Chapter 2
Discrete dynamics on the real line

We consider the discrete time dynamical system $x_{n+1} = f(x_n)$ for a continuous map $f : \mathbb{R} \to \mathbb{R}$.

Definitions

The *forward orbit* of x_0 is:

$$O^+(x_0) = \{x_0, f(x_0), f^2(x_0), \ldots\}$$

The *backwards orbit* of x_0 (defined only if f is invertible) is:

$$O^-(x_0) = \{x_0, f^{-1}(x_0), f^{-2}(x_0), \ldots\}$$

(Recall that in this course $f^2(x_0)$ denotes $f(f(x_0))$, NOT $(f(x_0))^2$ OR $f''(x_0)$.)

The point x_0 is called *periodic*, of period n, if $f^n(x_0) = x_0$. The least period $n > 0$ of x_0 is called the *prime period* of x_0. (Other authors sometimes call this the *period* of x_0.)

Lemma

Suppose x_0 has prime period k. Then $f^n(x_0) = x_0 \quad \Leftrightarrow \quad k|n$.

Proof

(i) If $k|n$ then $n = qk$ for some positive integer q, so $f^n(x_0) = f^{qk}(x_0) = f^k(f^k(\ldots f^k(x_0))) = x_0$ (since $f^k(x_0) = x_0$).

(ii) Given any n and k we may write $n = qk + r$ for some integers q and r with $0 \leq r < k$ (by the division algorithm). So if $f^n(x_0) = x_0$, we have $f^{qk+r}(x_0) = x_0$, that is, $f^r(f^k(\ldots f^k(x_0))) = x_0$. So $f^r(x_0) = x_0$ (since $f^k(x_0) = x_0$). But k is the minimal positive integer with $f^k(x_0) = x_0$, and $r < k$, so $r = 0$. Hence $k|n$. □

Definition

The orbit of x_0 is *eventually periodic of period n* (or x_0 is a *preperiodic* point), if $\exists K > 0$ such that $f^{n+k}(x_0) = f^k(x_0) \quad \forall k \geq K$.

Lemma

If f is invertible, every preperiodic point is periodic.

Proof

If x_0 is a preperiodic point for f then

$$f^{K+n}(x_0) = f^K(x_0)$$

Applying f^{-1} to both sides K times, we deduce

$$f^n(x_0) = x_0$$

Hence x_0 is periodic. □

Definitions

Let p be a fixed point of f. The *basin of attraction*, or *stable set* of p is the set of points

$$W^s(p) = \{x_0 : \lim_{n\to\infty} f^n(x_0) = p\}$$

that is to say the points x_0 such that the orbit $O^+(x_0)$ is *asymptotic* to p.

More generally let p have prime period k under f. The *basin of attraction*, or *stable set* of the k-cycle $\{p, f(p), f^2(p), \ldots, f^{k-1}(p)\}$ is the set of points

$$W^s(\{p, f(p), f^2(p), \ldots, f^{k-1}(p)\}) = \{x_0 : \lim_{n\to\infty} f^{nk}(x_0) = f^r(p) \text{ some } 0 \le r < k\}$$

Example 1 For $f(x) = x^3$ the basin of attraction of the fixed point $x = 0$ is the open interval $(-1, +1)$.

Example 2 For $f(x) = x^2 - 1$ the basin of attraction of the 2-cycle $\{-1, 0\}$ is the open interval between the points $\pm(1 + \sqrt{5})/2$, with a countable infinity of points removed (all the points which map onto the repelling fixed point $(1 - \sqrt{5})/2$ under a finite number of iterations of f).

Before we can prove anything about periodic points of functions in general we need to recall some basic facts.

Background facts from calculus and topology

1. $f : \mathbb{R} \to \mathbb{R}$ is *bijective* if it is both injective (1 to 1) and surjective (onto). In this case we define the inverse $f^{-1} : \mathbb{R} \to \mathbb{R}$ of f by $f^{-1}(y) = x$ if and only if $f(x) = y$.

2. $f : \mathbb{R} \to \mathbb{R}$ is *continuous* if $\lim_{x\to x_0} f(x) = f(x_0)$ for every x_0.

3. $f : \mathbb{R} \to \mathbb{R}$ is a *homeomorphism* if f is bijective and both f and f^{-1} are continuous. If both f and f^{-1} are of class C^r (i.e. they have continuous rth derivatives at every point) then we say f is a *C^r-diffeomorphism*.

4. Let $S \subset \mathbb{R}$. A point $x \in \mathbb{R}$ is called a *limit point* or an *accumulation point* of S if there exists a sequence of points x_n in S which converges to x.

5. S is called *closed* if it contains all its limit points.

6. S is called *open* if for each $x \in S$ there exists an open interval I with $x \in I \subset S$.

7. (Proposition) If S is open then its complement is closed, and vice versa.

8. The *closure*, $\overline{S}$, of S is the smallest closed set containing S.

9. S is *dense* in $\mathbb{R}$ if $\overline{S} = \mathbb{R}$. Examples: both the set of all rational numbers $\mathbb{Q}$ and its complement are dense in $\mathbb{R}$; the set of all periodic points of the map $z \to z^2$ on the unit circle S^1 is dense in S^1.

10. The composition $g \circ f$ of functions f and g from $\mathbb{R}$ to $\mathbb{R}$ is defined by $g \circ f(x) = g(f(x))$. (In this course we usually write gf for $g \circ f$.)
The derivative of the composite gf is given by $(gf)'(x) = g'(f(x))f'(x)$ ('chain rule').
Thus for a periodic cycle $x_0, x_1, \ldots, x_{n-1}$ (with $x_n = x_0$) we have $(f^n)'(x_0) = f'(x_0)f'(x_1)\ldots f'(x_{n-1})$.
This product is called the *multiplier* of the cycle.

11. *Mean Value Theorem.* If $f : [a, b] \to \mathbb{R}$ is C^1 then there exists a point $c \in (a, b)$ with $f'(c) = (f(b) - f(a))/(b - a)$.

12. *Intermediate Value Theorem.* If $f : [a, b] \to \mathbb{R}$ is continuous, and $f(a) = u$, $f(b) = v$, then given any w between u and v there exists a point $c \in (a, b)$ with $f(c) = w$.
As an application we have the lemma that every continuous function from a close interval $I = [a, b]$ to itself has a fixed point. (Proof: consider $g(x) = f(x) - x$; then $g(a) = f(a) - a \geq 0$ and $g(b) = f(b) - b \leq 0$.)

13. The image of a closed interval under a continuous map is a closed interval. (This follows from the fact that a continuous function on a closed interval is bounded and attains its bounds, together with the intermediate value theorem.)

We now have the calculus we need to analyse fixed points and periodic points.

Definition

A fixed point p of the continuous function $f : \mathbb{R} \to \mathbb{R}$ is called *attracting* if there exists an interval $I = (p - \delta, p + \delta)$ such that $I \subset W^s(p)$ (i.e. if $x \in I$ then $\lim_{n\to\infty} f^n(x) = p$.)

A fixed point p of the continuous function $f : \mathbb{R} \to \mathbb{R}$ is called *repelling* if there exists an interval $I = (p - \delta, p + \delta)$ such that for all $x \in I - \{p\}$ there exists $k > 0$ with $f^k(x) \notin I$.

If f is a C^1 function, a fixed point p of f is called *hyperbolic* if $|f'(p)| \neq 1$.

Theorem *If p is a hyperbolic fixed point of a C^1 function $f : \mathbb{R} \to \mathbb{R}$, then p is an attractor if $|f'(p)| < 1$ and a repeller if $|f'(p)| > 1$.*

Proof

By the mean value theorem, for any $x \in \mathbb{R}$, there exists a point c between x and p with

$$f(x) - f(p) = f'(c)(x - p)$$

i.e. with

$$f(x) - p = f'(c)(x - p)$$

Now suppose $|f'(p)| < 1$. Choose any k with $|f'(p)| < k < 1$. Since f' is continuous at p, there exists $\delta > 0$ such that if $c \in I = (p - \delta, p + \delta)$ then $|f'(c) < k$. So for any $x_0 \in I$, by applying the mean value theorem as above, we have

$$|f(x_0) - p| < k|x_0 - p|$$

Thus $x_1 \in I$ (where x_1 denotes $f(x_0)$) and we can apply the same reasoning to it to get

$$|x_2 - p| < k|x_1 - p|$$

where $x_2 = f(x_1) = f^2(x_0)$. Thus

$$|x_2 - p| < k^2|x_0 - p|$$

Repeating the same argument, inductively we deduce that for all n we have

$$|x_n - p| < k^n|x_0 - p|$$

But since $k < 1$ we know that $\lim_{n\to\infty} k^n = 0$. Hence $\lim_{n\to\infty} x_n = p$.

Next suppose $|f'(p)| > 1$. Choose any K with $|f'(p)| > K > 1$. Since f' is continuous at p, there exists $\delta > 0$ such that if $c \in I = (p - \delta, p + \delta)$ then $|f'(c) > K$. So for any $x_0 \in I$, by applying the mean value theorem as above, we have

$$|f(x_0) - p| > K|x_0 - p|$$

Now either $x_1 \notin I$ (where x_1 denotes $f(x_0)$), or we can apply the same argument to x_1 and deduce that

$$|x_2 - p| > K^2|x_0 - p|$$

We can repeat the argument, and if $x_{n-1} \in I$ we can deduce that

$$|x_n - p| > K^n|x_0 - p|$$

Since $\lim_{n\to\infty} K^n = \infty$ (as $|K| > 1$) eventually some x_n must lie outside I. □

Corollary *A periodic cycle is attracting if its multiplier has modulus less than* 1 *and repelling if its multiplier has modulus greater than* 1.

Proof Apply the theorem to f^k, where k is the prime period of the cycle. □

When a fixed point or periodic cycle has multiplier of modulus 1 we have to look at higher order terms in the Taylor expansion to find whether we have an attractor or repeller.

Examples

(a) $f(x) = x + x^2$ has a neutral fixed point (in fact a 'right shunt') at $x = 0$.
(b) $f(x) = x + x^3$ has a repeller at $x = 0$.
(c) $f(x) = x - x^5$ has an attractor at $x = 0$.

Diffeomorphisms of $\mathbb{R}$

From now on *diffeomorphism* will mean C^1*-diffeomorphism.* Thus a diffeomorphism f of $\mathbb{R}$ will have derivative $f'(x)$ defined at each $x \in \mathbb{R}$, this derivative will be continuous, and the inverse g of f will have the same property. Note that $g(f(x)) = x$, so applying the chain rule we have $g'(f(x)).f'(x) = 1$, so $g'(y) = 1/f'(x)$, where y denotes $f(x)$. Thus $f'(x)$ can never be zero. hence either $f'(x) > 0$ for all $x \in \mathbb{R}$ or $f'(x) < 0$ for all $x \in \mathbb{R}$.

Lemma *Every diffeomorphism f of $\mathbb{R}$ is either order-preserving ($x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$) or order-reversing ($x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$).*

Proof If $f'(x) > 0$ for all x then f is order-preserving by the mean value theorem, and if $f'(x) < 0$ for all x then f is order-preserving by the same theorem. □

Proposition *If f is an order-reversing diffeomorphism of $\mathbb{R}$ then f has exactly one fixed point.*

Proof

Since f is bijective and reverses order,

$$\lim_{x\to+\infty} f(x) = -\infty$$

$$\lim_{x\to-\infty} f(x) = +\infty$$

Hence $g(x) = f(x) - x$ goes to $-\infty$ as x goes to $+\infty$ and goes to $+\infty$ as x goes to $-\infty$.

Hence there exist points a and b with $g(a) > 0$ and $g(b) < 0$, and so, by the intermediate value theorem, there exists at least one point c having $g(c) = 0$, and therefore $f(c) = c$.

To show that c is unique, suppose that there exists another fixed point, d, and without loss of generality suppose that $c < d$. Then as f reverses order we have $f(c) > f(d)$. But $f(c) = c$ and $f(d) = d$, so this gives $c > d$, a contradiction. □

However an *order-preserving* diffeomorphism f of $\mathbb{R}$ can have any number of fixed points, from none (for example $f(x) = x + 1$) to an uncountable infinity (for example $f(x) = x$).

Next we consider periodic cycles of period $n > 1$.

Proposition *An order-preserving diffeomorphism f of $\mathbb{R}$ has no periodic points of prime order $n > 1$*

Proof

Suppose x_0 is not a fixed point. As usual, let x_1 denote $f(x_0)$ etc.

If $x_0 < x_1$ then since f preserves order we deduce that $x_1 < x_2$, and indeed that $x_0 < x_1 < x_2 < x_3 < \ldots$ so x_0 cannot be periodic.

If $x_0 > x_1$ then since f preserves order we deduce that $x_1 > x_2$, and indeed that $x_0 > x_1 > x_2 > x_3 > \ldots$ so x_0 cannot be periodic. □

It is easy to see that an *order-reversing* diffeomorphism f of $\mathbb{R}$ can have any number of prime period two orbits. For example $f(x) = -2x$ has no period two cycles and $f(x) = -x$ has an uncountable infinity of period two cycles. (Period 2 cyles appear as 'square boxes' in a graphical analysis.) However:

Proposition *An order-reversing diffeomorphism f of $\mathbb{R}$ has no periodic points of prime period $n > 2$*

Proof

First observe that if f is an order-reversing diffeomorphism (so $f' < 0$) then f^n is an order-preserving diffeomorphism if n is even (since $(f^n)' > 0$) and an order-reversing diffeomorphism if n is odd (since then $(f^n)' < 0$).

Now f a diffeomorphism with $f' < 0 \Rightarrow f^2$ an order-preserving diffeomorphism $\Rightarrow f^2$ has no period m points, $m > 1$, $\Rightarrow f$ has no period $2m$ points, $m > 1$. This completes the proof in the case that n is even.

And if n is odd, say $n = 2m+1$ with $m > 0$, then f a diffeomorphism with $f' < 0 \Rightarrow f^n$ an order-reversing diffeomorphism $\Rightarrow f^n$ has a unique fixed point $\Rightarrow$ the only fixed point of f^n is the unique fixed point of f, and so f has no periodic points of prime period n. □

Continuous maps from $\mathbb{R}$ to $\mathbb{R}$

Here periodic cycles of all sizes can occur. A remarkable theorem which indicates the complexity of the situation is the following result, first published in the Ukrainian Journal of Mathematics in 1964:

Sarkovskii's Theorem *Order the positive integers by:*

$$3 \rhd 5 \rhd 7 \rhd 9 \ldots \rhd 2 \times 3 \rhd 2 \times 5 \rhd 2 \times 7 \ldots \rhd 2^2 \times 3 \rhd 2^2 \times 5 \rhd \ldots \rhd \ldots \rhd 2^k \rhd 2^{k-1} \rhd \ldots \rhd 2 \rhd 1$$

If $f : \mathbb{R} \to \mathbb{R}$ is continuous and has an orbit prime period m, then it has orbits of prime period n for every n such that $m \rhd n$,

Before Sarkovskii's Theorem was widely known, a simpler version was discovered independently by Li and Yorke ('Period three implies Chaos', American Mathematical Monthly, Volume 82 (1975) pages 985-992). We shall prove Li and Yorke's Theorem. The main ingredient is the Intermediate Value Theorem. We start with an elementary results.

Proposition *If a continuous function $f : \mathbb{R} \to \mathbb{R}$ has a prime period 2 cycle then it has a fixed point.*

Proof

Let the period 2 cycle be $\{p, q\}$, with $p < q$. Let $g(x) = f(x) - x$. Then

$$g(p) = f(p) - p = q - p > 0$$

$$g(q) = f(q) - q = p - q < 0$$

Hence by the intermediate value theorem there exists a point $c \in (p, q)$ with $g(c) = 0$, that is $f(c) = c$. □

For Li and Yorke's Theorem we shall need two lemmas.

Lemma 1 *If f is continuous and I, J are closed intervals such that $f(I) \supseteq J$, then there exists a closed interval $I' \subset I$ such that $f(I') = J$.*

Proof

Let $I = [a, b]$ and $J = [c, d]$. Since $f(I) \supseteq J$, there exist points $p, q \in I$ with $f(p) = c$ and $f(q) = d$ (by the intermediate value theorem). Suppose $p < q$ (the case $p > q$ can be treated similarly). Take p' to be the right-most point in $[p, q]$ having $f(p') = c$, and then take q' to be the left-most point in $[p', q]$ having $f(q') = d$. Then there is no point x between p' and q' with $f(x) = c$ or $f(x) = d$. It follows (again by the intermediate value theorem) that $f([p', q']) = [c, d]$. □

Lemma 2

If f is continuous and I is any interval such that $f(I) \supseteq I$, then f has a fixed point in I.

Proof

Let $I = [a, b]$. Since $f(I) \supseteq I$ there exist points $c, d \in I$ with $f(c) = a, f(d) = b$. Let $g(x) = f(x) - x$. Then $g(c) = a - c \leq 0$ and $g(d) = b - d \geq 0$. By the intermediate value theorem there exists x between c and d with $g(x) = 0$, that is $f(x) = x$. □

Theorem (Li and Yorke)

If a continuous function $f : \mathbb{R} \to \mathbb{R}$ has a periodic orbit of prime period 3, then it has periodic orbits of prime period n for all positive integers n.

Proof

Let the orbit of prime period 3 be $a < b < c$ with $f(a) = b, f(b) = c$ and $f(c) = a$ (the only other possibility, $f(a) = c, f(c) = b, f(b) = a$ can be dealt with similarly).

Let $J_0 = [b, c]$, and let n be any integer ≥ 4.

$$f(J_0) \supseteq [a, c] \supseteq [b, c] = J_0$$

So by Lemma 1 there exists an interval $J_1 \subseteq J_0$ with

$$f(J_1) = J_0$$

Thus $f(J_1) \supseteq J_1$ and we can repeat the same reasoning with J_1 in place of J_0 to show that there exists an interval $J_2 \subset J_1$ with $f(J_2) = J_1$. Repeating the same reasoning, there exists an interval $J_3 \subset J_2$ with $f(J_3) = J_2$. Inductively this gives us intervals

$$J_{n-2} \subseteq J_{n-3} \subseteq \ldots \subseteq J_0$$

with

$$f^{n-2}(J_{n-2}) = J_0 = [b, c]$$

and hence

$$f^{n-1}(J_{n-2}) = f(J_0) \supseteq [a, c]$$

But $[a, c] \supset [a, b]$, so, by applying Lemma 1 to f^{n-1}, there exists an interval $J_{n-1} \subseteq J_{n-2}$ with

$$f^{n-1}(J_{n-1}) = [a, b]$$

Now

$$f^n(J_{n-1}) = f([a,b]) \supseteq [b,c] = J_0$$

As $J_{n-1} \subseteq J_0$ we deduce by Lemma 2 that f^n has a fixed point, say x_0, in J_{n-1}.

The orbit $x_0, x_1, \ldots, x_{n-1}$ of this point x_0 has $x_i \in J_0 = [b,c]$ for $0 \leq i \leq n-2$. It then has $x_{n-1} \in [a,b]$, and finally $x_n = x_0 \in [b,c]$. Hence this orbit cannot have period strictly dividing n.

It remains to show that f has orbits of prime periods $n = 2$ and $n = 1$. We leave these as exercises for the reader. $\square$