Chapter 1

Continuous time and discrete time dynamical systems

This course will be largely about *discrete time* dynamical systems, that is to say iterated maps $\mathbb{R} \to \mathbb{R}$, or $\mathbb{R}^2 \to \mathbb{R}^2$, or $\mathbb{C} \to \mathbb{C}$ etc, but we start off with a brief review of *continuous time* dynamical systems (ordinary differential equations). This will give us a historical perspective, and will also provide a good starting point for our study of the qualitative features of iterated maps in the following chapters.

Definition

A *continuous time dynamical system on* $\mathbb{R}^2$ is a pair of differential equations

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix}$$

where f_1 and f_2 are functions $\mathbb{R}^2 \to \mathbb{R}$. One can think of these as defining a *vector field* on $\mathbb{R}^2$, that is to say they assign a vector:

$$\begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix}$$

to each point $(x, y) \in \mathbb{R}^2$.

Examples

$$(1) \quad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \qquad\qquad (2) \quad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -y \\ -x \end{pmatrix}$$

'Solving the differential equation, with initial condition (x_0, y_0)*'* means finding a path $(x = x(t), y = y(t))$ with $x(0) = x_0)$ and $y(0) = y_0$, such that $\dot{x} = f_1(x, y)$ and $\dot{y} = f_2(x, y)$ at every point on the path. In other words as we travel along the path, at each point our velocity is the vector assigned to that point. This path is known as the *orbit* of the initial point (x_0, y_0).

Examples

(1) $$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

with initial condition $(x_0, y_0) \neq (0,0)$ has solution:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} x_0 e^t \\ y_0 e^t \end{pmatrix}$$

And

(2)
$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -y \\ -x \end{pmatrix}$$

with initial condition $(x_0, y_0) \neq (0,0)$ has solution:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} r\cos(t+\alpha) \\ r\sin(t+\alpha) \end{pmatrix}$$

where $r = \surd(x_0^2 + y_0^2)$ and $\alpha = \arctan(y_0/x_0)$.

Theorem (Existence and uniqueness of solutions of differential equations)

Let $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ *and* $\mathbf{x_0}$ *be such that:*

(i) $\mathbf{f}$ *is differentiable at* $\mathbf{x}$ *and* $\partial\mathbf{f}/\partial\mathbf{x}$ *is continuous at* $\mathbf{x}$ *for all* $\mathbf{x} \in \mathbb{R}^n$*, and*

(ii) $\mathbf{f}(\mathbf{x_0}) \neq \mathbf{0}$*.*

Then the equation $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x})$ *has a unique solution* $\mathbf{x}(t)$ *with initial condition* $\mathbf{x_0}$*, on some interval* $t \in [0, T]$ *with* $T > 0$*.*

This is a key theorem of 19th century analysis. We are not going to prove it, but it is important to understand why it is true, which is that one can 'join up the arrows of a vector field to make a path'. Note that we need $\mathbf{f}(\mathbf{x_0}) \neq \mathbf{0}$ since we need to know in what direction to start our path. Also note that in general we cannot say how large T is, because if we reach a point $\mathbf{x}$ where $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ we no longer have a unique direction in which to continue our path.

In practice we shall be interested in *qualitative* features of dynamical systems, rather than analytic formulae for solutions. By drawing 'phase portraits' of some examples we can see such features.

(i)
$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

has a *repelling fixed point* at $(0,0)$ since all the vectors in the vector field point directly away from this point.

(ii)
$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix}$$

has an *attracting fixed point* at $(0,0)$.

(iii)
$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$$

has a *neutral fixed point* at $(0,0)$ as the orbits of the system are circles centred at this point.

(iv) The system $r = r(1-r)$, $\dot{\theta} = 1$ (in polar coordinates) has a repelling fixed point at $(0,0)$, and the unit circle is an *attracting cycle*.

For continuous time dynamical systems in the plane there can be fixed points which are attracting, repelling or neutral, and there can be cycles of each of these three types. But the situation is much more complicated for continuous time dynamical systems in $\mathbb{R}^3$. Here we can see *strange attractors*.

The Lorenz equations

These are equations devised by Edward Lorenz in his 1963 model for atmospheric convection (his parameters (x, y, z) are not space co-ordinates):

$$\dot{x} = \sigma(y - x)$$

$$\dot{y} = rx - y - xz$$

$$\dot{z} = -bz + xy$$

$\sigma = 10, b = 8/3, r = 28$ were Lorenz's original parameter values. He ran the same computer program several times with almost the same initial conditions and found totally different long-term behaviour. The system has a 'strange attractor' which exhibits 'sensitive dependence on initial conditions'. This is an example of the so-called 'butterfly effect'. It is virtually impossible to give detailed weather forecasts more than a few days ahead as we can never know the initial conditions precisely enough to do so.

We shall not attempt to analyse the Lorenz equations in detail: instead we shall start with much simpler systems:

Continuous time dynamical systems on $\mathbb{R}$

The are ordinary differential equations, of the form $\dot{x} = f(x)$. Provided f is of class C^1 (that is to say differentiable, with continuous derivative) the existence and uniqueness theorem gives us a unique orbit $x = x(t)$ for each initial condition x_0 such that $f(x_0) \neq 0$, for t in some interval $[0, T]$.

Examples

(*1*) $\dot{x} = ax$ for some constant $a > 0$.

Examination of the phase portrait of this map shows a single repelling fixed point, at $x = 0$.

We can compute the analytic solution of the system:

$$\int \frac{dx}{x} = \int a dt \quad \Rightarrow \quad \ln(x) = at + c \quad \Rightarrow \quad x = x_0 e^{at}$$

(*2*) $\dot{x} = x(1 - x)$

Here the phase portrait exhibits a repeller at $x = 0$ and an attractor at $x = 1$.

Analytic solution:

$$\int \frac{dx}{x(1-x)} = \int dt \quad \Rightarrow \quad \int \left(\frac{1}{x} - \frac{1}{x-1}\right) dx = \int dt$$

$$\Rightarrow \quad \ln \frac{x}{x-1} = t + c \quad \Rightarrow \quad \frac{x}{x-1} = \frac{x_0}{x_0 - 1} e^t \quad \Rightarrow \quad x = \frac{x_0 e^t}{x_0 e^t - x_0 + 1}$$

Note that just because a differential equation 'has a solution' does not mean that it is easy to find a formula for the solution in terms of polynomials, trigonometric functions etc - indeed there may be no such formula!

Given any particular $f : \mathbb{R} \to \mathbb{R}$ it is easy to draw the phase portrait of the dynamical system $\dot{x} = f(x)$ and so obtain a *qualitative* description of the behaviour of the system, even if there is no analytic formula for the system: there are fixed points where $f(x) = 0$, the vector field points to the right where $f(x) > 0$, and it points to the left where $f(x) < 0$.

There is not much more to say in general about continuous time dynamical systems on $\mathbb{R}$, except that there are four types of fixed point (attractor, repeller, neutral 'right shunt' and neutral 'left shunt') and there are certain obvious rules about the order in which these occur along the real line because of the directions of the arrows at each type of fixed point (for example two attractors must be separated by a repeller). There can be more complicated behaviour with *discrete time* systems on $\mathbb{R}$.

Definitions

A *dynamical system* consists of a set of *states* (for example points of $\mathbb{R}$ or of $\mathbb{R}^2$), together with a time evolution rule.

(i) For *discrete* time $t \in \mathbb{Z}$ (or $t \in \mathbb{N} \cup \{0\}$) the evolution rule is of the form $x_{t+1} = f(x_t)$ (which we usually write $x_{n+1} = f(x_n)$).

(ii) For *continuous* time $t \in \mathbb{R}$ (or $t \in \mathbb{R}_{\geq 0}$) the evolution rule is of the form $\dot{x} = f(x)$.

Every continuous time dynamical system gives rise to a discrete time system by taking the 'time one map'.

Example

$$x = ax$$

This has solution $x(t) = x_0 e^{at}$, so

$$x(t+1) = x_0 e^{a(t+1)} = x(t)e^a$$

Thus the 'time one' map is:

$$x_{n+1} = e^a x_n$$

There is another way to obtain a discrete time system from a continuous time system, called the method of *Poincaré sections.*

Example

$$\begin{pmatrix} \dot{r} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} r(1-r) \\ 1 \end{pmatrix}$$

Consider the section $S = \mathbb{R}_{\geq 0}$ (the non-negative half of the x-axis). The *first return map* $S \to S$ is the map defined by sending each $x_0 \in S$ to the point of S where the orbit of x_0 under the system first returns to S.

In this example the system has an analytic solution:

$$r = \frac{r_0 e^t}{r_0 e^t - r_0 + 1}, \quad \theta = \theta_0 + t$$

so the first return map applied to $x \in S$ is:

$$x \to \frac{xe^{2\pi}}{xe^{2\pi} - x + 1}$$

since the the initial condition is $r_0 = x$, $\theta_0 = 0$ at $t = 0$, and the first return to S of the orbit of $x \in S$ occurs when $\theta = 2\pi$, i.e. at time $t = 2\pi$.

Poincaré sections take us from

continuous time dynamical systems on $(n+1)$*-dimensional spaces*

to

discrete time dynamical systems on n*-dimensional spaces*

For much of this course we shall be looking at discrete dynamical systems on $\mathbb{R}$ and $\mathbb{R}^2$, but the results we obtain can also be used to say things about continuous time dynamical systems in dimensions 2 and 3. (We remark that there is also a way to go in the other direction - a discrete time dynamical system can be 'suspended' to obtain a continuous time system on a space one dimension higher).

Here is an example of a family of discrete time systems on $\mathbb{R}$ we shall be looking at in more detail in later chapters:

The *logistic map* is the map $x_{n+1} = \mu x_n(1 - x_0)$ ($\mu > 0$ a constant). We examine its behaviour by '*graphical analysis*' (plotting the orbit of a point x_0 on the x-axis by drawing a vertical line upwards from x_0 till we hit the graph of $y = \mu x(1 - x)$, then proceeding horizontally till we hit the graph of $y = x$, then vertically downwards till we hit the x-axis at the next point of the orbit, x_1). We observe:

(1) For $\mu = 1$ the logistic map has an attracting fixed point at $x = 0$ and every $x_0 \in [0, 1]$ has orbit tending to this attracting fixed point.

(2) For $\mu = 2$ the point $x = 0$ has become a repelling fixed point, and every $x_0 \in (0, 1)$ has orbit tending to an attracting fixed point at $x = 1/2$.

We remark that whether $x = 0$ is an attractor or a repeller depends on the slope μ of the graph of $y = \mu x(1 - x)$ at $x = 0$. In the following Chapter we shall be looking more closely at this and prove that there is a general rule that if x_0 is a fixed point of a differentiable function f, then it is an attractor if $|f'(x_0)| < 0$ and a repeller if $|f'(x_0)| > 0$.