Chapter 7
Fractals and Dimension

Dimension

We say that a smooth curve has *dimension* 1, a plane has dimension 2 and so on, but it is not so obvious at first what dimension we should ascribe to the *Sierpinski gasket* or the von Koch snowflake or even to a *Cantor set*.

These are examples of *fractals* (the word is due to Mandelbrot in the 1970s, and is used to describe a "jagged" or "broken" object). Most of the fractals we shall deal with have some sort of "self-similarity" on different scales, often because they are constructed by some repetitive rule, and we can use this self-similarity to compute the dimension.

Let S be a subset of the unit square $I \times I$ in the plane, for example a smooth line segment. Divide the square up into into $1/n^2$ little squares, each of side length $1/n$. If S is a smooth line segment, the number of little squares that it meets will be (roughly) proportional to n. If S is a "2-dimensional shape" (for example the interior of a polygon or of a circle) the number of little squares that it meets will be (roughly) proportional to n^2. We shall say that S has *dimension* α if the number of little squares that S meets is proportional to n^α as n tends to infinity.

Definition

If S is a bounded subset of $\mathbb{R}^n$ we set $N(S,d)$ to be the number of boxes (little cubes) of side length d needed to cover S. Then the *(box-counting) dimension* of S is defined to be:

$$dim(S) = \lim_{d \to 0} \frac{\ln(N(S,d))}{\ln(1/d)}$$

The box-counting dimension is only defined if this limit exists. There is a more technical definition (which we shall see later) known as *Hausdorff dimension* which always exists for a bounded subset of $\mathbb{R}^n$, and which agrees with the box-counting dimension in most cases that the latter exists.

Example 1

Let Q be the unit square in $\mathbb{R}^2$. Then $N(Q, 1/n) = n^2$, so

$$dim(Q) = \lim_{n \to \infty} \frac{\ln(n^2)}{\ln(n)} = 2$$

Note that dimension is unchanged by the dimension of the space in which S is embedded. If we think of our square S as sitting in $\mathbb{R}^3$, and we compute its dimension by covering it with little cubes of side length $1/n$, we will still have $N(Q, 1/n) = n^2$ and so we will still get $dim(Q) = 2$.

Example 2

Consider the middle thirds Cantor set C. Here we have:

$$N(C, 1/3) = 2, \quad N(C, 1/9) = 4 \;\; \textit{and inductively } N(C, 1/3^n) = 2^n$$

Hence

$$dim(C) = \lim_{n\to\infty} \frac{\ln(2^n)}{\ln(3^n)} = \frac{\ln(2)}{\ln(3)} (= \log_3(2))$$

But not all Cantor sets have this dimension. There are Cantor sets in the real line of every dimension between 0 and 1 (including both extremes), and Cantor sets in the plane of every dimension between 0 and 2.

Example 3

Consider the Sierpinski gasket S of side length 1. It is easily seen that

$$N(S, 1) = 1, \quad N(S, 1/2) = 3, \quad N(S, 1/4) = 9 \;\; \textit{and inductively } N(S, 1/2^n) = 3^n$$

Hence

$$dim(S) = \lim_{n\to\infty} \frac{\ln(3^n)}{\ln(2^n)} = \frac{\ln(3)}{\ln(2)} (= \log_2(3))$$

We remark that for the Sierpinski gasket the observation that $N(S, 1/2^n) = 3^n$ is even more obvious if we use "little triangles of side length $1/2^n$" as our boxes, rather than little squares of side length $1/2^n$. In fact it can be shown that one gets the same answer for the box-counting dimension whatever shape of box one uses (provided of course that one uses the same shape on all scales).

Example 4

Let S be *von Koch Snowflake*, obtained by from the unit interval on the real line by replacing the middle third by the other two sides of an equilateral triangle of side length $1/3$, then repeating the same construction of each of resulting four line segments, and so on. Covering the snowflake with little triangles of side length $1/3^n$ we see that $N(S, 1/3^n) = 4^n$, and hence

$$dim(S) = \lim_{n\to\infty} \frac{\ln(4^n)}{\ln(3^n)} = \frac{\ln(4)}{\ln(3)} = 2\frac{\ln(2)}{\ln(3)}$$

Notes

1. The *length* of the von Koch snowflake S (also called the *one-dimensional measure of S*) is infinite, since $\lim_{n\to\infty}(4/3)^n = \infty$.

2. The von Koch snowflake is an example of a more general construction, where one start with the unit interval and replaces it with a copy of a *generator*, a shape made up of straight line segments each of side length $1/k$, then replaces each of these straight line segments by a small copy of the generator, and so on. We can also construct *space-filling curves*, that is to say curves which fill the whole unit square, in a similar way. Of course a space-filling curve has dimension 2.

Here is the definition of *Hausdorff dimension* (not for examination). Let S be a bounded subset of some $\mathbb{R}^n$. First we define the *Hausdorff p-dimensional measure of S*

$$M(S,p) = \sup_{\epsilon>0} \inf\{\sum_{i=1}^{\infty}(diam(A_i))^p : \bigcup_{i=1}^{\infty} A_i \supseteq S \text{ } and \text{ } diam(A_i) < \epsilon\}$$

It can be shown that for every S there exists a non-negative real number D such that

$$M(S,p) = \infty \ \forall p < D \text{ } and \text{ } M(S,p) = 0 \ \forall p > D$$

This value of D is called the *Hausdorff dimension* of S.

For example the von Koch snowflake has infinite 1-dimensional measure (length) and zero 2-dimensional measure (area). So its Hausdorff dimension is somewhere in between - in fact it is the same as its box-counting dimension, namely $2\ln(2)/\ln(3)$.

Iterated Function Systems

The middle thirds Cantor set C has some obvious self-similarities. Define maps f_1 and f_2 from $I = [0,1]$ into I by

$$f_1(x) = x/3 \qquad f_2(x) = x/3 + 2/3$$

Each of these maps contracts C onto a smaller copy within C. Iterating f_1 on its own we have simple dynamics (an attractor at $x = 0$), and iterating f_2 on its own we again have simple dynamics (an attractor at $x = 1$). But if we iterate *both together*, choosing f_1 or f_2 at random at each iteration, with probability $1/2$, we obtain the whole of C as an attractor for the system. The orbit of any initial point x_0 approaches closer and closer to C and moreover with probability 1 it approaches arbitrarily close to *every* point of C. Note that if the initial point x_0 is in C then its orbit remains in C. $\{f_1, f_2\}$ is an example of what Barnsley calls an *iterated function system*. It can be used to very efficiently plot the points of the middle thirds Cantor set. We simply start our computer off at the point $x_0 = 0$ and randomly apply f_1 and f_2

(or we start at any point $x_0 \in I$, but then we should not plot the first few dozen points at it might take that long for the orbit to get close to C).

Definition

An *iterated function system (IFS)* for a closed bounded subset S of $\mathbb{R}^n$ is a set of maps $\{f_1, f_2, \ldots, f_m\}$ such that each f_i is a map $\mathbb{R}^n \to \mathbb{R}^n$, and such that

(i) each f_i is a *contraction* (i.e. $\exists k < 1$ such that $||f(x) - f(y)|| \leq k||x - y||\forall x, y \in \mathbb{R}^n$), and

(ii) $\bigcup_{i=1}^{m} f_i(S) = S$.

The middle thirds Cantor set is an example, with f_1 and f_2 as above and the contraction ratio $k = 1/3$. Another example is the Sierpinski gasket with maps f_1, f_2 and f_3 contracting the triangle os side length 1 onto each of three triangles of side length 1/2. If (to make writing down explicit maps easier) instead of an equilateral triangle we take as our starting triangle the one with vertices at $(0,0), (1,0)$ and $(0,1)$ then we can define an IFS for the "45 degree Sierpinski gasket" by taking the maps:

$$f_1(x,y) = (x/2, y/2) \quad f_2(x,y) = f_1(x,y) + (0, 1/2) \quad f_3(x.y) = f_1(x,y) + (1/2, 0)$$

Here we have 3 maps, and each has contraction ration 1/2. Notice that if we have an IFS for a fractal S with the properties that the f_i all have the same contraction ratio λ, and the $f_i(S)$ do not overlap (if at worst they meet at their edges) then we can use a box counting argument to deduce that the dimension of S is $\ln(m)/\ln(1/\lambda)$, where m is the number of maps in the IFS.

If we have a picture which has appropriate self-similarities on different scales we can use the associated IFS as a very efficient method of reproducing the picture. A beautiful example is *Barnsley's fern* which can be reproduced using an IFS consisting of just four linear maps (or three if you don't mind not having the stalks). To get the best pictures it often helps to assign different probabilities to the different maps rather than to choose them at random with the same probability.

To get an idea of the mathematics behind the concept of an IFS, we need the idea of the "distance" between two closed bounded subsets A and B of $\mathbb{R}^n$. We set

$$d(A,B) = max_{x \in A} min_{y \in B} d(x,y)$$

where $d(x,y)$ is the Euclidean distance between x and y in $\mathbb{R}^n$. Note that $d(A,B) \neq d(B,A)$ in general: for example if $A \subset B$ but $A \neq B$ then $d(A,B) = 0$ but $d(B,A) \neq 0$. We define the *Hausdorff distance* between A and B to be:

$$D(A,B) = max\{d(A,B), d(B,A)\}$$

It is easily shown that $D(A, B) = 0 \Leftrightarrow A = B$.

Let $\mathcal{H}(\mathbb{R}^n)$ denote the set of all closed bounded subsets of $\mathbb{R}^n$. Equip $\mathcal{H}(\mathbb{R}^n)$ with the Hausdorff metric (defined above). Then $\mathcal{H}(\mathbb{R}^n)$ is a *complete metric space* (i.e. all Cauchy sequences have limits).

The Contraction Mapping Lemma

If X is a complete metric space and $f : X \to X$ is a continuous map such that there exists a constant $k < 1$ with $d(f(x), f(y)) \leq kd(x, y)$ for all $x, y \in X$, then f has a unique fixed point, and all orbits of f tend to that fixed point.

This is a very well known and very useful result. It is not deep or difficult, and while we shall not prove it here, all that is needed to prove it is to know that a geometric series with ratio $k < 1$ has a finite sum. The Contraction Mapping Lemma provides a mathematical justification of the technique of using an IFS for a set S to draw a picture of S. Given an IFS $\{f_1, \ldots f_m\}$ on $\mathbb{R}^n$, we simply observe that the map

$$F : Y \to f_1(Y) \cup f_2(Y) \cup \ldots \cup f_m(Y)$$

defines a contraction mapping on the complete metric space $\mathcal{H}(\mathbb{R}^n)$. It follows from the Contraction Mapping Lemma that F has a unique attractor S. This set S is the attractor of the IFS.