parameter λ as the closure of a countable union of copies of the quadratic Mandelbrot Set. Complete combinatorial descriptions are presented in detail in the simplest case, that of $n=4$, in [BF2].

4. Matings between transcendental functions and a Fuchsian group

By letting the n in $J \circ Cov_0^{T_n}$ tend to infinity we obtain the first examples of matings between a group and a transcendental entire function [BF3]. Let $R(z) = \sin^2(\pi\sqrt{z})$ (this is an appropriately normalized version of $\lim_{n\to\infty} R_n$). Adopting the convention that $Cov^R(\infty) = \{\infty\}$, we find that the graph of $J \circ Cov_0^R$ is the union of the graphs of $(2:2)$ subcorrespondences f_j, defined by

$$w \in f_j(z) \Leftrightarrow (Jw)^2 - 2(z+j^2)(Jw) + (z-j^2)^2 = 0 \quad (j \in \mathbb{N})$$

and that the $(2:2)$ correspondence f_1 generates $J \circ Cov_0^R$ (by an analogue of Theorem 3.2).

Let H_∞ denote the subgroup of $PSL(2,\mathbb{C})$ generated by $z \to z+2$ and $z \to -1/z$. By an analogue of Theorem 3.3, under appropriate conditions the correspondence $J \circ Cov_0^R$ is a mating between H_∞ and a holomorphic map having the combinatorial behaviour of a scalar multiple μR of the function R. For such matings it is shown in [BF3] that:

- Λ_+ is a bounded subset of $\mathbb{C}$ and is the closure of the union of a countable set of copies of the filled Julia set of a single quadratic polynomial;
- when this quadratic is hyperbolic, the interior of Λ_+ is homeomorphic to the Fatou set of μR.

In [BF3] we present striking computer plots of Λ_+ and establish a relationship between the points of $\partial\Lambda_+$ and the 'dynamical rays' in the Julia set $J(\mu R)$ of the transcendental function μR, which is a Cantor bouquet based at ∞. This enables us to picture $\partial\Lambda_+$ as obtained from $J(\mu R)$ by removing the essential singularity at ∞ and retracting dynamical rays to points.

The computer plots of matings and parameter spaces in the final version of [BF2] *were drawn using a program written by X.Buff following the Kyoto Complex Dynamics Workshop in 2003. Adapting this program to produce the plots for* [BF3] *was a very useful experience for the RA.*

Review of Project Plan

Two developments led to a change in emphasis to the research as originally proposed:

(i) The discovery early in the project of new classes of matings between higher degree polynomials and Hecke groups, which lead to a better understanding of matings in general through the structure theorems reported in Section 3 above.
(ii) As a theory of conformal measures for matings was developed, an increasing need to prove the pinching theorems reported in Section 1.

As a consequence two new objectives were added, and one of the original objectives (developing the 'thermodynamic formalism') was not pursued, due to lack of time. Amended objectives:

• (Original) To construct conformal measures on the limit set L of a correspondence f, starting from those arising from matings of the modular group and hyperbolic quadratic maps.
• (Original) To analyse the conformal measures to obtain metric results for the limit sets of correspondences, such as the dimension of L is equal to the critical exponent of f defined in a suitable manner, and that the measures are essentially equivalent to Hausdorff measures.
• (Original) To generalise the framework to incorporate other correspondences, for example those arising from parabolic quadratic maps and other Kleinian groups such as Hecke groups and other representations of the free product of cyclic groups of order 2 and n.
• (New) To construct correspondences which are matings between higher degree polynomials and Hecke groups, and between transcendental entire functions and Hecke groups, and to investigate the combinatorics of the regular and limit sets of these matings.
• (New) To prove uniform convergence theorems for pinching deformations of correspondences which are matings, and apply these to prove results concerning continuity of Hausdorff dimension for limit sets.

Dissemination of Results

Publications resulting from the project are listed later in this report. Talks on various aspects of the results were presented by the PI and/or the RA at:

2002: Dynamics Days - Asia Pacific (Hangzhou); ICM (Beijing);
2003: Kyoto Complex Dynamics Workshop; IHP (Paris) Dynamical Systems Semester;
2004: Warwick International Workshop on Holomorphic Dynamics; Roskilde (Denmark) Workshop on Holomorphic Motions;

The RA visited the University of Barcelona for a research collaboration in the summer of 2003, and IHES from September to December 2003 *(a valuable opportunity for the RA to interact with participants in the IHP Dynamical Systems Semester).* She participated in the Snowbird (Utah, USA) Conference for the 25th Anniversary of the Mandelbrot Set in 2004, and gave talks at the BMC in April 2005 and the QMUL Complex Dynamics Day in May 2005. The PI and RA also presented the results of the project in seminars at Warwick, Manchester, Southampton, Cambridge, Sheffield, East Anglia, Barcelona, Paris and Marseille.

Management of Resources

Spending on salaries, equipment and consumables was as budgeted in the proposal. There was a small underspend on travel, thanks to unanticipated contributions from hosts.

On-going and Future Research

The results of the project show that it is possible to apply techniques developed for the measure-theoretic and deformation-theoretic analysis of rational maps and Kleinian groups to analogous questions for holomorphic correspondences which are matings. The project has also developed the structural theory of matings and produced interesting new classes of examples. There are many detailed questions to follow up - too numerous to list here. On a broader front, a major challenge is to identify more general classes of holomorphic correspondences to which the theory developed in the project may be applied.

Publications resulting from the grant

All available at `http://www.maths.qmul.ac.uk/~sb` *or* `~mari`

[BF1] Shaun Bullett and Marianne Freiberger, *Hecke groups, polynomial maps and matings*, Int. J. Mod. Phys. B, 17 (2003) 3922-3931

[BF2] Shaun Bullett and Marianne Freiberger, *Holomorphic correspondences mating Chebyshev-like maps with Hecke groups*, to appear in Ergodic Theory and Dynamical Systems (34pp)

[BF3] Shaun Bullett and Marianne Freiberger, *A family of matings between transcendental entire functions and a Fuchsian group*, accepted (subject to revision) for publication in the Noel Baker Memorial Volume (ed. P.Rippon), CUP (19pp)

[BH] Shaun Bullett and Peter Haïssinsky, *Pinching holomorphic correspondences*, submitted to Commentarii Math. Helv. (31pp)

[F1] Marianne Freiberger, *Matings between Kleinian groups isomorphic to $C_2 * C_5$ and quadratic polynomials*, Conform. Geom. Dyn. 7 (2003), 11-33

[F2] Marianne Freiberger, *Conformal measures and matings between Kleinian groups and quadratic polynomials*, submitted to Fundamenta Mathematica (35pp)

References

[1] S. Bullett and C. Penrose, *Mating quadratic maps with the modular group*, Inventiones Mathematicae **115** (1994), 483–511

[2] S. Bullett and W.J. Harvey, *Mating quadratic maps with Kleinian groups via quasiconformal surgery*, Electronic Research Announcements of the AMS 6 (2000), 21–30

[3] A. Douady and J.H. Hubbard, *On the dynamics of polynomial-like mappings*, An. de l'École Norm. Sup. 18 (1985) 287–343

[4] P. Haïssinsky, *Pincement de polynômes*, Comment. Math. Helv. 77 (2002), no. 1, 1–23

[5] P. Haïssinsky and Tan Lei, *Matings of geometrically finite polynomials*, Fund. Math., 181 (2004), 143–188

[6] B. Maskit, *Parabolic elements in Kleinian groups*, Ann. Math. 117 (1983) 659–668

[7] CT McMullen, *Hausdorff dimension and conformal dynamics II: Geometrically finite rational maps*, Comment Math. Helv. 75 (2000), no 4, 535–593

[8] S Morosawa, Y Nishimura, M Taniguchi, T Ueda, *Holomorphic Dynamics*, Cambridge Studies in advanced mathematics No 66, CUP 2000