Mating Circle-Packing Groups with Quadratic Maps

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Abstract

We exhibit holomorphic correspondences which are *matings* between quadratic maps $q_c : z \to z^2 + c$ and circle-packing representations of the free product $C_2 * C_3$ of cyclic groups of orders 2 and 3. We show that as the complex parameter c passes through the value $c = 0.25$ the limit set of the correspondence undergoes a discontinuous 'implosion' into circle-packings, analogous to the implosion of the Julia set of $z \to z^2 + c$ into Cantor sets. We describe the combinatorics of these circle-packings and of the corresponding subset of parameter space.

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Contents

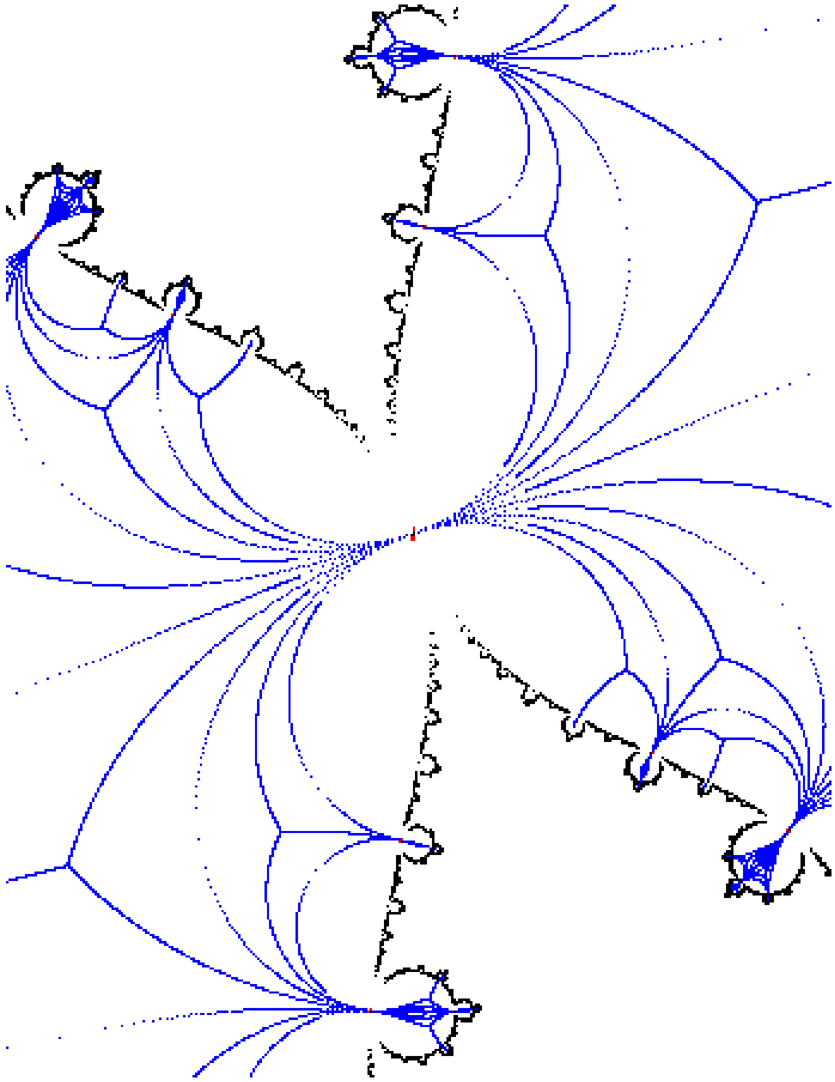

Figure 9. Mating of $z \to z^2$ with the circle-packing representation of $C_2 * C_3$ of rotation number 1/4. This is a correspondence in the family (1) (see section 1), with parameter values

$$a = -\frac{1}{26} + \frac{15\sqrt{3}}{26}i; \quad 3k = 1 + a + a^2$$

1 Introduction

Let r be a representation of the free product $C_2 * C_3$ of cyclic groups of orders 2 and 3 as a Kleinian group, that is to say as a discrete subgroup of $PSL(2, \mathbf{C})$. If the action of r (on the Riemann sphere) is faithful, and has non-empty ordinary set, then either the ordinary set is connected and its complement (the limit set) is a Cantor set, or the ordinary set is a disjoint union of countably many round open discs and its complement is the closure of the union of their boundary circles: in this case we shall say that r is a *circle-packing* representation of $C_2 * C_3$. The Teichmüller space of representations of $C_2 * C_3$ in $PSL(2, \mathbf{C})$ up to conjugacy in the latter, is parameterized by a single complex variable, for example by the cross-ratio of the fixed points of generators of C_2 and C_3. The faithful discrete representations with connected ordinary set correspond to a once-punctured topological disc $\mathcal{D}$ (geometrically a once-punctured 'cauliflower') in parameter space. These representations are all quasiconformally conjugate to one another. The *circle-packing* representations of $C_2 * C_3$ are obtained from them by 'pinching hyperbolics to parabolics', and occur at a dense set of cusps on the boundary $\partial\mathcal{D}$ of the cauliflower. One example is the modular group $PSL(2, \mathbf{Z})$, which is well-known to be freely generated by the Möbius transformations $z \to -1/z$ and $z \to -1/(z+1)$, of orders 2 and 3 respectively, and which we can therefore regard as a representation of $C_2 * C_3$ in $PSL(2, \mathbf{C})$. It is a circle-packing representation: its ordinary set has two components, the open upper and lower halves of the complex plane, and its limit set is a single circle, the real axis together with the point at infinity.

In order to *mate* Kleinian groups with rational maps we have to move into a larger category containing both. A *holomorphic correspondence* on the Riemann sphere $\hat{\mathbf{C}}$ is a relation $f \subset \hat{\mathbf{C}} \times \hat{\mathbf{C}}$ defined by a polynomial equation $p(z, w) = 0$. We can regard f as the graph of a multivalued map $f : z \to w$ with multivalued inverse $f^{-1} : w \to z$. If p has degree m in z and degree n in w and has no square factors, then f is an $m : n$ correspondence. A *mating* of a quadratic map $q_c : z \to z^2 + c$ tha faithful discrete representation r of $C_2 * C_3$ having connected ordinary set, is a $2 : 2$ holomomorphic correspondence f such that $\hat{\mathbf{C}}$ is partitioned into two completely invariant sets, on one of which the action of f resembles that of $C_2 * C_3$ on its ordinary set, and on the other of which the action resembles that of q_c and q_c^{-1} on two copies of the filled Julia set $K(q_c)$. Before we can express this resemblance formally we need to point out one more property of representations of $C_2 * C_3$. Write G for the unique extension of $C_2 * C_3$ by an involution χ which anticommutes with the generators σ of C_2 and ρ of C_3. The group $C_2 * C_3$ has index two in G and any representation r of $C_2 * C_3$ extends uniquely to a representation of G (see Section 2 below). Write $\Omega(r(G))$ for the ordinary set of the Kleinian group $r(G)$. We are now ready for the formal definition of a mating. Let r be a faithful discrete representation r of $C_2 * C_3$ of the first type, and $q_c : z \to z^2 + c$ be a quadratic map with connected Julia set.

Definition

A *mating* of r with q_c is a $2:2$ *holomorphic correspondence* f on the Riemann sphere $\hat{\mathbf{C}}$ such that $\hat{\mathbf{C}}$ is partitioned into a completely invariant open set $\Omega = \Omega(f)$ and its complement $\Lambda = \Lambda(f)$ with the following properties:

• Λ is the disjoint union of two sets Λ_+ and Λ_-, on which the $2:2$ correspondence $f : \Lambda \to \Lambda$ decomposes into the following parts:

$f : \Lambda_- \to \Lambda_-$, a $2:1$ correspondence (a map of degree two);
$f : \Lambda_+ \to \Lambda_+$, a $1:2$ correspondence (the inverse of a map of degree two);
$f : \Lambda_- \to \Lambda_+$, a $1:1$ correspondence (a bijection).

• There is a homeomorphism from the filled Julia set K of q_c to Λ_- conjugating $q_c|_K$ to $f|_{\Lambda_-}$ and a homeomorphism from K to Λ_+ conjugating $q_c|_K$ to $f^{-1}|_{\Lambda_+}$, each of these homeomorphisms having zero Beltrami derivative almost everywhere on K (and so in particular being conformal on the interior of K).

• The $2:2$ correspondence $f : \Omega \to \Omega$ acts properly discontinuously and there is a conformal homeomorphism h from the orbifold Ω/f to the orbifold $\Omega(r(G))/G$ compatible with the actions of f and G respectively in the following sense: there exists a (completely invariant) set of curves $\mathcal{C}$ in Ω and a fundamental domain D for the action of χ on $\Omega(r(G))$ which is invariant under $C_2 * C_3$, such that h lifts to a conformal homeomorphism $(\Omega - \mathcal{C}) \to D$ conjugating f to $\{\sigma\rho, \sigma\rho^{-1}\}$.

In [6] it is shown that for every representation r of the first type (connected ordiary set) and quadratic map q_c with connected Julia set, there exists such a mating in the two parameter family of correspondences $z \to w$ defined by:

$$\left(\frac{az+1}{z+1}\right)^2 + \left(\frac{az+1}{z+1}\right)\left(\frac{aw-1}{w-1}\right) + \left(\frac{aw-1}{w-1}\right)^2 = 3k \tag{1}$$

The two degrees of freedom provided by the (complex) parameters a and k correspond to our freedom to vary both r and c. Below we explain how to construct matings analogous to those of [6] when r is of the second type, a circle-packing representation of $C_2 * C_3$. These matings are also represented by correspondences in the family (1), having values of the parameters a and k on the *boundary* of the region corresponding to r of the first type. The first examples were presented in [1], in the special case of the modular group $PSL(2, \mathbf{Z})$. But $PSL(2, \mathbf{Z})$ is a circle-packing representation of $C_2 * C_3$ having just two components in its ordinary set, and for most circle-packing representations there are infinitely many components, making the dynamical behaviour considerably more intricate. We describe the combinatorics of such matings and present computer plots of examples. We follow on to explore the behaviour of these matings as c passes through the value 0.25 at which the Julia set of q_c becomes disconnected (in a 'parabolic implosion' described by Douady and co-workers [10, 11]). In our family of matings, as c passes through this value the set $\Lambda(f)$ 'implodes', and $\Omega(f)$ 'invades' the interior of $\Lambda(f)$, but unlike the Julia set of q_c the limit set $\Lambda(f)$ remains connected, collapsing (discontinuously) onto a fine tracery of circles in what was previously its interior. We describe the combinatorics of this process, present more computer plots, and conclude with a conjectural combinatorial description of a slice of parameter space.

The circle-packing representations of $C_2 * C_3$ are indexed by *rotation numbers*, rational numbers ν between 0 and 1. In the interests of simplicity of both computation and exposition we restrict all of our computer plots and many of our combinatorial descriptions to the case $\nu = 1/4$, the first 'non-trivial' rotation number. Moreover in the interest of keeping the paper to a readable length we also restrict ourselves to describing the phenomena that occur, and at best we offer sketch proofs (see subsection 3.8 for the main conjecture concerning existence of the matings we are concerned with here). Finally we remark that the choice of $C_2 * C_3$ is also made for simplicity of exposition: there is analogous behaviour for a general free product of finite cyclic groups $C_m * C_n$, mated with an appropriate polynomial of degree $(m-1)(n-1)$ [5].

2 Faithful discrete representations of $C_2 * C_3$ in $PSL(2, \mathbf{C})$.

Up to conjugacy each representation r of $C_2 * C_3$ in $PSL(2, \mathbf{C})$ is determined by a single complex parameter, the cross-ratio between the fixed points on $\hat{\mathbf{C}}$ of the action of the generator σ of C_2 and those of the generator ρ of C_3. Such a representation comes equipped with a (unique) involution χ which exchanges the two fixed points of σ and also those of ρ, so that $\chi\sigma = \sigma\chi$ and $\chi\rho = \rho^{-1}\chi$ [4]. On the Poincaré 3-disc χ is simply rotation through π around the common perpendicular to the axes of σ and ρ. Write G for the group $< \sigma, \rho, \chi >$.

The faithful discrete actions r with connected regular set $\Omega(r(G))$ form a single quasiconformal conjugacy class, the class of representations for which one can find simply-connected fundamental domains for σ and ρ with interiors together covering the whole Riemann sphere, that is to say the conditions of the simplest form of the Klein Combination Theorem are satisfied [4]. Such fundamental domains may be constructed as illustrated in figure 1.

Here P and P' are the fixed points of ρ, Q and Q' are the fixed points of σ, R is a fixed point of (the involution) $\chi\rho$ and S is a fixed point of $\chi\sigma$. The lines l, m and n, joining R to S, Q to S and R to P, are chosen such that they are smooth and remain non-intersecting in the quotient orbifold $\Omega(r(G))/G$. The region bounded by $n, \rho n, \chi n$ and $\chi\rho n$ is a fundamental domain for ρ, and the region exterior to the loop made up of $m, \sigma m, \chi m$ and $\chi\sigma m$ is a fundamental domain for σ. The intersection of these two regions is a fundamental domain for the (faithful) action of $C_2 * C_3$ on $\Omega(r(G))$, and the half Δ of this intersection bounded by $n, l, m, \sigma m, \chi l$ and ρn is a fundamental domain for the action of G. The union of all translates of Δ under elements of $C_2 * C_3$ is a topological disc D which is a fundamental domain for the action of χ on $\Omega(r(G))$. The complement $\Lambda(r(G))$ of $\Omega(r(G)) = D \cup \chi(D)$ in $\hat{\mathbf{C}}$ is a Cantor set.

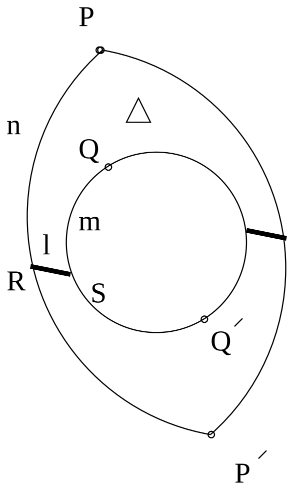

Figure 1 A fundamental domain Δ for the group $G = <\sigma, \rho, \chi>$.

The orbifold $\Omega(r(G))/G$ is a sphere with one cone point of angle $2\pi/3$ and three of angle π. According to [13] we can deform r to a representation on the boundary of $\mathcal{D}$ by pinching any simple closed geodesic on $\Omega(r(G))/G$ which lifts to a union of disjoint arcs in $\Omega(r(G))$. Contracting these arcs to points deforms the corresponding hyperbolic element of $r(G)$ into a parabolic element, and 'joins up' the Cantor set $\Lambda(r(G))$ to form a circle-packing. To give a meaning to the words 'simple closed geodesic' in an orbifold context, one should pass to a finite index torsion-free subgroup and so replace the orbifold by a Riemann surface finitely covering it. However in our case it is sufficient to pass to the subgroup H of G generated by the products $\sigma\rho$ and $\sigma\rho^{-1}$ (which contains the order 3 element ρ). The orbifold $\Omega(r(G))/H$, a four-fold covering of $\Omega(r(G))/G$, still has cone points, but there are just two of them, and both are of angle $2\pi/3$. Topologically, $\Omega(r(G))/H$ is a torus. Since both cone points are of odd order, a curve ending at or passing through either does not lift to an arc in $\Omega(r(G))$. Thus we are reduced to looking for sets of non-intersecting simple closed geodesics on the torus, avoiding the cone points and invariant under σ and χ. It is an elementary exercise to check that for each rational $0 < p/q \leq 1$ there is just one such set of geodesics of rotation number $\nu = p/q$ in the usual language of geodesics on tori, and that it consists of just one simple closed curve on $\Omega(r(G))/H$ when q is even and two (parallel) curves when q is odd (where p/q is written in lowest terms). In figure 2 we illustrate examples with $\nu = 1/1$ (which we shall usually write as $\nu = 0$ since ν is only defined modulo $\mathbf{Z}$) and with $\nu = 1/2, 1/3, 1/4, 1/6$. In the figures the fixed points of ρ and σ are marked, and in each case the involution χ is rotation through an angle of π about the centre point of the picture.

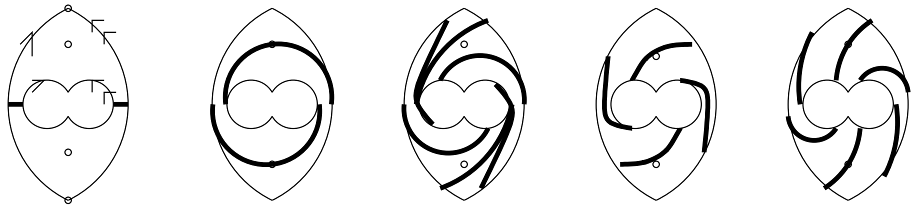

Figure 2. Geodesics with rotation numbers $1/1(=0), 1/2, 1/3, 1/4$ and $1/6$ on the torus $\Omega(r(G))/H$.

Pinching the geodesic of rotation number $\nu = 0$ to a point simply gives us the familiar representation of $C_2 * C_3$ as $PSL(2, \mathbf{Z})$, with limit set the real axis together with infinity. Pinching the geodesic having $\nu = 1/2$ gives us the same limit set (up to conformal conjugation): the only difference is that σ now acts as $z \to 1/z$ in place of $z \to -1/z$ and so interchanges the two components of $\Omega(r(G))$. For all other rotation numbers the ordinary set $\Omega(r(G))$ has *infinitely* many components after pinching, and the limit Cantor set $\Lambda(r(G))$ 'joins up' to become the closure of an *infinite* union of circles. In figure 3 we illustrate the first few circles in the limit set in the case of $\nu = 1/4$. The disc D_0 (a component of the regular set $\Omega(r(G))$) is stabilised by the subgroup of G generated by ρ and the involution $\chi\sigma\rho^{-1}\sigma$. This subgoup (itself isomorphic to $C_2 * C_3$) acts as a copy of $PSL(2, \mathbf{Z})$, with limit set the single circle ∂D_0. The regular set $\Omega(r(G))$ is the union of countably many discs, each stabilised by a conjugate of the stabiliser of D_0, and the limit set $\Lambda(r(G))$ is the closure of the union of the boundaries of these discs.

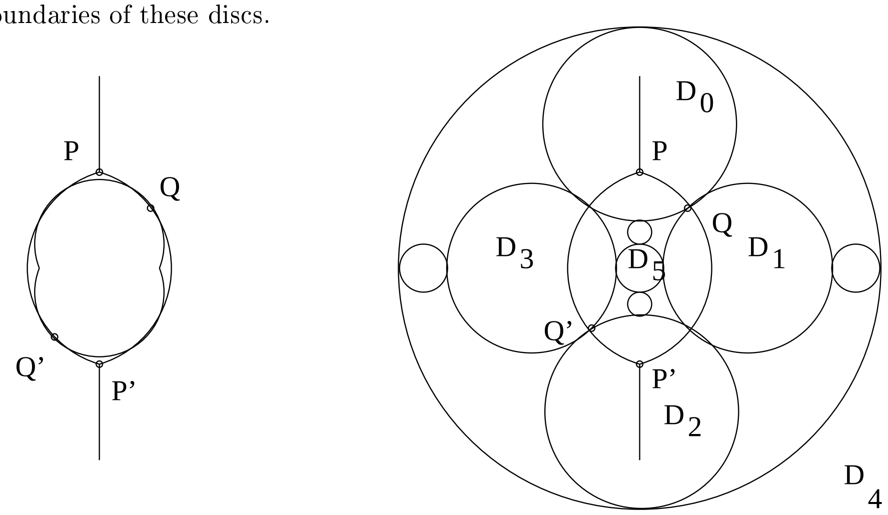

Figure 3
(i) Rotation number $1/4$ (after pinching). The region between the inner and outer curves is a fundamental set for the action of $H =< \sigma\rho, \sigma\rho^{-1} >$:any one of the four components is a fundamental domain for the action of $G =< \sigma, \rho, \chi >$.
(ii) The first few discs of the ordinary set of the group illustrated in (i). The 'principal cycle' is $D_0 \to D_1 \to D_2 \to D_3 \to D_0$ with maps:

$\sigma\rho, \sigma\rho^{-1} : D_0 \to D_1$
$\sigma\rho^{-1} : D_1 \to D_2$
$\sigma\rho, \sigma\rho^{-1} : D_2 \to D_3$
$\sigma\rho^{-1} : D_3 \to D_0$
The disc D_4 maps onto D_0 via $\sigma\rho^{-1}$ and onto D_2 via $\sigma\rho$. The discs D_1 and D_3 map onto D_5 via $\sigma\rho$ and $\sigma\rho^{-1}$ respectively.

The pattern of discs shown for $\nu = 1/4$ generalises as follows for other rational rotation numbers, which we shall now write as $p/2q$ with (p, q) coprime (but p not necessarily odd). In each case the disc D_0 is mapped to itself by ρ and a certain word w of length $2q$ in $\sigma\rho, \sigma\rho^{-1}$. This word can be derived from the *Sturmian* word of rotation number p/q by a process analogous to that described for circle-packing representations of $C_2 * C_4$ in [4]. The $2q$ images of D_0 under initial subwords of w form a cycle, each touching the preceding disc and the subsequent disc. We shall refer to this as the *principal cycle* of discs in the circle-packing. Note that D_0 is mapped to D_1 by both $\sigma\rho$ and $\sigma\rho^{-1}$, and D_q is also mapped to D_{q+1} by both these group elements, but apart from these each D_i is mapped to D_{i+1} by just one of $\sigma\rho, \sigma\rho^{-1}$. Note also that the involution χ stabilises the principal cycle, sending each D_i to D_{i+q}.

It is easy to compute generators of $C_2 * C_3$ for the rotation number $1/4$ circle-packing representation explicitly. Take ρ to be the rotation through $2\pi/3$ around fixed points $z = -1/2 \pm i\sqrt{3}/2$, and as fundamental domain D_ρ for ρ take the intersection of the round discs having centres -1 and 0 and radius 1. Now take a line through $z = -1/2$ at angle $\pi/4$ to the real axis, and let P, P' be its intersections with the boundary ∂D_ρ of D_ρ. Thus,

$$P = \left(\frac{\sqrt{7}-3}{4}\right) + \left(\frac{\sqrt{7}-1}{4}\right) i \qquad P' = \left(\frac{-1-\sqrt{7}}{4}\right) - \left(\frac{\sqrt{7}-1}{4}\right) i$$

It is a straightforward exercise to verify that the involution σ having P and P' as its fixed points sends ∂D_ρ to a curve $\sigma(\partial D_\rho)$ which meets ∂D_ρ only at P and P' and that σ and ρ therefore yield the rotation number $1/4$ circle-packing representation of $C_2 * C_3$.

3 Mating $q_c : z \to z^2 + c$ with representations of $C_2 * C_3$.

In this section we recall the definition and construction of a mating when the ordinary set of the representation $r : C_2 * C_3 \to PSL(2, \mathbf{C})$ is connected [6], and we adapt it to the situation where a hyperbolic element of the group has been pinched to a parabolic.

Given a quadratic map $q_c : z \to z^2 + c$, where for simplicity we assume $c \in \mathcal{M}$, the Mandelbrot set, and given a discrete representation of $C_2 * C_3$ with connected regular set Ω we construct an annulus corresponding to each [6].

3.1 An annulus associated to the quadratic map

There is a holomorphic conjugacy (due to Böttcher) from $z \to z^2$ to q_c on a neighbourhood of ∞, fixing the point ∞ and tangent to the identity map there [8]. An *equipotential* for q_c is the image of a circle $\{Re^{2\pi it} : 0 \leq t < 1\}$ under this conjugacy. It is a smooth Jordan curve parameterized by *external angle* t. The region bounded by such an equipotential is a simply-connected domain V, mapped $2:1$ by q_c onto a larger domain $U \supset V$ which also has boundary an equipotential parametrised by external angle. Denote the annulus $U - V$ by A and its inner and outer boundaries by $\partial_1 A$ and $\partial_2 A$ respectively. The map q_c sends $\partial_1 A$ two-to-one onto $\partial_2 A$. There is an involution $i : z \to -z$ on V sending each $z \in V$ to the other point which has the same image in U under q_c, and there are many choices possible of an orientation-reversing smooth involution j on $\partial_2 A$, a canonical choice being given by $t \to 1-t$ on external angles.

3.2 An annulus associated to the Kleinian group

Recall the fundamental domain Δ constructed for the group $G =< \sigma, \rho, \chi >$ in Section 2, and illustrated in figure 1. Let B denote the annulus consisting of the three copies $\Delta \cup \rho\Delta \cup \rho^{-1}\Delta$ of Δ, with the boundary identifications (induced by χ) indicated in figure 4.

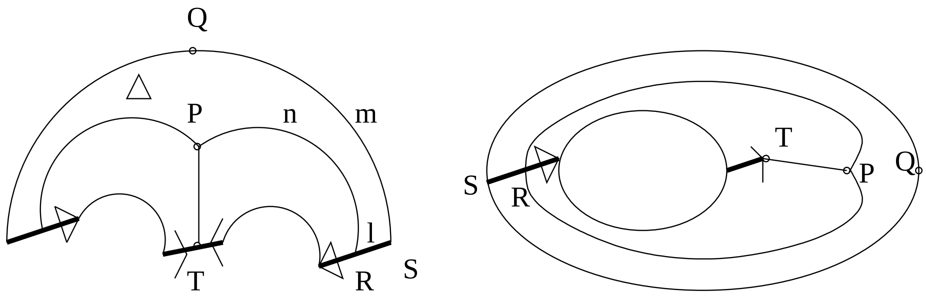

Figure 4 The set $\Delta \cup \rho(\Delta) \cup \rho^{-1}(\Delta)$ and its quotient the annulus B.

3.3 A bijection between annuli

The rotation ρ maps $\Delta \cup \rho\Delta \cup \rho^{-1}\Delta$ to itself in the obvious way, with a single fixed point at P, but since ρ *anticommutes* with χ this action does not descend to a rotation on the quotient annulus B. Instead it descends to the action of a $2:2$ correspondence g on B, mapping each $z \in B$ to the pair $\{\rho z, \rho^{-1} z\}$ (or rather to their equivalence classes under the action of χ). The points P and T are singular for g, having *unique* images P and R respectively, but all other points of B have two distinct images under g. Generic orbits of g have cardinality three, the correspondence g sending each point of an orbit to the other two, and the image of Δ in B is a 'fundamental domain' for the action. The boundary of B is divided into three segments (two inner and one outer, figure 4), each of which is mapped to the other two by g. Thus when its domain is restricted to the inner boundary $\partial_1 B$, and its range is restricted to the outer boundary $\partial_2 B$, the correspondence g defines a two to one map. When restricted to a correspondence from the inner boundary to itself, g defines a (fixed point free) bijection. Moreover since the involution σ commutes with χ it descends to an involution (which we also denote σ) on the outer boundary $\partial_2 B$ of B, having fixed points Q and S. Observe that since the composition $\sigma \circ g : \partial_1 B \to \partial_2 B$ is an orientation preserving two to one map, and the bijection $g : \partial_1 B \to \partial_1 B$ is the covering involution of this map, the annulus B carries the same data as that furnished on the annulus A by the quadratic map q_c:

Lemma 1 *[6] There exists a quasiconformal homeomorphism h from A to B which restricts to a smooth homeomorphism from ∂A to ∂B conjugating the boundary maps $(q_c : \partial_1 A \to \partial_2 A,\ j : \partial_2 A \to \partial_2 A)$ to the boundary maps $(\sigma \circ g : \partial_1 B \to \partial_2 B,\ \sigma : \partial_2 B \to \partial_2 B)$.*

We use h to transport the $2:2$ correspondence g from B to A and abuse notation to the extent of also denoting $h \circ g \circ h^{-1}$ by g.

3.4 Construction of matings when $\Omega(r(G))$ is connected

Theorem 1 *[6] For every quadratic map $q_c : z \to z^2 + c$ with $c \in \mathcal{M}$ and every faithful discrete representation r of $C_2 * C_3$ in $PSL(2, \mathbf{C})$ having connected regular set, there exists a polynomial relation $p(z, w) = 0$ defining a $2:2$ correspondence $z \to w$ which is a mating of q_c with r.*

We outline the construction which provides the proof of Theorem 1 (see [6] for details), and which we subsequently adapt to the circle-packing case.

First glue together U and a second copy U' of U, via the boundary involution j, to obtain a sphere $U \cup U'$, equipped with an involution, which we also denote j, exchanging U with U' and restricting to the original j on the common boundary. Inside U' we have a simply-connected subdomain V', corresponding to $V \subset U$. Let $q'_c = j \circ q_c \circ j : V' \to U'$ denote the quadratic map corresponding to $q_c : V \to U$ and A' denote the annulus $U' - V'$. To define a $2:2$ correspondence f on $U \cup U'$ we fit together:

- $q_c : V \to U$ (a $2:1$ correspondence);
- $(q'_c)^{-1} = j \circ q_c^{-1} \circ j : U' \to V'$ (a $1:2$ correspondence);
- $j \circ i : V \to V'$ (a $1:1$ correspondence), and
- $j \circ g : A \to A'$ (a $2:2$ correspondence),

where $g : A \to A$ is the $2:2$ correspondence constructed earlier. It is easily checked that the restrictions of f defined above fit together to define a continuous $2:2$ correspondence f on the whole Riemann sphere. The next step is to define an ellipse field on A by using Lemma 1 to transport the standard complex structure from the annulus B. Using j one can extend this ellipse field to A' and pulling back via q_c^{-1} and $q_c'^{-1}$ one can extend it to an ellipse field on the whole of $\hat{\mathbf{C}} - (K(q_c) \cup K(q'_c))$, which transforms equivariantly under the action of the $2:2$ correspondence f. We extend this ellipse field to the whole of $\hat{\mathbf{C}}$ by using the standard complex structure on $K(q_c) \cup K(q_c')$. Finally, by applying the Measurable Riemann Mapping Theorem, we obtain a complex structure respected by f. The union of the images of Δ under $C_2 * C_3$ is the fundamental domain D for the action of χ referred to in the definition of a mating, and the grand orbit under f of the image (under h^{-1}) in A of the curve $l \subset \partial\Delta$ provides the set of curves $\mathcal{C}$ such that $\Omega - \mathcal{C}$ is conformally homeomorphic to D [6].

Note that when c is outside the Mandelbrot set the construction goes through in exactly the same way, provided we choose for the boundary of U an equipotential lying outside the the critical value c, so that both this equipotential and its pre-image under q_c are Jordan curves, bounding an annulus A. We can no longer use the definition of Section 1 for a mating, since we no longer have a nice set of curves $\mathcal{C}$ in Ω along which to cut to transform it to a disc. However we may use the construction above (in the present subsection) to define what we mean by a *mating* in this case, though technically it would be better to describe it as a *capture* [16, 17, 20] since the critical value now lies in $\Omega(f)$. When c is not in the Mandelbrot set the choice of which equipotential for q_c we use to define U makes a difference, and we must regard this choice as part of the data describing the mating/capture.

3.5 Matings in the circle-packing case

We now 'pinch' the mating described in the previous subsection, along a certain union of arcs associated to a circle-packing, and then use the object so constructed to define what we mean by a *mating* in this case. Once again, for simplicity assume that c lies in the Mandelbrot set, and define the annuli A and B just as before. Now lift the geodesic of rotation number ν on the torus $\Omega(r(G))/H$ to a union of arcs on $\Delta \cup \rho(\Delta) \cup \rho^{-1}(\Delta)$ and mark B by the quotient of this union under χ.

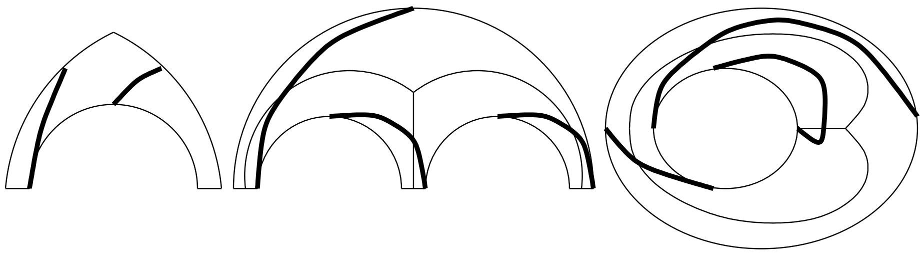

Figure 5. The geodesic for $\nu = 1/4$ on Δ, on $\Delta \cup \rho(\Delta) \cup \rho^{-1}(\Delta)$ and on B.

Figure 5 illustrates how the markings on B are obtained in the case $\nu = 1/4$, and in figure 6 we illustrate the markings on B for a range of different rotation numbers. Using the bijection h we may now carry the marked geodesic across to A and consider its inverse images under q_c: the first few 'layers' for $\nu = 1/4$ are illustrated in figure 7, and also the resulting space when the geodesic is 'pinched': note the similarity between the marked lines and the 'basilica lamination' of the unit disc, which when pinched gives a model for the Julia set of $z \to z^2 - 1$. The marked lines are not in general segments of external rays in U, but we can follow them back down to $\partial K(q_c)$ and label them by the external angle of the end which lands there (or of the external angles of the two ends which land there).

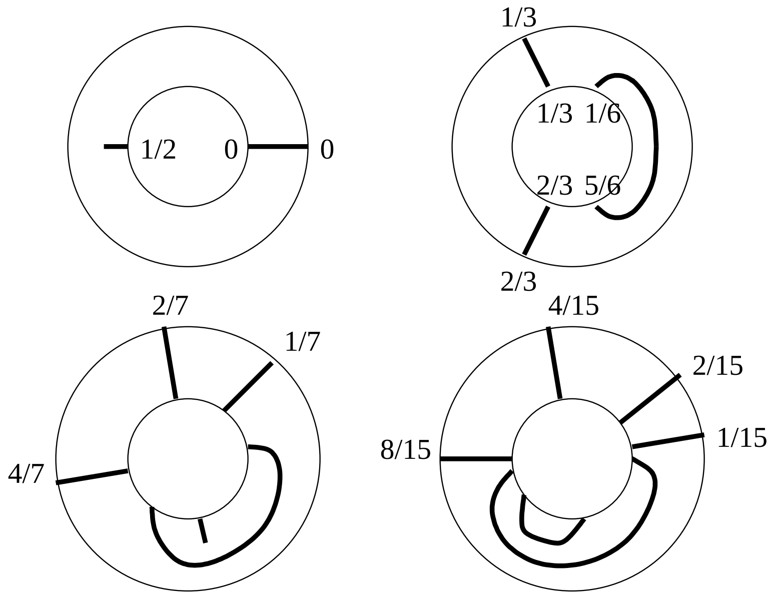

Figure 6. The annulus B marked with the geodesics of rotation numbers $0, 1/4, 1/6$ and $1/8$ respectively.

In figure 6 we have attached such labels to the q lines which continue right up to the external boundary of B. These labels correspond to the unique ordered periodic orbit of $t \to 2t$ of rotation number 2ν on the circle $\mathbf{R}/\mathbf{Z}$.

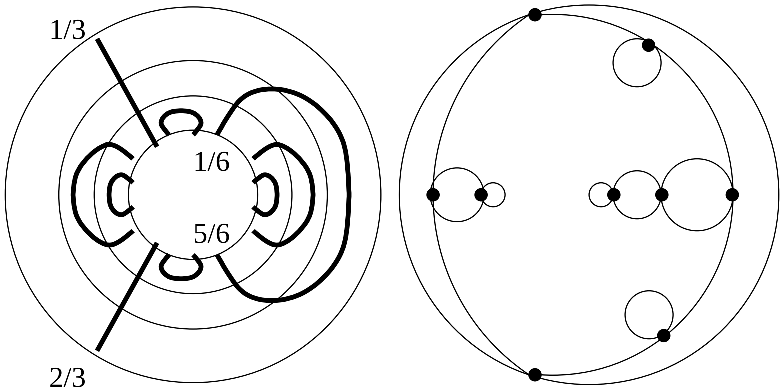

Figure 7. $A \cup q_c^{-1}(A) \cup q_c^{-2}(A)$ marked with the geodesic corresponding to $\nu = 1/4$, and with that geodesic pinched.

We glue together two copies U and U' of a disc bounded by an equipotential for $q_c : z \to z^2 + c$ just as before, except that we have to be careful about how we match up the markings at the common boundary. This is where the labelling by the ordered periodic orbit of $t \to 2t$ comes in: we glue ∂U to $\partial U'$ in such a way that the 'longest gap' in the periodic orbit marked on ∂U is glued to the 'shortest gap' on $\partial U'$. Next we shrink the marked curves to points (noting that we can only do this and finish up with a topological sphere if none of the curves is a loop, and no two curves have the same end points). The resulting space and correspondence is our combinatorial/topological model: we define a *mating* of the quadratic map q_c with the circle-packing representation of rotation number ν to be a *holomorphic* correspondence with this underlying combinatorial/topological dynamics.

Note. We shall say that c and ν are *compatible* if and only if c does not lie in the 2ν-limb (or wake) of the Mandelbrot set. Matings are not possible between q_c and r_ν for incompatible c and ν since the marked curves to be pinched then include closed loops.

3.6 Regular and limit sets of a circle-packing mating

Let $r_\nu(G)$ denote the circle-packing representation of $G =< \sigma, \rho, \chi >$ of rotation number $\nu = p/2q$ (where p, q are co-prime) and let $\Omega(r_\nu(G))$ denote its regular set, a disjoint union of discs. Let the $2:2$ correspondence $f : z \to w$ defined on the Riemann sphere $\hat{\mathbf{C}}$ be a *mating* of $r_\nu(G)$ with q_c as defined in the previous subsection, and let $(P_1, ..., P_q)$ denote the (unique) ordered orbit of q_c of rotation number p/q on $J(q_c) = \partial K(q_c)$. Then $\hat{\mathbf{C}}$ is partitioned into sets Λ and Ω, each totally invariant under the correspondence, with the following properties:

(i) Λ is the union of two connected sets, Λ_+ and Λ_-, the first of which is invariant under forward iteration of the correspondence $z \to w$, and the second of which is invariant under backward iteration; Λ_+ and Λ_- are each homeomorphic to a quotient $L(q_c)$ of the filled quadratic Julia set $K(q_c)$ by an identification map which is $1:1$ except on pre-images of $P_1, ..., P_q$ (where it is $2:1$); the $(1:2)$ map $z \to w$ on the interior of Λ_+ is conformally conjugate to q_c^{-1} acting on the interior of $K(q_c)$ and the $(2:1)$ map $z \to w$ on the interior of Λ_- is conformally conjugate to q_c acting on the interior of $K(q_c)$; both these conjugacies extend continuously to the intersection $\Lambda_+ \cap \Lambda_-$, sending this intersection to $P_1, ..., P_q$.

(ii) Let $\Omega_1(f), ..., \Omega_q(f)$ denote the components of Ω whose boundaries meet (in pairs) at the q points of $\Lambda_+ \cap \Lambda_-$ (call these the *principal cycle of components*), let $\Omega_1(r_\nu(G)), ..., \Omega_{2q}(r_\nu(G))$ denote the $2q$ components of the *principal cycle* of components of the representation $r_\nu(G)$, and let χ be the *canonical involution* (see Section 2). There exists a conformal homeomorphism

$$\left(\bigcup_1^{2q} \Omega_i r_\nu(G) \right) / \chi \to \bigcup_1^{q} \Omega_i(f)$$

under which the actions of those of the maps $\{\sigma\rho, \sigma\rho^{-1}\}$ on the various components which stabilise $\bigcup_1^{2q} \Omega_i r_\nu(G)$ are sent precisely to those of the branches of

f which stabilise $\bigcup_1^q \Omega_i(f)$. This homeomorphism induces a conformal equivalence between the space of grand orbits of $G =< \sigma, \rho, \chi >$ on $\Omega(r_\nu(G))$ and the space of grand orbits of f on $\Omega(f)$.

Properties (i) and (ii) above could be used to define what is meant by a *mating* of a quadratic map with a circle-packing representation of $C_2 * C_3$ as an alternative to the 'pinched geodesic' method we adopted in the previous subsection. Note that for c outside the Mandelbrot set $\mathcal{M}$ either approach gives a viable definition of a mating/capture, provided we choose as boundary of U an equipotential lying outside the critical value c.

3.7 Conjectured existence of matings in the circle-packing case

In view of Theorem 1 (Section 3.4), and the evidence of computer experiments for the case $\nu = 1/4$ (Section 4 below), it is natural to make the following conjecture.

Conjecture *Given any circle-packing representation r_ν of $C_2 * C_3$ and quadratic map with $c \in \mathcal{M} \setminus \mathcal{M}_{2\nu}$ (that is to say outside the 2ν-limb of $\mathcal{M}$) there exist unique values of a and k such that the correspondence (1) is a mating of r_ν with q_c.*

There are two approaches to *proving* this:

I. Let $c_{2\nu}$ denote the value of c for which $z \to z^2 + c$ has a parabolic fixed point of rotation number 2ν. For example $c_0 = 0.25$ and $c_{1/2} = -0.75$. By 'cut and paste' arguments in the cases that $2\nu = 0$ and $2\nu = 1/2$ one can show that if there exists a rational map realising a mating between the quadratic maps q_c and $q_{c_{2\nu}}$ then there exists a correspondence realising a mating between q_c and r_ν. Details will be presented in [7]. For other values of ν it should be possible to apply a similar method, but the technical details are more complicated because the surgery involves insertion of intervals as well as cutting and pasting.

II. Take a mating of q_c with a representation of $C_2 * C_3$ having connected ordinary set and deform it to a mating of q_c with r_ν by altering the conformal structure on a neighbourood of the 'marked geodesic' to be pinched. One would hope to prove that:

(i) If the one-parameter family of correspondences in this deformation tends uniformly to a non-degenerate limit correspondence then this limit correspondence is a mating of q_c with r_ν.

(ii) A deformation can be chosen in such a way as to give a one-parameter family tending uniformly to a non-degenerate correspondence if and only if c is not in the limb $\mathcal{M}_{2\nu}$ of $\mathcal{M}$.

The technical difficulties in the second approach appear to lie particularly in step (ii). These difficulties, and hopefully their resolution, will be discussed elsewhere.

For c outside $\mathcal{M}$ the natural conjecture to make is that there exists a mating of r_ν with q_c if and only if c is outside the 2ν-wake of $\mathcal{M}$. (This conjecture may need some refining to deal with the fact that for c outside $\mathcal{M}$ the data for a mating includes a choice of equipotential.)

3.8 Klein's Combination Theorem; equations for families of correspondences

So far we have concerned ourselves only with the existence of matings of certain kinds, and not with explicit equations for correspondences realising these matings. To arrive at such equations we observe that the $2:2$ correspondence $j \circ f$, where f and j are as constructed in the previous subsections, is *symmetric* (that is to say $z \to w \Leftrightarrow w \to z$) and each of its (generic) grand orbits contains just three points. It follows that there exits a rational function Q of degree 3 such that $j \circ f = Gal^Q$ (in the notation of [5]); in other words $j \circ f$ is of the form

$$z \to w \Leftrightarrow \frac{Q(w) - Q(z)}{w - z} = 0.$$

The numbers and types of singular points of $j \circ f$ are such that Q must have one double critical point and two single ones. The involution j is itself of the form Gal^P for a rational function P of degree 2, namely any function for which j is the covering involution. Thus

$$f = Gal^P \circ Gal^Q.$$

There are now two obvious ways to write down equations for f. If we normalise Q to the form $z \to z^3 - 3z$ (which has a double critical point at ∞ and single ones at ± 1) we obtain the family of equations:

$$(2) \qquad f : z \to w \Leftrightarrow z^2 + z(Jw) + (Jw)^2 = 3$$

where $J(= Gal^P)$ is an involution (parametrised by its fixed points), and if we normalise P to the form $P(z) = z^2$ we obtain the family of equations:

$$(1) \qquad f : z \to w \Leftrightarrow \left(\frac{az+1}{z+1}\right)^2 + \left(\frac{az+1}{z+1}\right)\left(\frac{aw-1}{w-1}\right) + \left(\frac{aw-1}{w-1}\right)^2 = 3k$$

For a correspondence in this family to be a mating in the sense of Section 1, we require that there exist transversals D_P and D_Q to P and Q respectively, with interiors together covering the whole of $\hat{\mathbf{C}}$ (the 'Klein Combination Condition') [5]. Transversals D_P, D_Q satisfying this condition are illustrated in each normalisation in figure 8.

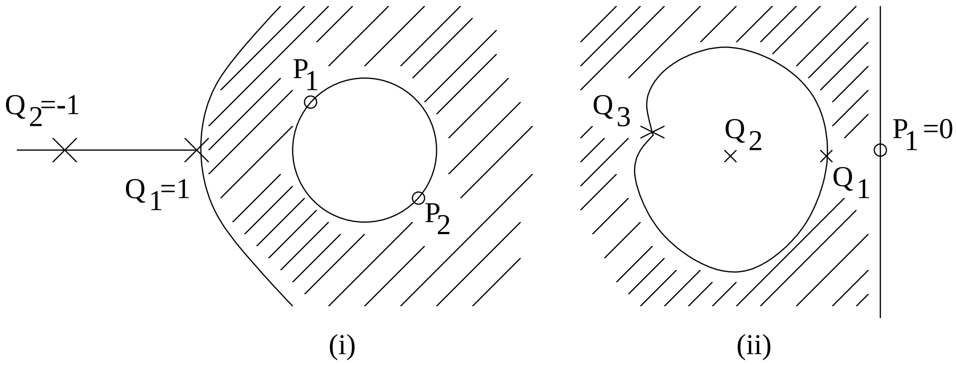

Figure 8 $\Delta = D_P \cap D_Q$ in the two normalisations.

(i) Q has critical points $Q_1 = 1, Q_2 = -1$ and $Q_3 = \infty$ (double); P has critical points at P_1, P_2; D_Q is the region to the right of the curve shown (its boundary is asymptotic to angles $\pm\pi/3$ at ∞) and D_P is the region exterior to the circle.

(ii) P has critical points $P_1 = 0$ and $P_2 = \infty$; Q has critical points at Q_1, Q_2 and $Q_3 = -1$ (double); D_P is the left-hand half-plane and D_Q is the region outside the heart-shaped closed curve (which has angle $2\pi/3$ at Q_3).

The interpretation of this picture in terms of the surgery construction of matings (Section 3.4) is that we should take D_P as our disc U and D_Q as our set $U' \cup h^{-1}(\Delta)$. The intersection $D_P \cap D_Q$ (a copy of Δ) is a 'fundamental domain' for the action of the correspondence on Ω, by the version of Klein's Combination Theorem proved in [5].

When a geodesic of rotation number $p/2q$ on the orbifold $\Omega(f)/f$ is pinched as in Section 3.5, the boundaries of D_P and D_Q are pinched together at q points. The matings corresponding to the circle-packing representation with rotation number 1/1 (=0) were considered in [1]. The algebraic condition corresponding to pinching the rotation number 0 geodesic is the requirement that one of the fixed points of the involution Gal^P also be a fixed point of Gal^Q. Choosing the fixed point $z = 0$ of Gal^P to be that point imposes the condition $k = 1$ on the correspondence (1). It was conjectured in [1] that it is possible in the slice $k = 1$ of (1) to find a mating of $z \to z^2 + c$ with $PSL(2, \mathbf{Z})$, for any c in the Mandelbrot set $\mathcal{M}$.

In Sections 4 and 5 of the paper we shall consider in detail the case of rotation number $\nu = 1/4$, where we shall find that explicit computations are still relatively easy to make but that the phenomena observed are considerably more complicated than those described in [1]. The case of $\nu = 1/4$ is a good example to study as the dynamical behaviour is typical in many ways of that for all rational values of ν other than 0 and 1/2.

4 Dynamics of matings between q_c and $r_{1/4}$ when $c \in \mathcal{M}$.

The correspondence f defined by (1) is a mating between q_c and the circle-packing representation $r_{1/4}(G)$ of $G = <\sigma, \rho, \chi>$ if and only if there exist transversals D_P and D_Q the interiors of which have union $\hat{\mathbf{C}} \setminus \{P_1, P_2\}$ where P_1 and P_2 are the fixed points of Gal^P, and P_1 and P_2 are mapped to one another by the correspondence. For the correspondence (1) the fixed points of Gal^P are 0 and ∞, and it is an elementary exercise to verify that these are mapped to one another by f if and only if

$$a^2 + a + 1 = 3k. \tag{3}$$

Observe that when a lies on the unit circle and k satisfies (3), the correspondence $f : z \to w$ defined by (1) commutes with $z \to 1/\bar{z}$ (reflection in the unit circle). Consider values of a on the upper half of the unit circle, that is to say $a = e^{i\theta}$ with $0 < \theta < \pi$. In this section we shall see that as θ runs through the interval $0 < \theta \leq arccos(-1/17)$ the correspondence is a mating of $r_{1/4}(G)$ with q_c, having c real and running through the interval $-0.75 < c \leq 0.25$. In Section 5 we shall examine the 'circle-packing implosion' that occurs at $\theta = arccos(-1/17)$.

For $a = e^{i\theta}$ equation (3) gives $k = \lambda e^{i\theta}$, with λ real. If $0 < \theta < 2\pi/3$ then $\lambda > 0$ and the Klein combination condition $D_P^\circ \cup D_Q^\circ = \hat{\mathbf{C}} \setminus \{P_1, P_2\}$ may be satisfied by taking the set D_Q as shown in figure 8(i) and D_P the complement of a disc having centre on the real axis and touching the boundary of D_Q at the (complex conjugate) fixed points $a/\sqrt{k}, 1/\sqrt{k}$ of Gal^P. So for these values of θ we have a 'mating' between $r_{1/4}(G)$ and some q_c, with c real because of reflection symmetry. But when does $c \in \mathcal{M}$, the Mandelbrot set ? Clearly this is true if and only if Λ_- contains a *critical point* of the restriction of f to a $2:1$ map $\Lambda_- \to \Lambda_-$. Such a point is a critical point of the *correspondence* f in a sense which we now define.

4.1 Critical points and values

The *forward critical values* (or *backward singular points*) of a $2:2$ correspondence f are the values w such that $f^{-1}(w)$ consists of a single point z, in our case that is to say the values of w such the equation (1) has a single solution z. These are given by

$$\left(\frac{aw-1}{w-1}\right)^2 = 4\left(\left(\frac{aw-1}{w-1}\right) - 3k\right).$$

We denote these points by

$$w_0 = \frac{1-2\sqrt{k}}{a-2\sqrt{k}} \quad \text{and} \quad w_0' = \frac{1+2\sqrt{k}}{a+2\sqrt{k}}.$$

The *forward critical points* are the corresponding values of z, namely

$$z_0 = \frac{-\sqrt{k}-1}{a+\sqrt{k}} \quad \text{and} \quad z_0' = \frac{\sqrt{k}-1}{a-\sqrt{k}}$$

respectively. By applying the time-reversing involution $z \leftrightarrow -z$ we see that the *backward critical values* (or *forward singular points*), the points z such that $f(z)$ consists of a single value w, are

$$z_1 = \frac{-1-2\sqrt{k}}{a+2\sqrt{k}} \quad \text{and} \quad z_1' = \frac{2\sqrt{k}-1}{a-2\sqrt{k}}.$$

The corresponding *backward critical points* are

$$w_1 = \frac{1-\sqrt{k}}{a-\sqrt{k}} \quad \text{and} \quad w_1' = \frac{\sqrt{k}+1}{a+\sqrt{k}}.$$

4.2 Mating with $z \to z^2$

When the forward critical point z_0 is fixed, the branch of f which fixes z_0 is locally conjugate to $z \to z^2$. Hence $z_0 \in \Lambda_-$ (since it cannot now lie in Ω), and f is a mating between $r_{1/4}(G)$ and $z \to z^2$.

$$\begin{aligned} z_0 = w_0 \Leftrightarrow \frac{-\sqrt{k}-1}{a+\sqrt{k}} &= \frac{1-2\sqrt{k}}{a-2\sqrt{k}} \\ \Leftrightarrow a &= \frac{4k+\sqrt{k}}{2-\sqrt{k}} \end{aligned}$$

which, combined with the contact condition $a^2+a+1=3k$ yields

$$(4k+\sqrt{k})^2 + (4k+\sqrt{k})(2-\sqrt{k}) - (3k-1)(2-\sqrt{k})^2 = 0$$

and after dividing through by $(\sqrt{k}+1)^2$ this gives us

$$13k - 10\sqrt{k} + 4 = 0$$

and thus $\sqrt{k} = (5+3\sqrt{3}i)/13$ and

$$a = \frac{-1+15\sqrt{3}i}{26} \quad k = \frac{2(-1+30\sqrt{3}i)}{169}.$$

The dynamical behaviour at these parameter values is plotted in figure 9 (at the beginning of this article). The Riemann sphere is partitioned into two completely invariant regions Λ (the union of the closures Λ_- and Λ_+ of the innermost region of each of the upper and lower 'lobes' in the figure) and Ω the union of the interiors of the two outer regions and the interiors of all the little discs visible in each of the lobes. The correspondence maps the various regions to one another as indicated in figure 10.

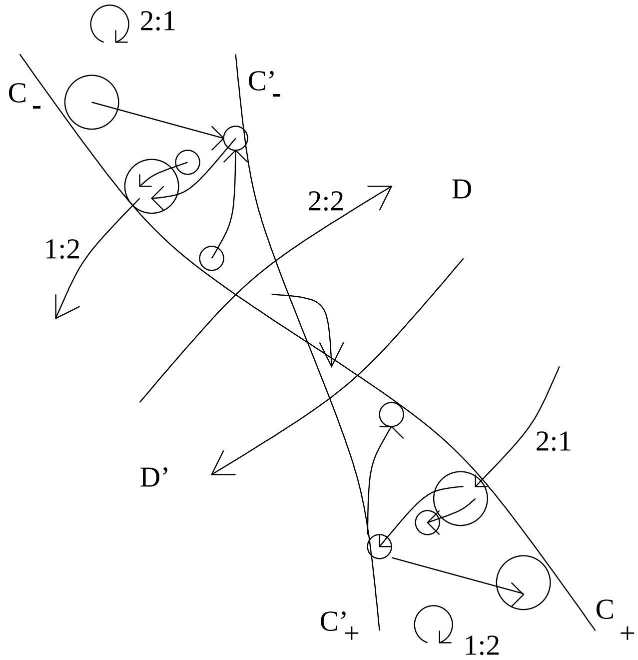

Figure 10. The principal regions (for $c = 0$). $\hat{\mathbf{C}}$ is partitioned into Ω (a union of open discs, including the two marked D and D') and its complement $\Lambda_- \cup \Lambda_+$ (where Λ_- is the upper of the two regions, mapped $2:1$ to itself). The 'exit disc' is the disc mapped $1:2$ onto D' and the 'entry disc' is the image of D under a $2:1$ map. All other maps marked are $1:1$ (except for $D' \to D$ which is a pair of $1:1$ maps).

On the interior of Λ_- the $2:1$ map is conjugate to $z \to z^2$ on the unit disc, by the Böttcher conjugacy extended from a neighbourhood of the (fixed) critical point z_0. This allows us to assign an angle to each point on the boundary of Λ_- and identify the various chains of discs as corresponding to those of the 'basilica' Julia set (that for $z \to z^2 - 1$). Thus $\partial\Lambda_-$ is obtained from the circle $\mathbf{R}/\mathbf{Z}$ by contracting the leaves of the 'basilica' lamination, figure 11 (except for the leaf from $1/3$ to $2/3$, the points which correspond to $z = 0$ and $z = \infty$ respectively on the dynamical plane).

There is a $1:1$ branch of the correspondence sending not only Λ_- to Λ_+, but the whole of the upper lobe apart from the exit disc to the whole of the lower lobe except the entry disc: thus each of the little discs has exactly two pre-image discs and two image discs, apart from the exit disc, each point of which has both its images in D', and the entry disc, each point of which has both its pre-images in D.

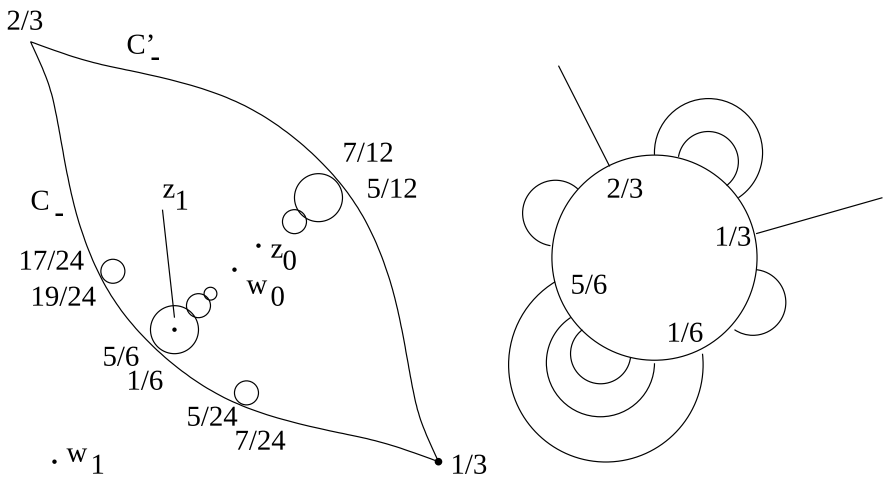

Figure 11.

(i) Some particular points and curves. Here w_0 has z_0 as unique pre-image; z_1 has w_1 as unique image.

(ii) The first few leaves of the 'basilica lamination', which, when pinched, give (i).

The left and right hand boundaries of the upper lobe, the curves C_- and C'_- respectively (figure 11), correspond to the boundaries of the periodic components of the basilica filled Julia set, that is to say (with the parametrisation we have already imposed on $\partial\Lambda_-$ using the Böttcher map),

- C'_- consists of all sequences in the symbols $0'$ and $1'$, where $0' = 01$ and $1' = 10$, with the identifications

$$.a_1a_2...a_n0'\overline{1'} \sim .a_1a_2...a_n1'\overline{0'}$$

and the usual numerical order $(0' < 1')$;

- C_- consists of the sequences in C'_- preceded by a single 0 or a single 1.

The dynamics of the correspondence on these sequences is that of the shift, so C_- is mapped $2:1$ onto C'_- and C'_- is mapped $1:1$ onto C_-. Similarly on the lower lobe C'_+ is mapped $1:2$ onto C_+ and C_+ is mapped $1:1$ onto C'_+. The boundary restriction of the correspondence restricted to $D' \to D$ (where it is $2:2$) is completed by the restriction $C_- \to C_+$ of the $1:1$ branch mapping the upper lobe (minus exit disc) to the lower lobe (minus entry disc): strictly speaking this restricted branch is not a bijection from C_- to C_+ since it splits C_- open in the middle and sends its two ends to the same point in the middle of C_+, but it is $1:1$ apart from at these points.

The double iterate $f \circ f$ of the correspondence (1), when restricted to $D \to D$ is a $2:2$ correspondence which splits into two homeomorphisms together defining an action of $PSL(2, \mathbf{Z})$ conformally conjugate to the standard action on the upper half-plane. Under this conjugacy the action of the right shift on symbol sequences in $0'$ and $1'$, with insertion of $1'$ or $0'$, is conjugate to the action of $x \to x+1, x \to x/(x+1)$ on the positive real line. The dynamics on D' is conformally conjugate to that on D. Each of the other discs is a copy of D quotiented by a (standard) involution.

As we vary the parameters a and k (maintaining the condition $a^2+a+1=3k$) the critical point z_0 is no longer fixed but provided it remains within Λ_- the latter set remains homeomorphic to the modified filled Julia set $L(q_c)$ of a polynomial q_c having $c \in \mathcal{M}$. Indeed we conjecture that all $L(q_c)$ for $c \in \mathcal{M} \setminus \mathcal{M}_{1/2}$ can be achieved in this way. Here $\mathcal{M}_{1/2}$ denotes the '1/2-limb' of $\mathcal{M}$, that is to say the part of $\mathcal{M}$ to the left of the parameter rays of argument 1/3 and 2/3 (which both land at the root point $c=-0.75$ of the component of $\mathcal{M}^\circ$ corresponding to period 2). Note that we cannot expect to be able to mate our circle-packing representation of $C_2 * C_3$ with q_c having $c \in \mathcal{M}_{1/2}$, for the obvious combinatorial reason that this would require us to shrink lamination leaves connecting the same pairs of points both inside and outside the circle and our dynamical space would no longer be a sphere.

4.3 Mating with $z \to z^2 + 0.25$

The mating with $z \to z^2+0.25$ occurs when the fixed point equation

$$\left(\frac{az+1}{z+1}\right)^2 + \left(\frac{az+1}{z+1}\right)\left(\frac{az-1}{z-1}\right) + \left(\frac{az-1}{z-1}\right)^2 = 3k$$

has two double solutions rather than four simple ones. Clearing denominators the fixed point equation reads

$$3(a^2-k)z^4 + (a^2-8a+1+6k)z^2 + 3(1-k) = 0$$

so the condition for repeated roots (of form $\pm z$) is

$$(a^2-8a+1+6k)^2 = 36(a^2-k)(1-k).$$

This equation has a (repeated) trivial solution $a=1$, and dividing through by $(a-1)^2$ yields

$$a^2 - 14a + 1 + 48k = 0.$$

Now setting $a=e^{it}$ the contact condition $3k=a^2+a+1$ yields $k=\lambda a$ where $\lambda = (2\cos t+1)/3$ and the equation above becomes

$$17e^{2it} + 2e^{it} + 17 = 0$$

which has as solutions

$$a = e^{it} = -\frac{1}{17} \pm \frac{12\sqrt{2}}{17}i.$$

The fixed points z have

$$z^2 = -\frac{3(a-1)}{2(2a+1)}$$

which yields (for the value of a above with positive imaginary part)

$$z^2 = \frac{-1 - 2\sqrt{2}i}{3}$$

and hence

$$z = \pm\left(-\frac{1}{\sqrt{3}} + \frac{\sqrt{2}}{\sqrt{3}}i\right) \sim \pm(-0.58 + 0.82i).$$

To complete the calculations at the value of a above (with positive imaginary part) we see that

$$k = \frac{1}{3}(2\cos t + 1)(\cos t + i\sin t) = -\frac{5}{289} + \frac{60\sqrt{2}}{289}i$$

and hence

$$\sqrt{k} = \frac{2\sqrt{10}}{17} + \frac{3\sqrt{5}}{17}i$$

so the point w_0 having a unique pre-image z_0 is

$$w_0 = \frac{1 - 2\sqrt{k}}{a - 2\sqrt{k}} = \frac{(7 - 16\sqrt{10}) + (28\sqrt{5} + 4\sqrt{2})i}{81} \sim -0.55 + 0.84i$$

and the corresponding point z_0 (the 'critical point') is

$$z_0 = \frac{-\sqrt{k} - 1}{a + \sqrt{k}} = \frac{(2 - 2\sqrt{10}) + (4\sqrt{2} + \sqrt{5})i}{9} \sim -0.49 + 0.88i.$$

The points z_0, w_0 and the fixed point are all on the unit circle and are arranged as indicated in figure 11. The dynamics of the correspondence restricted to a $2:1$ map on the interior of the upper lobe, plotted in figure 12 is conjugate to that of $z \to z^2 + 0.25$ on the interior of its filled Julia set. As we shall see below, the dynamical behaviour changes dramatically at this parameter value.

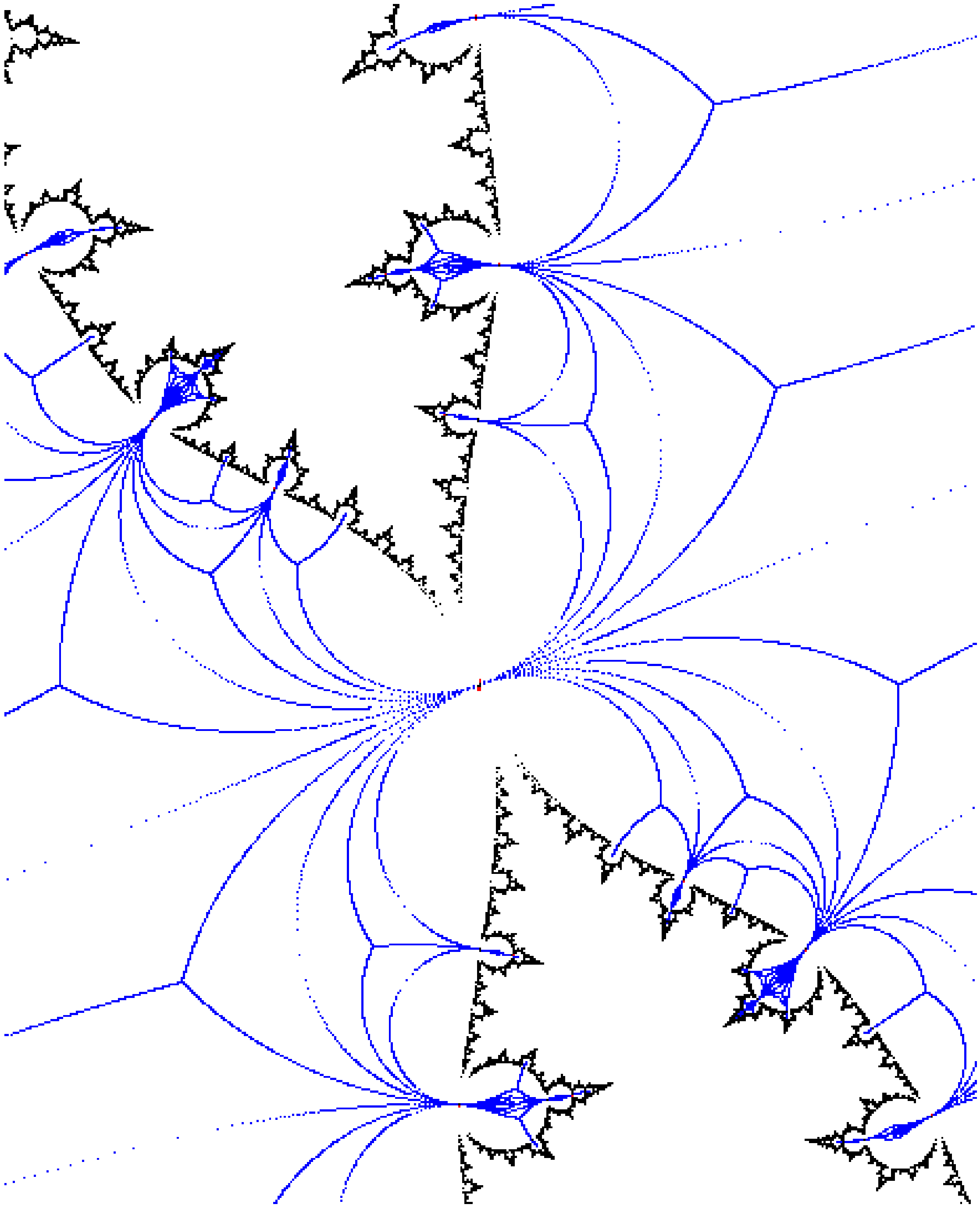

Figure 12. Correspondence (1), subject to condition (3), with $a = e^{i\theta}$ where $\cos\theta = -1/17$. This is a mating of $r_{1/4}(G)$ with $z \to z^2 + 0.25$.

5 A circle-packing implosion at $c = 0.25$

5.1 Real values of the parameter c

We first examine the behaviour as c passes through 0.25 but remains on the real axis. This corresponds to restricting the value of a to the unit circle.

As c passes through the value 0.25 the fixed point splits into a complex conjugate pair of repelling fixed points, Douady's 'egg-beaters'. Propelled by these egg-beaters the forward orbit of the critical point z_0 now reaches the 'outside world' - moreover the number of iterations required for this escape decreases monotonically as we perturb further. Examples are plotted in figure 13.

Consider the point z_1 in the upper lobe which has unique image w_1 in the outside world. This is the point

$$z_1 = \frac{-2\sqrt{k} - 1}{a + 2\sqrt{k}}$$

and its image is

$$w_1 = \frac{1 - \sqrt{k}}{a - \sqrt{k}}.$$

Let a_n be the value of the parameter a, on the unit circle, such that when we iterate the correspondence restricted to the upper lobe n times the critical point z_0 is sent to z_1 (and therefore escapes on the next iteration). At the parameter value a_2 the dynamical behaviour of the correspondence is as illustrated in figure 13c. Key features are:

• There is a countable union of open (topological) discs contained in the upper lobe such that the forward orbit of any point in this union eventually reaches the 'exit disc' and then leaves the lobe.

• The union of these discs in the upper lobe together with the corresponding union in the lower lobe and the two outside regions form the regular set of the correspondence. The complement of this regular set is the limit set of the correspondence. It has empty interior.

• The dynamics of the correspondence remains unchanged on the principal cycle of components (in other words the quotient by χ of the action of the group G on the principal cycle of $\Omega(r_{1/4}(G))$). The other discs are each either a quotient of one of these components by a standard involution respecting the dynamics (for example the 'entry' and 'exit' discs are such) or a branched double cover of one of these quotients (the component containing the critical point is one such: the others are its inverse images) with the corresponding double covering dynamics.

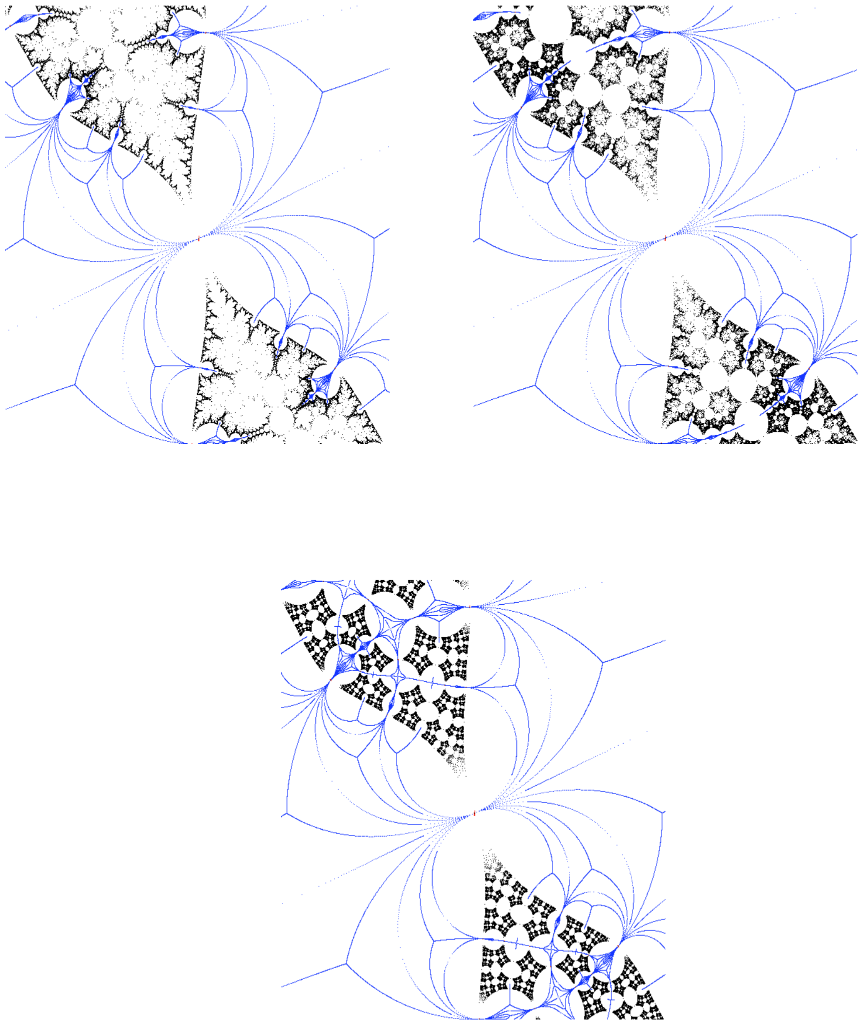

Figure13. The correspondence (1) subject to condition (3), with $a = e^{i\theta}$ where
(a) $\cos\theta = -0.062$ (top left);
(b) $\cos\theta = -0.07$ (top right);
(c) $\cos\theta = -0.09$ (bottom).

The combinatorics of one of these circle packings can be most efficiently described by means of surgery moves on the basilica lamination, each move taking place as the critical value moves across one of the arcs marked in figure 7. Our description is analogous to that of the corresponding rational maps of degree two in the family of matings of the quadratic maps $z \to z^2 - 1$ with the quadratic map $z \to z^2 + c$ (see [20] for this family, and also see [16, 17, 15] for comprehensive descriptions of the behaviour of rational maps of degree two).

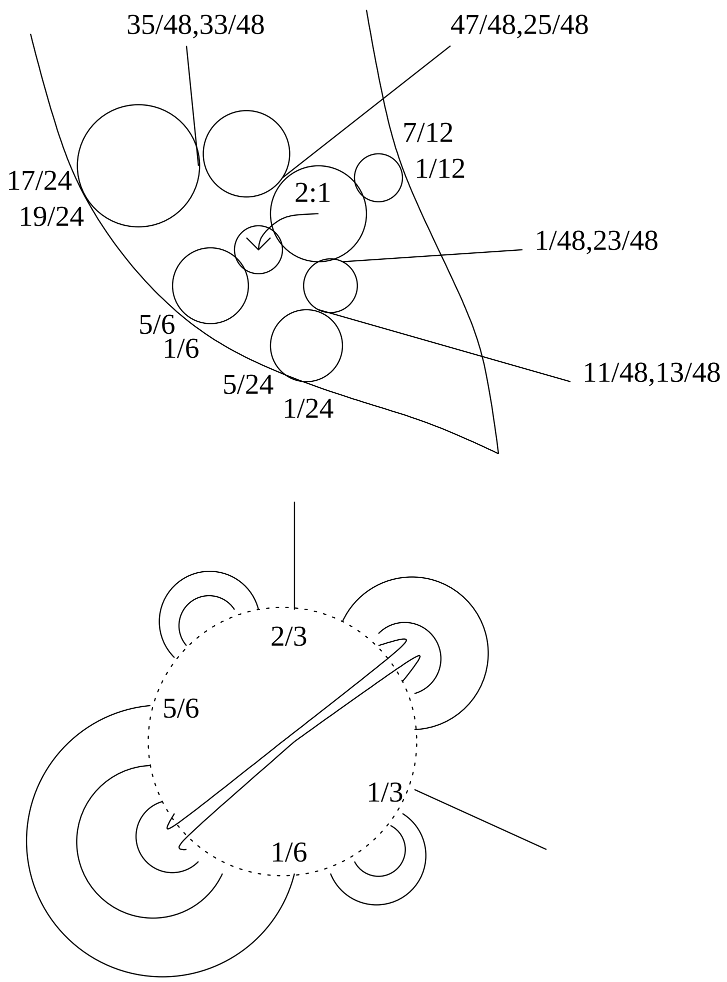

Figure 14

(i) the circle-packing corresponding to $n = 2$ (see figeure 13c); (ii) the corresponding lamination.

In figure 14 we illustrate these moves for the case $n = 2$. We parameterize the unit circle as $\mathbf{R}/\mathbf{Z}$: the binary expressions of these real numbers, sequences of 0's and 1's, describe their trajectories under the doubling map. We first cut the unit circle at the points $0, 1/2, 1/4, 3/4, ...$, that is to say all dyadic rationals. This corresponds dynamically to splitting the fixed point $0 \sim 1$ into a pair of points, and doing the same to each of its preimages: combinatorially it corresponds to undoing the real number identifications $...0\bar{1} \sim ...1\bar{0}$ to leave a Cantor set. We then construct the first few leaves of the basilica lamination in the standard way, but (for $n = 2$) instead of linking $1/48$ to $11/48$ and $23/48$ to $25/48$ we link $1/48$ to $23/48$ and $47/48$ to $25/48$. We then construct leaves for all the inverse images (under the doubling map) of the leaves already constructed.

Note that in order to draw these inverse image leaves in the plane we need to make use of the 'gaps' we have created by the first step of cutting the circle open at 0 and its inverse images.

Combinatorially the space we create in this way (in the case $n = 2$) is that obtained from the set of all sequences of 0's and 1's by making the identifications

$$0.000\overline{01} \sim 0.011\overline{10},$$

$$0.100\overline{01} \sim 0.111\overline{10}$$

and their consequences (i.e. identifications of sequences which have these as tails). Topologically the space we create by shrinking the leaves of the lamination to points is a circle packing.

As we move the value of the parameter a around the unit circle in the positive sense, away from the value corresponding to $c = 0.25$, the circle packing passes through a sequence of different topological types, one for each n (decreasing). The transitions occur each time the critical point z_0 has as one of its forward images the point $z_2 = -(a+2)/(2a+1)$ (which is the pre-image of 0 other than ∞: it is the point corresponding to $1/6 \sim 5/6$ on the boundary of the upper lobe). Equivalently, thinking in terms of the construction of these matings along the lines of Section 3.4, the transitions occur when the boundary of U is chosen to be an equipotential passing through a forward image $q_c^n(c)$ of the critical value c (where the values of a we are considering on the unit circle correspond to the U obtained by taking some fixed real $c > 0.25$ and equipotentials of decreasing size).

The last such transition occurs when z_0 maps directly to z_2 (equivalently when the boundary of U passes through the critical value c of the quadratic map $z \to z^2 + c$), after which the limit set becomes a single (topological) circle - see (b) below. We compute this and certain other special values of a (on the unit circle) and describe the dynamical behaviour of the correspondence at each of these values:

(a) **'Group resolution'**

A particularly interesting case is when the critical value w_0 (a point with unique pre-image) coincides with the point z_1 (which has unique image). This occurs when the critical point z_0 maps directly to z_1 in one step, in other words when $a = a_1$. The correspondence is then 'critically resolvable' in the sense of [2], that is to say that when we lift it to a graph correspondence we obtain a pair of Mobius transformations.

The values of a and k when $w_0 = z_1$ are

$$a = \frac{-1 + 3\sqrt{7}i}{8} \quad k = \frac{-1 + 3\sqrt{7}i}{32}.$$

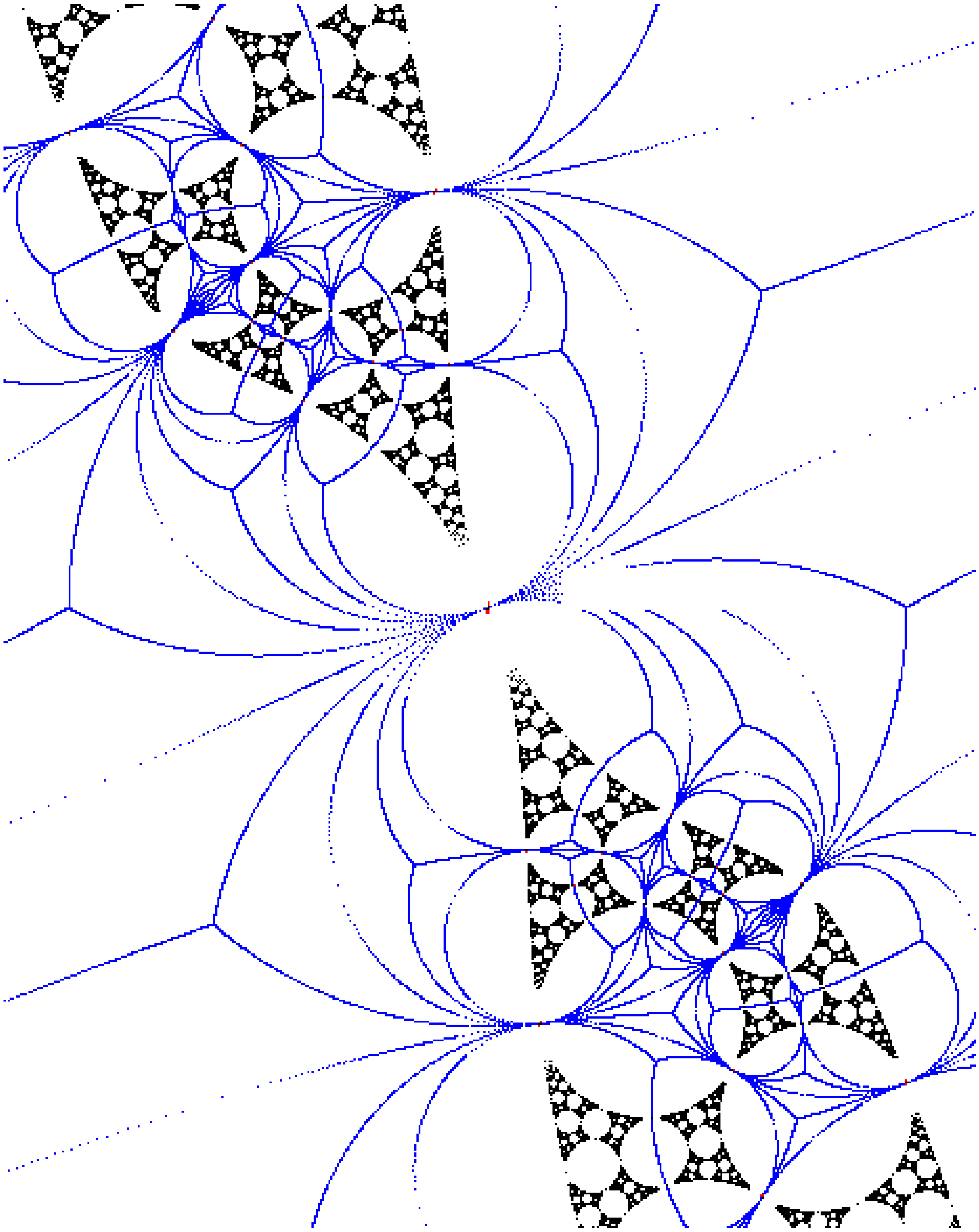

Figure 15. Correspondence (1) with $a = e^{i\theta}$ where $\cos\theta = -1/8$.

The correspondence for these parameter values is plotted in figure 15. Note that the only two discs on which the dynamics of the correspondence is a proper quotient of the dynamics on a principal component are the 'entry' and 'exit' discs respectively (those containing the points z_1 and $-z_1$). A computation yields that:

$$z_0 = \frac{(1-\sqrt{7})+(1+\sqrt{7})i}{4}$$
$$z_1 = \frac{-\sqrt{7}+3i}{4}$$
$$w_1 = \frac{(-1-\sqrt{7})+(1-\sqrt{7})i}{4}.$$

Note that z_0 and w_1 lie on orthogonal lines through the orgin. If we take the double covering of the sphere ramified at z_1 and $-z_1$ we obtain the circle-packing representation of the group $C_2 * C_3$ of rotation number $1/4$ discussed at the end of Section 2. In the lift the group acts by $z \to w = \sigma\rho(z), z \to w = \sigma\rho^{-1}(z)$. To pass from this lift back to the quotient one simply applies the identification $z \sim \chi z$ to the group picture.

(b) **Transition to a single circle**

Let z_2 denote the point of $\partial\Lambda_-$ corresponding to the point $1/6 \sim 5/6$ of the basilica lamination. Thus ∞ and z_2 are the two values of $f^{-1}(0)$, and an elementary calculation yields

$$z_2 = -(a+2)/(2a+1).$$

The final transition occurs when the critical point z_0 maps directly to z_2. Equivalently this is when $f^{-1}(z_2)$ is a single point, that is to say when

$$\left(\frac{az_2-1}{z_2-1}\right)^2 = 4\left(\left(\frac{az_2-1}{z_2-1}\right)^2 - (a^2+a+1)\right)$$

which, when we substitute our value for z_2, gives

$$11a^4 + 28a^3 + 30a^2 + 28a + 11 = 0.$$

Setting $u = a + 1/a$ this equation yields

$$u = \frac{-14+6\sqrt{3}}{11}$$

and hence

$$a^2 + \left(\frac{14-6\sqrt{3}}{11}\right)a + 1 = 0.$$

Thus a is on the unit circle and has real part

$$Re(a) = \frac{-7+3\sqrt{3}}{11} \sim -0.164$$

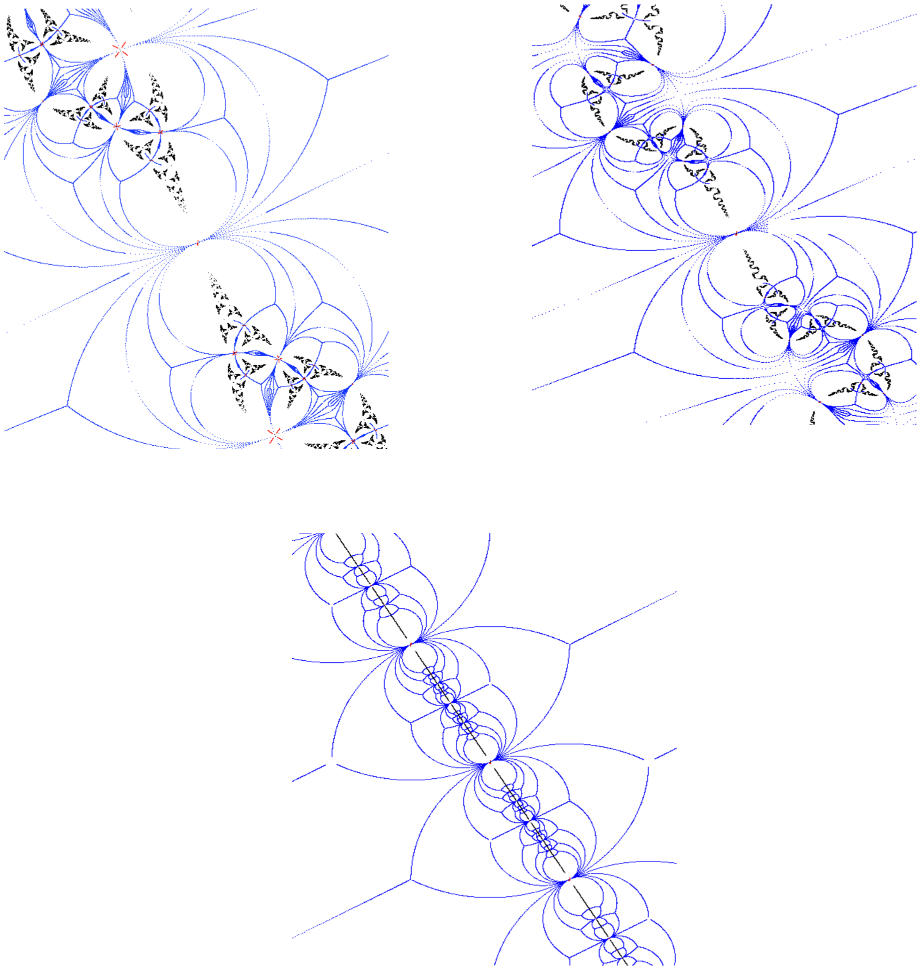

Figure 16. Correspondence (1) subject to condition (3), with $a = e^{i\theta}$ where:
(a) $\cos\theta = -0.164$ (top left);
(b) $\cos\theta = -0.2$ (top right);
(c) $\cos\theta = -0.5$ (bottom).

See figure 16a for a computer plot at this value. Once a has moved beyond this value the $1/12, 11/12$ and $5/12, 7/12$ identifications are split apart and we have $1/12, 5/12$ and $11/12, 7/12$ identifications in their place. The lamination is as illustrated in figure 17.

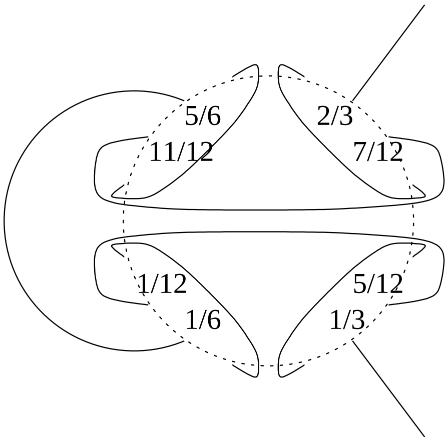

Figure 17.

We now find that the limit set is a single topological circle. To see this, observe that our limit set now consists of the space of sequences of 0's and 1's with the identifications imposed by setting

$$1/6 = .0\overline{01} \sim .1\overline{10} = 5/6$$

and identifying every word which finishes $0\overline{01}$ with the same word with its ending $0\overline{01}$ replaced by $1\overline{10} = 5/6$. The space of sequences is now totally ordered by setting

$$.a_1a_2...a_n0... \leq .a_1a_2...a_n1... \quad n \quad \text{even}$$

$$.a_1a_2...a_n0... \geq .a_1a_2...a_n1... \quad n \quad \text{odd}.$$

We leave it as an exercise for the reader to check that this order is well-defined at the level of equivalence classes and is transitive. The least element is $.\overline{01} = 1/3$ and the greatest is $.\overline{10} = 2/3$. These correspond to the points $z = 0$ and $z = \infty$ in the dynamical plane. The lower lobe of the limit set similarly degenerates into a (topological) interval, and thus overall the limit set is a single topological circle.

The regular set (the outside regions) now consists of just two discs, each sent two-to-two to the other by the correspondence. Note that until the transition to a single circle the action of the correspondence on its principal cycle of components does not change (it is conjugate to the quotient by χ of the action of G on its regular set). However this last transition allows singular points into the principal cycle of components and the action of the correspondence on this cycle is no longer 'group-like'. Thus the correspondence ceases to be a mating in the sense of our earlier definition: in terms of the mating construction what has happened is that the critical value c of $q_c : z \to z^2 + c$ has crossed the boundary of U (or equivalently we must now choose as boundary for U an equipotential inside c) and thus $V = q_c^{-1}(U)$ has changed its topology from a single disc to a pair of discs, invalidating our method of construction. It remains the case that the grand orbit space $\Omega(f)/f$ is conformally homeomorphic to $\Omega(r_{1/4}(G))/G$, but this does not tell us very much, since all circle-packing representations of $C_2 * C_3$ have grand orbit spaces conformally homeomorphic to one another. For the 'quasifuchsian' correspondence that our correspondence has now become (one for which the complement of the regular set $\Omega(f)$ is a topological circle) the 'unfolding' of the grand orbit space $\Omega(f)/f$ is controlled by new data describing the relative positions of the critical points in the unfolding.

As we perturb a further along the unit circle the action of the correspondence on its regular set changes: the two critical points in the set (one forward and one backward) change their relative positions and each time they lie on a common grand orbit the action of the correspondence on its regular set undergoes a (topological) change. The limit set behaviour remains unchanged (figure 16b) but we reach a parameter value (figure 16c) where the forward critical point maps directly to the backward critical point, so that the pair become a double point of the graph of the correspondence and as a result the correspondence consists of pair of Möbius transformations. At this value the behaviour of the correspondence is that of a Fuchsian group on the whole Riemann sphere - or perhaps it would more accurately be described as an anti-Fuchsian group since since it is conjugate to a group generated by real two-by-two matrices of determinant -1.

(c) **The Fuchsian case**

The Fuchsian case occurs when

$$a = \frac{-1 + i\sqrt{3}}{2},$$

which corresponds to $k = 0$. The correspondence now has equation

$$\left(\frac{az+1}{z+1}\right)^2 + \left(\frac{az+1}{z+1}\right)\left(\frac{aw-1}{w-1}\right) + \left(\frac{aw-1}{w-1}\right)^2 = 0$$

and splits into two Möbius transformations $z \to w$ defined by

$$\frac{aw-1}{w-1} = e^{\pm 2\pi i/3}\left(\frac{az+1}{z+1}\right).$$

These we can write as $z \to w = \sigma\rho(z)$ and $z \to w = \sigma\rho^{-1}(z)$ where σ is the involution $z \to -z$ and ρ is an elliptic transformation of order three having fixed points -1 and $-1/a$. This is easily seen to be a Fuchsian representation of the free product $C_2 * C_3$, having limit set the straight line $z = \{te^{2\pi i/3} : t \in \mathbf{R}\}$. Conjugating by a clockwise rotation through $2\pi/3$ to transform this line to the real axis, the correspondence becomes the pair of Möbius transformations

$$z \to \frac{1}{z-1}, \qquad z \to \frac{1-z}{z}.$$

Note that each of these transformations has determinant -1 and so exchanges the upper and lower half planes and that together they map the real intervals

$[0, \infty]$ two-to-one onto $[0, \infty]$;
$[-\infty, 0]$ one-to-two onto $[-\infty, 0]$, and
$[0, \infty]$ one-to-one onto $[-\infty, 0]$.

We remark that at $a = -1/2 + i\sqrt{3}/2$ our alternative normalization (2) of the equation of the correspondence becomes invalid, since $k = 0$ and the fixed points of the involution J would coincide at ∞.

As we now move a on round the unit circle the limit set of the correspondence remains a topological circle but the correspondence again does not act as a group on its regular set. The limit set becomes increasingly deformed, but to analyse what is going on we shall need to consider new transversals D_P and D_Q.

(d) **Pinching the topological circle**

For $a = e^{i\theta}$ (on the unit circle) with $Re(a) < -1/2$, the parameter $k = (a^2 + a + 1)/3$ has the form $k = -\mu e^{i\theta}$ with $\mu = -(1 + 2\cos(\theta))/3$ real and positive. Thus normalization (2) of the correspondence gives us an involution J having fixed points

$$\frac{1}{\sqrt{k}} = \frac{e^{-i\theta/2}}{i\sqrt{\mu}}, \qquad \frac{a}{\sqrt{k}} = \frac{e^{i\theta/2}}{i\sqrt{\mu}}.$$

These are symmetrically placed about the imaginary axis rather than the real axis (which was the case for $Re(a) > -1/2$) and in place of the earlier transversal D_Q for $Q(z) = z^3 - 3z$ illustrated in figure 8(i) we consider a new transversal with boundary the outer curve illustrated in figure 18.

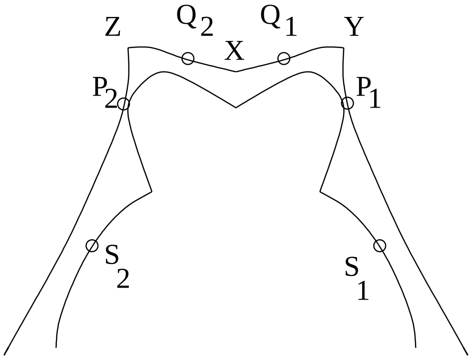

Figure 18 An alternative transversal for Q.

Here the curves on the right and left-hand sides are the images under Gal^Q of the part of the imaginary axis from the point X upwards. The points marked P_1, P_2 are the fixed points of J, and Y, Z are the images of X under Gal^Q. Recall that the critical point $Q_1 = 1$ of Q has images $1, -2$ under Gal^Q and the critical point $Q_2 = -1$ has images $-1, 2$. The parts of the right-hand and left-hand half-planes lying outside D_Q are the images of D_Q under the two 'branches' of Gal^Q. Note that the critical points Q_1 and Q_2 must both lie on the boundary of D_Q (for it to be a transversal) and that while we are free to move the point

X along the imaginary axis we must at the same time move Y and Z to remain its images under Gal^Q. We are also free to choose the precise curve from X to Q_1 making part of the boundary of D_Q, but this choice determines the whole of the boundary from Z to Y, by the application of Gal^Q and symmetry about the imaginary axis. The inner curve in figure 18 is the image of the outer curve under J, and the points labelled S_1 and S_2 are the images of Q_1 and Q_2.

For there to exist a transversal D_P for P such that $D_P^\circ \cup D_Q^\circ = \hat{\mathbf{C}}$, modulo the fixed points of $J = Gal^P$, it is both necessary and sufficient that the image under J of the boundary of D_Q lie inside D_Q (again modulo the fixed points). This ceases to be possible, however judicious our choice of the segment of boundary of D_Q from Q_1 to X, once the point S_1 (the image under J of the critical point Q_1) crosses the right hand boundary of D_Q.

This occurs when $J(1)$ has the form $c + id$ with $(c+id)^3 - 3(c+id)$ pure imaginary, that is when $c^2 - 3d^2 = 3$.

Since the involution J having the fixed points given above is

$$J(z) = \frac{z\cos(\theta/2) + i\sqrt{\mu}}{i\sqrt{\mu}z - \cos(\theta/2)}$$

an elementary calculation yields

$$J(1) = \frac{\mu(1-\cos^2(\theta/2)) - i\sqrt{\mu}(\mu+1)\cos(\theta/2)}{\mu(\mu+\cos^2(\theta/2))}$$

and as $\mu = (1 - 4\cos^2(\theta/2))/3$ we deduce

$$J(1) = \frac{3(1 - 4\cos^2(\theta/2) - 4i\sqrt{\mu}\cos(\theta/2))}{1 - 4\cos^2(\theta/2)}$$

which is of the form $c + id$ with $c = 3$ and

$$d = \frac{-12\sqrt{\mu}\cos(\theta/2)}{1-4\cos^2(\theta/2)} = \frac{-4\sqrt{3}\cos(\theta/2)}{\sqrt{1-4\cos^2(\theta/2)}}.$$

Thus $c^2 - 3d^2 = 3$ if and only if

$$\frac{48\cos^2(\theta/2)}{1-4\cos^2(\theta/2)} = 2$$

that is to say, if and only if $\cos(\theta/2) = \pm 1/2\sqrt{7}$, that is $\cos(\theta) = -13/14$. Thus our example occurs at $a = -13/14 \pm 3\sqrt{3}i/14$.

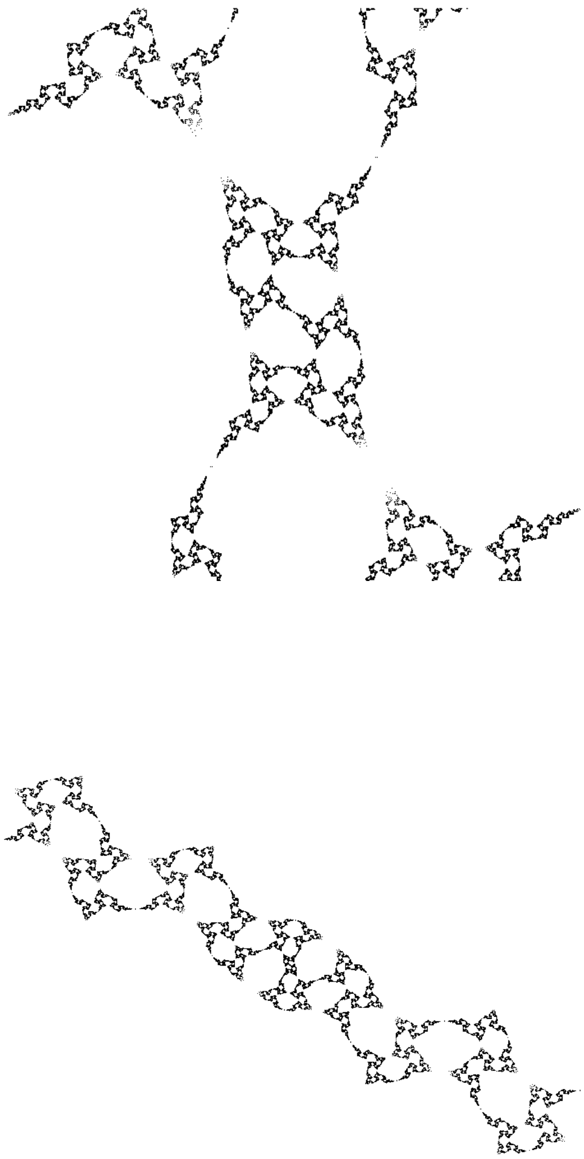

Figure 19. Correspondence (1), subject to condition (3), with $a = e^{i\theta}$ where $\cos\theta = -13/14$.

A computer plot of the limit set of the correspondence for this value of a is displayed in the top half of figure 19 (and a 'zoomed out' version in the bottom half). Note that these plots are drawn in the normalization (1) so that we can compare them with the earlier plots. Observe that the regular set has now been pinched, so that the limit set is no longer a topological circle but is once again a (topological) circle packing.

Two further remarks:

• At the value of the parameter a at which pinching occurs, the four points Q_1, S_1, Q_2, S_2 (in that cyclic order) form a period four orbit of $f = Gal^P \circ Gal^Q$. As we approach this value of a what happens is that $\Delta = D_P \cap D_Q$, which is already pinched at the fixed points P_1, P_2 of J, has four more curve segments in it pinched to points. This further pinching corresponds to shrinking some union of curves in the regular set of the quasifuchsian group. It appears to be the 'flexibility' allowed by varying the relative positions of backward and forward critical points in the 'unfolding' of the grand orbit space that permits a second 'geodesic' to be pinched. The details remain mysterious as yet, but it would seem reasonable to conjecture that the boundary of the part $\mathcal{K}$ of parameter space where transversals D_P, D_Q can be found satisfying the (contact) Klein combination condition, contains a dense of points corresponding to shrinking some well-defined set of 'geodesics'.

• The points Q_1 and Q_2 are critical values of the correspondence f; thus the two discs of the regular set that touch at Q_1 map to a single disc under the inverse of the corresponding critical branch of f: the image of Q_1 under this inverse branch is an inward pointing cusp in the image disc - a fact that implies that the disc boundaries are no longer quasicircles (as they appear to be in the earlier circle-packings). We remark that while in both transitions (b) and (d) the critical point and its image critical value lie on the limit set, in the first case an infinitesimal neighbourhood of the critical point (in the limit set) has the form of a 'cross', with image a 'line', whereas in the second case it has the form of a 'line' with image a 'half-line'.

5.2 Complex values of c.

When we move a off the unit circle (corresponding to moving c off the real axis) other circle-packing configurations become possible for the limit set. Figure 20 displays plots of three examples. To arrive at such configurations from inside the critical non-escape locus the critical value need not necessarily pass between the 'primary egg-beaters' (the two points with external angles .000... and .111... obtained by splitting the point $0 \sim 1$) but can pass through a 'gateway' opened up between a pair of 'egg-beaters' at any pre-image of this pair. For example the critical value may pass between the pair $.1111\bar{0}$ and $.1110\bar{1}$ obtained by splitting the point with angle 15/16. A schematic example (not corresponding to any of the examples in figure 20) is illustrated in figure 21.

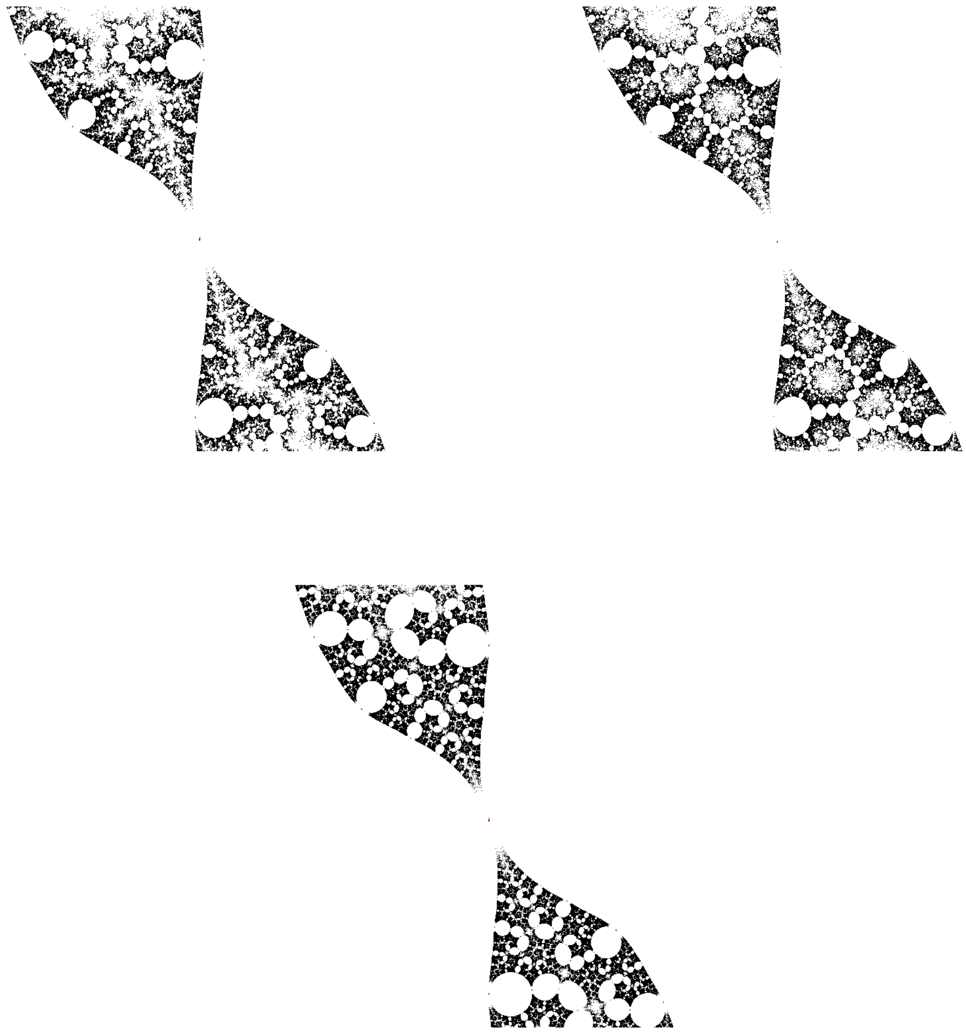

Figure 20. Correspondence (1), subject to condition (3), with:
(a) $a = -0.69 + 1.0i$ (top left);
(b) $a = -0.07 + 1.0i$ (top right);
(c) $a = -0.08 + 1.0i$ (bottom).

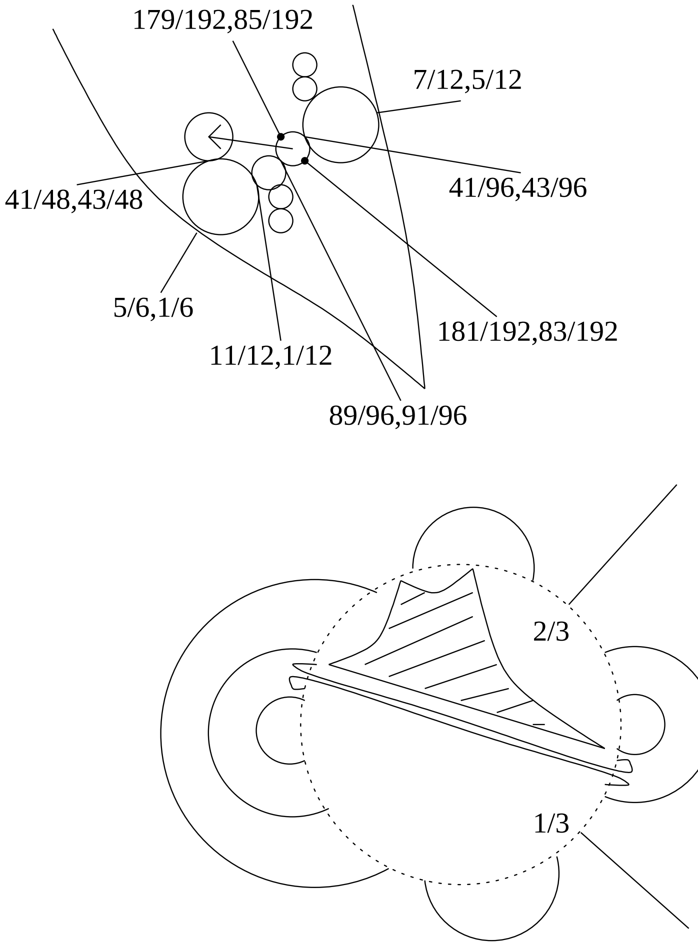

Figure 21.
(i) A circle-packing mating. The map between discs indicated is $2:1$.
(ii) The associated lamination. The 'new leaves' are from $85/192$ to $179/192$ and from $83/192$ to $181/192$. The 'ideal quadrilateral' illustrated has vertices at $14/15, 13/15, 11/15$ and $7/15$, the points of the ordered orbit of $t \to 2t$ of rotation number $3/4$.

Here the critical value has passed between these 'egg-beaters' and now lies in a disc adjacent to, but on one side of, the escape disc: the critical value reaches the escape disc in three further iterations. The lamination picture of the circle-packing obtained in this case is also illustrated in figure 21. The reader should be warned that the situation is a little more complicated than that for the 'implosion' at $c = 0.25$, in that for c to arrive at a boundary point of the Mandelbrot set $\mathcal{M}$ of external angle $p/2^k$, starting from the main component of $\mathcal{M}^\circ$, we have to pass through a sequence of components, and each transition from one component to the next corresponds to certain pinching in a combinatorial model for $\Lambda_-(\sim \Lambda_+)$. Thus this pinching has already been done before we start the cascade of surgery moves corresponding to joining the 'gateway' $p/2^k$ to its opposite 'gateway' $p/2^k + 1/2$. An example is illustrated in figure 21. The 'implosion' of the 15/16 'gateway' takes place at a value of c in the 3/4-limb of $\mathcal{M}$: the ideal quadrilateral illustrated collapses to a point when one crosses from the main component of $\mathcal{M}^0$ into this limb.

There should also be interesting phenomena associated to 'geometric limits'. Consider sequences of circle-packings corresponding to placing the critical value at the 'centres' of sequences of discs tending to the boundary point of angle 0: one possibility is given by the sequence of parameter values $a_1, a_2, ...a_n, ...$ defined earlier in the 'real' case, but there are other possibilities, for example one may take a sequence of discs each adjacent to those containing these a_n. There are an infinity of different transitions through the value $c = 0.25$ once we have moved off the real axis. We conjecture that if one proceeds towards $c = 0.25$, from outside, in a suitably coherent manner keeping the dynamics 'in phase' (e.g. along the sequence of parameter values $a_1, a_2, a_3, ...$ defined earlier) the dynamical picture tends to a limit circle-packing which is a 'decorated' version of the picture for $c = 0.25$. This is what one would expect by analogy with the Cantor set 'implosion' at $c = 0.25$ for the maps $z \to z^2 + c$ [10, 11].

6 The combinatorics of a slice of parameter space.

Consider the one (complex) dimensional slice of (a, k) parameter space satisfying the contact condition $a^2 + a + 1 = 3k$ and in particular within this slice consider the subset $\mathcal{K}$ for which there are (contact) Klein combination domains, and the subset $\mathcal{L}$ of $\mathcal{K}$ for which the correspondence is a mating of $r_{1/4}(G)$ with a quadratic map $z \to z^2 + c$. We do not know the exact shape of $\mathcal{K}$ (though see the remarks at the end of Section 5.1(d)), or indeed of $\mathcal{L}$, but within $\mathcal{L}$ is the *non-escape locus* consisting of the set of parameter values $\mathcal{N}$ for which the orbit of the critical point z_0 does not escape to the outside world: these correspond to matings where $c \in \mathcal{M}$, the classical Mandelbrot set.

We assert that the interior of $\mathcal{N}$ is homeomorphic to the interior of the $\mathcal{M}$ with its 1/2-limb $\mathcal{M}_{1/2}$ amputated, but that $\mathcal{N}$ itself is a quotient of $\mathcal{M} \setminus \mathcal{M}_{1/2}$ obtained by making certain identifications of pairs of boundary points, namely those corresponding to leaves of the basilica lamination which lie on the same side of the $1/3, 2/3$ leaf as 0 (~ 1). Thus 1/6 is identified with 5/6, and 1/12 is identified with 11/12 and so on. A shorthand way to describe this is that we have a 'mating of $\mathcal{M} \setminus \mathcal{M}_{1/2}$ with the basilica'. This structure is to be expected: the angle of the point of the boundary at which the critical value passes out of Λ_+ is the angle of the corresponding point on $\mathcal{M}$. But if such a point is a point of contact of two of the discs making up Ω then it corresponds to two different external angles on $\mathcal{M}$.

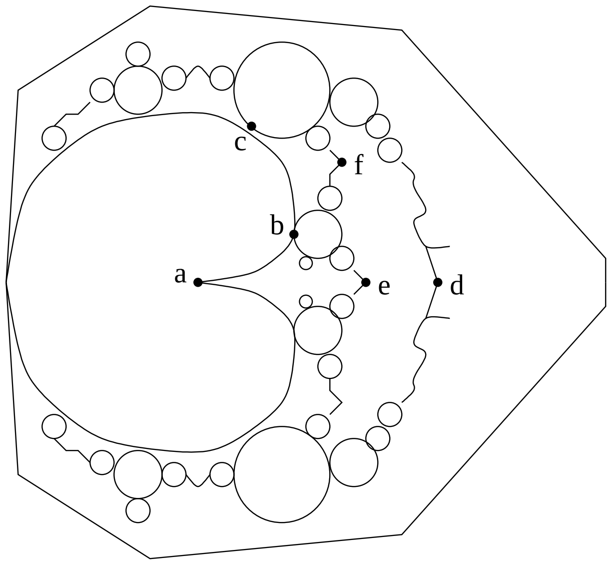

Figure 22. Parameter space (schematic), where the parameter is $i(1-a)/(1+a)$. Marked points are those corresponding to external angles a:$0 \sim 1$; b:$1/15 \sim 2/15$; c:$1/7 \sim 2/7$; d:$1/6 \sim 5/6$; e:$1/12 \sim 11/12$; f:$5/48 \sim 7/48$.

Figure 22 is a schematic diagram of $\mathcal{N} \subset \mathcal{L} \subset \mathcal{K}$, with a parametrisation chosen so that values of a on the unit circle correspond to points on the horizontal symmetry axis of the figure . Note that the point of $\mathcal{N}$ corresponding to the point of $\mathcal{M}$ having external angle $1/6$ ($c = i$) is bent round to touch that corresponding to external angle $5/6$ ($c = -i$). Similarly the point with external angle $1/12$ is bent round to touch that having external angle $11/12$ and so on. We believe that each component of $\mathcal{L} - \mathcal{N}$ is a (topological) disc and if two parameter values lie in the same component, the limit sets of the corresponding correspondences are circle-packings of the same topological type. Thus if we proceed along the horizontal symmetry line away from the point corresponding to $c = 0.25$ we pass through each topological type of the circle packings corresponding to our a_n, with n monotonically decreasing, until we reach the single circle case. The outer layer $\mathcal{K} - \mathcal{L}$ corresponds to a certain class of quasifuchsian correspondences.

To conclude we remark that everything goes through in a similar fashion for $\nu = p/2q$ in place of $\nu = 1/4$: we just replace the 'basilica' lamination of the disc by a certain lamination associated to the ordered periodic orbit of $t \to 2t$ of rotation number p/q on the circle $\mathbf{R}/\mathbf{Z}$. But the reader should be warned that for example for $\nu = 1/6$ the subset of the disc to be pinched is a certain collection of lines, and not the whole triangles one collapses to points to obtain the 'Douady rabbit' of the quadratic maps case: see [7] for details.

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