Matings in Holomorphic Dynamics

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Abstract

Matings between pairs of holomorphic dynamical systems occur in several different contexts. Polynomial maps can be mated with polynomial maps to yield rational maps, Fuchsian groups can be mated with Fuchsian groups to yield quasifuchsian Kleinian groups, and certain polynomial maps can be mated with certain Fuchsian groups to yield holomorphic correspondences. We consider some of the examples, methods of construction, results and open questions in all three areas. In a final section we present yet another class of examples, matings between Fuchsian groups and holomorphic correspondences. We show how the modular group, which is 'rigid' as a group, has a deformation space in this context, and we describe some of the features of this space.

1 Introduction

We shall be considering *holomorphic dynamical systems* of several kinds, all acting on the Riemann sphere $\hat{\mathbb{C}}$ or some subset of it:

• Fuchsian (respectively Kleinian) groups: discrete subgroups of $PSL_2(\mathbb{R})$ (respectively $PSL_2(\mathbb{C})$) acting by Möbius transformations.
• Iterated rational maps: maps of the form $z \to p(z)/q(z)$ where $p(z)$ and $q(z)$ are polynomials.
• Iterated holomorphic correspondences: multivalued 'maps' of the form $z \to w$ where z and w satisfy some polynomial relation $P(z,w)$=0.

The basic idea of a *mating* is as follows. Suppose we are given two holomorphic dynamical systems F_i ($i = 1, 2$), each acting on a closed simply-connected subset

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A_i of $\hat{\mathbb{C}}$ in such a way that the boundary ∂A_i is invariant under F_i. Moreover suppose that there is a homeomorphism $h : \partial A_1 \to \partial A_2$ which *topologically conjugates* $F_1|_{\partial A_1}$ to $F_2|_{\partial A_2}$. We can then glue the two systems together to form a system on $X = A_1 \cup_h A_2$. If X is homeomorphic to a sphere, we call the new system a *topological mating* between F_1 and F_2, and we can ask whether there is a holomorphic dynamical system F on $\hat{\mathbb{C}}$ which is topologically conjugate to this glued system via a homeomorphism which is holomorphic on the interiors of A_1 and A_2. If such an F exists we say that $F_1|_{A_1}$ and $F_2|_{A_2}$ are *matable* and that F is a *mating* between them.

For the different types of holomorphic dynamical system we take appropriate variations on this basic idea. For example, suppose we wish to mate a polynomial map p_1 with a polynomial map p_2 of the same degree n. In general the filled Julia sets A_1 and A_2 of p_1 and p_2 will not have homeomorphic boundaries, but if A_1 and A_2 are both locally connected, then rather than using a homeomorphism to glue them together we may use the smallest closed equivalence relation generated by regarding each $p_i|_{\partial A_i}$ as a quotient of the map $(z \to z^n)|_{S^1}$ (where S^1 denote the unit circle). And when mating a quadratic polynomial map p with the modular group $PSL_2(\mathbb{Z})$ we shall see that we have to take a wedge of *two* copies of the filled Julia of p for our set A_1 in order to make the dynamics on the boundary match that of the modular group on its limit set $\hat{\mathbb{R}}$. But, once we have made these minor adjustments to definitions, the surprise is how often pairs of holomorphic dynamical systems turn out to be matable, and also the types of the holomorphic dynamical systems that realise these matings. When they exist:

• Matings between Fuchsian groups acting on the complex upper half-plane with limit set $\hat{\mathbb{R}}$ are realised by *quasifuchsian* groups (discrete subgroups of $PSL_2(\mathbb{C})$ which have limit sets which are *quasicircles*).
• Matings between polynomial maps acting on their filled Julia sets are realised by rational maps.
• Matings between Fuchsian groups acting on the complex upper half-plane with limit set $\hat{\mathbb{R}}$ and polynomial maps acting on their filled Julia sets are realised by holomorphic correspondences.

The word 'mating' was introduced by Douady and Hubbard, who in the early 1980s observed the phenomenon of mating between polynomials, in computer pictures of Julia sets of rational maps, and set out to prove its existence. This observation led to a great deal of subsequent work, by Douady and Hubbard themselves and by others. The same term is applied below in the context of a much older topic in Kleinian group theory, that of Bers' Simultaneous Uniformization Theorem [2], and to a rather newer topic, that of iterated holomorphic correspondences which fit together the dynamics of polynomial groups and Fuchsian groups, the first examples of which were exhibited in [6]. I make no attempt to present a comprehensive survey of all these areas, but rather try to introduce the main ideas, indicating the parallels between the examples of matings in the different areas, and mentioning some of the major results, methods and outstanding questions. More detailed information concerning combinations of holomorphic dynamical systems, and the contents of 'Sullivan's Dictionary' between the theories of iterated rational maps and Kleinian groups, can be found in [32] and [30].

2 The Measurable Riemann Mapping Theorem

We first introduce a technical tool that will be a common theme in our construction of matings in different contexts. Recall that the *Riemann Mapping Theorem* asserts that if U is a bounded open simply-connected subset of $\mathbb{C}$, then there exists a conformal homeomorphism $\phi : U \to \mathbb{D}$, where $\mathbb{D}$ denotes the open unit disc in $\mathbb{C}$. The Riemann mapping ϕ is unique up to post-composition by conformal homeomorphisms of $\mathbb{D}$, that is to say fractional linear maps which send the unit disc $\mathbb{D}$ to itself.

The *Measurable Riemann Mapping Theorem* asserts the analogous result when the initial data includes a measurable function $\mu : U \to \mathbb{C}$, and rather than a conformal homeomorphism ϕ from U to $\mathbb{D}$, what we seek is a *quasiconformal* homeomorphism $f : U \to \mathbb{D}$ which at almost every point $z \in U$ has *complex dilatation* $\mu(z)$.

Before we state this theorem formally, we first recall that a homeomorphism f between open sets in $\mathbb{C}$ is said to be K-*quasiconformal* if it sends infinitesimally small round circles to infinitesimally small ellipses which have ratio of major axis length to minor axis length less than or equal to K. The Beltrami differential

$$\mu(f) = \frac{\partial f}{\partial \bar{z}} \Big/ \frac{\partial f}{\partial z}$$

is known as the *complex dilatation* of f. It is a standard result that f is K-quasiconformal, with $K = (1+k)/(1-k)$, if and only if $\mu(f)$ is defined almost everywhere and has essential supremum $||\mu||_\infty = k < 1$.

The Measurable Riemann Mapping Theorem is due to Morrey [29], Bojarski, Ahlfors and Bers [1]. The statement below is taken from [15] (Théorème 5): it is expressed in terms of functions defined on the whole of $\mathbb{C}$, but can be adapted to suit other situations, for example when the domain of μ is a bounded simply-connected open subset U of $\mathbb{C}$ and we seek a quasiconformal homeomorphism $f : U \to \mathbb{D}$ or indeed when the domain of μ is the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and we seek a quasiconformal homeomorphism $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$.

The Measurable Riemann Mapping Theorem

Let μ be any L^∞ function $\mathbb{C} \to \mathbb{C}$ with $||\mu||_\infty = k < 1$. Then there exists an orientation-preserving quasiconformal homeomorphism $f : \mathbb{C} \to \mathbb{C}$ which has complex dilatation $\mu(f)$ equal to μ almost everywhere on $\mathbb{C}$. This homeomorphism is unique if we require that $f(0) = 0$ and $f(1) = 1$. Furthermore if μ depends analytically (respectively continuously) on a parameter λ then the homeomorphism f also depends analytically (respectively continuously) on λ.

3 Matings between Fuchsian groups

Let G_1 be a *geometrically finite Fuchsian group*, that is to say a discrete subgroup of $PSL_2(\mathbb{R})$, acting by fractional linear maps on the upper half $\mathcal{U}$ of the complex plane, with a fundamental domain having a finite number of sides. Suppose the limit set of this action is $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. The group G_1 also acts on the lower half plane $\mathcal{L}$ and the limit set of this action is also $\hat{\mathbb{R}}$. Let G_2 be another geometrically finite discrete subgroup of $PSL_2(\mathbb{R})$, isomorphic to G_1 as an abstract group, and such that the action of G_2 on $\mathcal{U}$ is topologically conjugate to that of G_1.

Bers' Simultaneous Uniformization Theorem

Given subgroups G_1 and G_2 of $PSL_2(\mathbb{R})$ with the properties described above, there exists a discrete subgroup G of $PSL_2(\mathbb{C})$ the action of which on the Riemann sphere $\hat{\mathbb{C}}$ has the following properties:
(i) The limit set of the action is a quasicircle $\Lambda \subset \hat{\mathbb{C}}$.
(ii) On one component, Ω_1, of $\hat{\mathbb{C}} \setminus \Lambda$ the action of G is conformally conjugate to the action of G_1 on $\mathcal{U}$.
(iii) On the other component, Ω_2, of $\hat{\mathbb{C}} \setminus \Lambda$ the action of G is conformally conjugate to the action of G_2 on $\mathcal{L}$.

In the situation described by (i),(ii) and (iii) we may call the *Kleinian* group G (discrete subgroup of $PSL_2(\mathbb{C})$) acting on $\hat{C}$ a *mating* between the Fuchsian group G_1 acting on $\mathcal{U}$ and the Fuchsian group G_2 acting on $\mathcal{L}$. The action of G is a *holomorphic realization* of the dynamical system obtained by gluing together the actions of G_1 on $\mathcal{U}$ and G_2 on $\mathcal{L}$ by means of a (topological) homeomorphism from the boundary $\partial\mathcal{U}$ of $\mathcal{U}$ to the boundary $\partial\mathcal{L}$ of $\mathcal{L}$ which conjugates the action of G_1 on $\partial\mathcal{U}$ to that of G_2 on $\partial\mathcal{L}$.

Bers' Simultaneous Uniformization Theorem is a consequence of the Measurable Riemann Mapping Theorem, as we shall now see. Since G_1 is geometrically finite, the orbit space $\mathcal{L}/G_1$ is a Riemann surface with a finite number of marked cone points and puncture points. The orbit space $\mathcal{L}/G_2$ is a Riemann surface with a combinatorially identical set of data. It follows by standard Riemann surface theory that there exists a quasiconformal diffeomorphism $h : \mathcal{L}/G_1 \to \mathcal{L}/G_2$, sending marked points to marked points. The complex dilatation μ of h, when composed with the orbit projection, yields a G_1-equivariant L^∞ function $\mu : \mathcal{L} \to \mathbb{C}$, which we may extend to the whole of $\hat{\mathbb{C}}$ by defining $\mu(z)$ to be zero on $\hat{\mathbb{C}} \setminus \mathcal{L} = \overline{\mathcal{U}}$. An equivalent description, in terms of measurable fields of infinitesimal ellipses, is that the field of ellipses defined by μ on $\mathcal{L}/G_1$ is pulled back to a G_1-equivariant ellipse field on $\mathcal{L}$ and extended to the rest of $\hat{\mathbb{C}}$ by the standard (round) circle field on $\overline{\mathcal{U}}$. By the measurable Riemann Mapping Theorem there now exists a quasiconformal diffeomorphism $\phi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ having complex dilatation μ. But, as is easily verified by the chain rule, each element of $G = \phi G_1 \phi^{-1}$ has complex dilatation zero, and so maps infinitesimal round circles to infinitesimal round circles. Thus G is a group of conformal automorphisms of $\hat{\mathbb{C}}$, that is to say a subgroup of $PSL_2(\mathbb{C})$. The limit set of G

is the set $\Lambda = \phi(\hat{\mathbb{R}})$, which is a quasicircle by definition, since it is the image of a round circle under ϕ. Moreover ϕ provides a conformal conjugacy between the actions of G_1 on $\mathcal{U}$ and G on $\Omega_1 = \phi(\mathcal{U})$, and $\phi \circ h^{-1} : \mathcal{L} \to \Omega_2 = \phi(\mathcal{L})$ provides a conformal conjugacy between the actions of G_2 on $\mathcal{L}$ and G on $\Omega_2 = \phi(\mathcal{L})$, where here $h : \mathcal{L} \to \mathcal{L}$ denotes the lift of our quasiconformal diffeomorphism $h : \mathcal{L}/G_1 \to \mathcal{L}/G_2$.

3.1 A family of examples: once-punctured torus groups

Consider discrete representations in $PSL_2(\mathbb{R})$ of the free group F_2 on generators X and Y. Let A and B be elements of $PSL_2(\mathbb{R})$ representing X and Y, and restrict attention to the case that A and B are hyperbolic and their commutator $ABA^{-1}B^{-1}$ is parabolic. A generic representation of this kind has fundamental domain a quadrilateral in the upper half-plane, with all four vertices on the (completed) real line and all four sides geodesics in the hyperbolic metric, that is to say arcs of semicircles orthogonal to the real line: the group elements A and B identify pairs of opposite sides of this quadrilateral, and the orbit space is a punctured torus. The cross-ratio of the four vertices of a fundamental domain is a conjugacy invariant of the group (as a subgroup of $PSL_2(\mathbb{R})$). Any two representations G_1 and G_2 of this kind (which in general will have different different cross-ratios) provide examples of groups to which we may apply Bers' Theorem. We consider G_1 acting on the upper half-plane and G_2 acting on the lower. Their actions on their common limit set, the completed real axis, will be topologically conjugate, but in general this conjugacy will not be smooth (figure 1). Bers' theorem tells us that we can realise the topological mating obtained by gluing these two actions together as a *holomorphic* dynamical system, a Kleinian group (discrete subgroup of $PSL_2(\mathbb{C})$) which has as its limit set a quasicircle.

On the boundary of this family of examples we find some phenomena that we shall also observe in matings between polynomials and in matings between groups and polynomials. Start with a mating of the kind describe above, and choose a simple closed geodesic l on the punctured torus Ω_1/G_1: such geodesics are in one-to-one correspondence with $\mathbb{Q}\cup\{\infty\}$, the p/q geodesic corresponding to a straight line of slope p/q on a square lattice (figure 2). Following Maskit [22], we may deform G_1, keeping G_2 fixed, by an isotopy contracting the length of this geodesic to zero. The effect on the upper half-plane is to contract to a point each component of the union of the lifts of l to $\mathcal{U}$. We obtain a Kleinian group $G \subset PSL_2(\mathbb{C})$ which has limit set Λ a proper quotient of $\hat{\mathbb{R}}$. On the component Ω_2 of its ordinary set, the action of G remains conformally conjugate to that of G_2 on $\mathcal{L}$. However Ω_1 has now been pinched into (infinitely many) components, each topologically a disc, and the limit set Λ of G is now the closure of a countable infinity of topological circles. A group G of this kind is called a *singly cusped b-group.*

One can obtain a *doubly cusped* group G by contracting a geodesic of slope p'/q' on $\mathcal{L}/G_2$ at the same time as contracting a geodesic of slope p/q on $\mathcal{U}/G_1$. Such a group G can be considered as a*mating* between two singly-cusped groups. Note

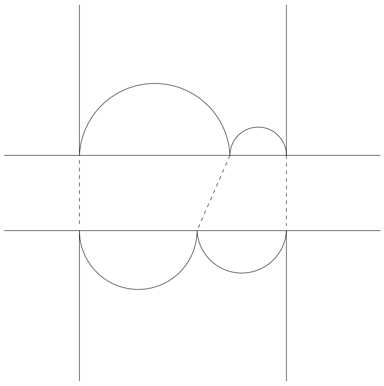

Figure 1: Fundamental domains for different actions of a punctured torus group on the upper and lower half planes: one can construct a conjugacy between the two actions on the real axis by sending the vertices of one fundamental domain to those of the other, as shown, and extending equivariantly to all the vertices of the respective tilings.

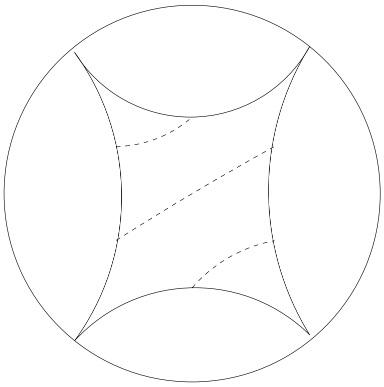

Figure 2: Fundamental domain for a punctured torus group, drawn on the Poincaré disc, with the geodesic for $p/q = 1/2$ marked.

that the cases $p'/q' = -p/q$ are excluded as in these the ends of each curve to be contracted in $\mathcal{U}$ are also the ends of a curve to be contracted in $\mathcal{L}$, so when we perform the contraction certain simple closed curves on $\hat{\mathbb{C}}$ are pinched to points and we no longer have a topological sphere.

Corresponding to a lamination of *irrational* slope ν on the torus $\mathcal{U}/G_1$ is what is known as a *singly degenerate* representation $G \subset PSL_2(\mathbb{C})$, which has ordinary set with just one component (the image of $\mathcal{L}$) and limit set a dendrite. We remark that whereas a singly cusped G is geometrically finite, this is no longer the case for a singly degenerate G. One can further modify a singly degenerate G by contracting a geodesic of rational slope on $\mathcal{L}/G_2$, in which case the dendrite becomes a circle-packing. Finally, corresponding to a lamination of irrational slope ν on $\mathcal{U}/G_1$ and a lamination of irrational slope ν' on $\mathcal{L}/G_2$ (with $\nu' \neq -\nu$) there is a *doubly degenerate* representation, with limit set the entire Riemann sphere $\hat{\mathbb{C}}$. We can think of such a G as a *mating* between two singly degenerate (dendrite) representations. In the next section we shall see analogous behaviour in examples of matings between polynomial maps.

The first examples of doubly degenerate Kleinian groups were due to Jørgensen [21] (see also his 1975 preprint, published as [20]). A well-illustrated account of this phenomenon can be found in the book *Indra's Pearls* [31]. The *Classification Theorem for Kleinian Once-punctured Torus Groups* [28] states that every once-punctured torus group is uniquely expressible as a mating of Fuchsian groups or as one of the pinched types listed above. The proof of this classification was a major achievement of 3-dimensional hyperbolic manifold theory, completed by Minsky's proof of Thurston's *Ending Lamination Conjecture* in this case [28]. McMullen [24] proved that the limit set Λ of a Kleinian once-punctured torus group, degenerate or otherwise, is always a quotient of the circle $\hat{\mathbb{R}}$, with the action of G on Λ a quotient of the unperturbed action of G_1 on $\hat{\mathbb{R}}$, and that as a corollary Λ is always locally connected (this is definitely not the case for Julia sets of polynomials [26]). McMullen proved earlier, [23], that cusped groups occur at a dense set of points on the boundary of moduli space.

4 Matings between polynomial maps

We restrict attention to quadratic polynomials to simplify descriptions of matings and also because a more extensive set of results is known for these than for polynomials of higher degree.

Recall that every quadratic polynomial with coefficients in $\mathbb{C}$ is conjugate to a unique polynomial of the form $q_c : z \to z^2 + c$, and that the *Mandelbrot set* M is defined to be the set of all values of $c \in \mathbb{C}$ for which the Julia set of q_c is connected. Equivalently M is the set of all values of c such that the orbit $(q_c^n(0))_{n\geq 0}$ of the critical point 0 is bounded. The *filled Julia set* $K(q_c)$ of q_c is the set of all $z \in \mathbb{C}$ which have bounded orbits under q_c. The *Julia set* of q_c is the boundary of $K(q_c)$. More generally, for a rational map $z \to f(z) = p(z)/q(z)$, the *Julia set* of f is the set of points $z \in \hat{\mathbb{C}}$ at which the set of forward iterates $(f^n)_{n\geq 0}$ of f fails to be a normal family [26].

When $c \in M$, there is a conformal bijection ϕ_c between the basin of attraction of the fixed point at infinity, $\hat{\mathbb{C}} \setminus K(q_c)$, and the open unit disc $\mathbb{D}$, conjugating the map q_c on $\hat{\mathbb{C}} \setminus K(q_c)$ to the map $z \to z^2$ on $\mathbb{D}$. The map ϕ_c is unique, once we require it to be tangent to the map $z \to 1/z$ at ∞. Its inverse ϕ_c^{-1} is known as the *Böttcher coordinate* for $\hat{\mathbb{C}} \setminus K(q_c)$, and the image under ϕ_c^{-1} of a radial line $\{re^{2\pi i\theta} : 0 \leq r < 1\}$ is called the *external ray* of argument θ for q_c.

By the criterion of Carathéodory, the Böttcher coordinate extends to a continuous map from the unit circle S^1 onto $J(q_c)$ if and only if $J(q_c)$ is locally connected. It is well known [26] that a sufficient condition for $J(q_c)$ to be locally connected is that q_c is *hyperbolic*, that is to say every critical point of q_c lies in the basin of attraction of some periodic orbit. Given two hyperbolic quadratic polynomials q_c and $q_{c'}$ with locally connected Julia sets, we glue a copy K of $K(q_c)$ to a copy K' of $K(q_{c'})$ along their boundaries, using the smallest closed equivalence relation for which $\phi_c^{-1}(e^{2\pi it}) \sim \phi_{c'}^{-1}(e^{-2\pi it})$ for each $t \in [0, 1)$. If $(K \cup K')/_\sim$ is a topological sphere, the map induced on this sphere by gluing q_c on K to q_c' on K' is called a *topological mating* of q_c with $q_{c'}$. If this map is topologically conjugate to a *rational map* f on $\hat{\mathbb{C}}$, via a conjugacy which is conformal on the interiors of K and K', we say the q_c and $q_{c'}$ are *matable*, and that f is a *mating*.

We shall not concern ourselves in detail with the case that $J(q_c)$ is not locally connected, but remark that whether or not $J(q_c)$ is locally connected it is possible to define a *formal mating* by gluing together copies of q_c on $\mathbb{C}$ and $q_{c'}$ on $\mathbb{C}$ along a *circle at infinity*, using the identification $\infty.e^{2\pi it} \to \infty.e^{-2\pi it}$. One can then quotient by the smallest closed equivalence relation for which any two points on the same ray are equivalent, and ask whether the resulting space is a topological sphere.

We shall consider several methods which have used to show that polynomials are matable.

4.1 Thurston's Criterion and the Quadratic Mating Theorem

A (topological) branched covering map $f : S^2 \to S^2$ is said to be *post-critically finite* if the forward orbits of all branch points are finite. For example a topological mating of two polynomial maps for which all critical points are periodic is such a map. Thurston's Theorem [14] provides a criterion for a post-critically finite branched covering map to be *equivalent* to a rational map, and a proof of uniqueness of this rational map (up to Möbius conjugation) when it does exist. For a statemnt of the criterion, and the definition of 'equivalent' in this context see [14]: what concerns us here is that given a topological mating between two post-critically finite hyperbolic polynomials, once we have decided whether they are *topologically matable* (that is whether $(K \cup K')/_\sim$ is a topological sphere)

Thurston's Theorem provides a method to decide whether they are *matable*, that is to say whether there is *rational* map realising the mating. The question of whether $(K \cup K')/_\sim$ is a topological sphere turns out to be purely combinatorial: in the case of hyperbolic quadratic polynomials q_c and $q_{c'}$, it can be shown that the space $(K \cup K')/_\sim$ is a sphere if and only if c and c' are not in conjugate limbs of the Mandelbrot set. If c and c' are in conjugate limbs, then certain pairs of external rays in the *formal* mating have both end points in common, so collapsing them pinches certain simple closed curves on S^2 to points and yields a space that cannot be a topological sphere. For quadratic polynomials, it can be proved by an application of Thurston's Theorem that there is no obstruction to realising a topological mating by a rational map. Using this technique Tan Lei and Mary Rees proved results which together gave a complete solution to the question of matability for quadratic polynomials with periodic critical orbits.

The Quadratic Mating Theorem (Tan Lei [38], M.Rees [33]) *Two quadratic polynomials q_c and $q_{c'}$ with periodic critical orbits are matable if and only if c and c' do not belong to conjugate limbs of the Mandelbrot set.*

Using *quasi-conformal surgery*, this theorem can be extended to all *hyperbolic* quadratic polynomials. Shishikura [37] further extended it to the case where the critical points of q_c (or of both q_c and $q_{c'}$) are *preperiodic*, and so one (or both) of $K(q_c)$ and $K(q_{c'})$ is a dendrite. An explicit case where both are dendrites is examined in detail by Milnor in [27], which is also an excellent source for further information on matings between quadratic polynomials.

4.2 Quadratic-like Maps and Quasi-conformal Surgery

To construct matings between polynomials which are not both post-critically finite, we can no longer use Thurston's Theorem. However there are cases in which we can either construct a mating directly, using conformal surgery, or we can deform an existing mating to obtain the one we seek.

Before describing an example of the use of surgery, we first observe that there is a sense in which any quadratic polynomial q_c with connected Julia set can be regarded as a mating between itself and $q_0 : z \to z^2$, since the basin of ∞ of q_c is a copy of the filled Julia set of q_0 (the unit disc $\mathbb{D}$) and on this basin the map q_c is conjugate to the map q_0 on $\mathbb{D}$. A more general notion than that of a polynomial map is that of a *polynomial-like map* [13]. This is a proper holomorphic surjection $p : V \to U$ where U and V are simply-connected open sets in $\mathbb{C}$ with $U \supset \overline{V}$. Such a map p has a well-defined *filled Julia set* $K(p) = \bigcap_{n \geq 0} p^{-n}(\overline{V})$ and a well-defined *degree d*. The *Straightening Theorem* of Douady and Hubbard [13] asserts that given any polynomial-like map p there is a genuine polynomial map P which is *hybrid equivalent* to p in the sense that there is a quasiconformal conjugacy h between p and P on neighbourhoods of $K(p)$ and $K(P)$ such that h is conformal on the interior of $K(p)$. In the case that $K(p)$ is connected, we can think of P as a *mating* between the polynomial-like map p and the polynomial $z \to z^d$ on their respective filled Julia sets.

Let $int(M_0)$ denote the interior of the *main cardioid* in the Mandelbrot set, that is to say $int(M_0)$ is the set of values c for which q_c has an attracting fixed point other than ∞. These are also the values of c for which the Julia set $J(q_c)$ is a quasicircle, and for $c \in int(M_0)$ the map q_c is conjugate on a neighbourhood of $J(q_c)$ to q_0 on a neighbourhood of the unit circle. Because it is a nice illustration of how one can use the Measurable Riemann Mapping Theorem to construct matings of polynomials in a very similar manner to the way we constructed matings of Fuchsian groups, we adapt Douady and Hubbard's method to show that for any $c \in M$ and $c' \in int(M_0)$ the map q_c can be mated with $q_{c'}$.

Theorem 1 *For every $c \in M$ and $c' \in int(M_0)$ the quadratic maps q_c and $q_{c'}$ are matable.*

Proof

We first associate an annulus A to q_c. An *equipotential* for q_c is the image of a circle $\{Re^{2\pi it} : 0 \leq t < 1\}$ under ϕ_c^{-1} (where ϕ_c is the Böttcher coordinate conjugating q_c on $\hat{\mathbb{C}} \setminus K(q_c)$ to q_0 on $\mathbb{D}$). It is a smooth Jordan curve parameterized by *external angle* t. The region bounded by such an equipotential is a simply-connected domain V, mapped two-to-one by q_c onto a larger domain $U \supset \overline{V}$ which also has boundary an equipotential parametrised by external angle. Let A denote the annulus $U \setminus \overline{V}$, and denote its inner and outer boundaries by $\partial_1 A$ and $\partial_2 A$ respectively. The map q_c sends $\partial_1 A$ two-to-one onto $\partial_2 A$. Next, to $q_{c'}$ we associate an annulus B which lies *inside* $K(q_{c'})$ (whereas A lies *outside* $K(q_c)$). This is where we use the hypothesis that $c' \in M_0$. We take a small round circle $\mathcal{C}$ around the attracting fixed point of $q_{c'}$ and let $\partial_2 B = q_{c'}^{-n}(\mathcal{C})$ and $\partial_1 B = q_{c'}^{-(n+1)}(\mathcal{C})$, for a sufficiently large value of n. Since $J(q_{c'})$ is a topological circle, on which the action of $q_{c'}$ is the doubling map, $\partial_2 B$ and $\partial_1 B$ are simple closed curves bounding an annulus B, and $q_{c'}$ maps $\partial_1 B$ two-to-one onto $\partial_2 B$.

By the (classical) Riemann Mapping Theorem there exists a conformal homeomorphism h from the outer component of $\hat{\mathbb{C}} \setminus A$ (the component containing ∞) to the inner component of $\hat{\mathbb{C}} \setminus B$. Since the boundaries $\partial_2 A$ and $\partial_2 B$ of the two regions are smooth curves, h extends to a diffeomorphism $\partial_2 A \to \partial_2 B$.

Lemma 1 *The diffeomorphism $h : \partial_2 A \to \partial_2 B$ extends to a quasiconformal homeomorphism h from A to B which on boundaries conjugates $q_c : \partial_1 A \to \partial_2 A$ to $q_{c'} : \partial_1 B \to \partial_2 B$.*

This Lemma can be proved as follows. Cut the annulus A open along a smooth path linking a point $z_0 \in \partial_2 A$ to one of the two points $q_c^{-1}(z_0) \in \partial_1 A$ (for example cut A along the ray of argument zero), and similarly cut the annulus B along a smooth curve linking $h(z_0) \in \partial_2$ to one of the two points $q_{c'}^{-1} h(z_0) \in \partial_1 B$. By arbitrarily choosing a smooth diffeomorphism between these two curves, we reduce the problem of finding a quasiconformal homeomorphism $h : A \to B$, with the prescribed properties, to the problem of extending a specified smooth diffeomorphism between the boundaries of two rectangles to a quasiconformal homeomorphism between their interiors. But a diffeomorphism between smooth curves is *quasisymmetric* and it is well-known that every quasisymmetric homeomorphism between the boundaries of two rectangles extends to a quasiconformal homeomorphism of interiors - indeed there is a *conformally natural* extension [12].

Given the quasiconformal homeomorphism h provided by the Lemma, define a branched covering map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ by setting:

- $f(z) = q_c(z)$ for z in the inner component of $\hat{\mathbb{C}} \setminus A$;
- $f(z) = h^{-1} q_{c'} h(z)$ for $z \in A$ and for z in the outer component of $\hat{\mathbb{C}} \setminus A$.

By construction f is continuous. The dilatation of h defines an ellipse field on A, which we can then pull back inductively under q_c to define an ellipse field on $\bigcup_{n \geq 0} q_c^{-n}(A)$. Extend this to an ellipse field μ on the whole of $\hat{\mathbb{C}}$ by setting it to be the standard field of infinitesimal round circles elsewhere. By construction this ellipse field is invariant under pull-back by f. The Measurable Riemann Mapping Theorem now yields a diffeomorphism $g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ which has dilatation μ, and conjugating f by g yields a *conformal* branched covering map $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$.

4.3 Pinching, and other techniques for realising matings

The interior, M_0, of the main cardioid of the Mandelbrot set may be parametrised by the multiplier $re^{2\pi i \theta}$ $(r < 1)$ of the attracting fixed point. Following along a rational ray $\theta = p/q$ to the boundary of M_0 corresponds in the dynamical plane to pinching to a single point a 'q-pronged star' centred on the attracting fixed point, and also pinching each of its iterated inverse images. For example Douady's rabbit may be obtained from a quadratic map with an attracting (but not superattracting) fixed point by contracting 3-pronged stars. This process is analogous to the pinching of geodesics described earlier in the case of Kleinian groups in that one is pinching a simple closed curve on the space of grand orbits, but the proofs are technically more difficult in the absence of the 3-dimensional hyperbolic geometry available in that case: one can contract a star to a smaller version by defining an appropriate ellipse field on a suitable neighbourhood and then applying the Measurable Riemann Mapping Theorem to 'straighten' the ellipses into circles, but the final step of contracting to a point is a technically difficult exercise because it involves taking a limit of K-quasiconformal homeomorphisms with K tending to infinity. Cui [11] and Haïssinsky [17, 18] developed powerful methods of proof, and Haïssinsky and Tan Lei [19] have applied these to show that in many cases matings between hyperbolic polynomials may be deformed into matings between parabolic polynomials.

Another technique for proving the existence of matings is to identify a candidate rational map and show that it is indeed a mating. By applying Yoccoz puzzle-piece methods to identify candidates, Yampolsky and Zakeri [39] were able to demonstrate the existence of matings between Siegel disc quadratic maps. There are many other topics concerning matings between polynomial maps that we do not have space to go into here, for example questions concerning *shared matings* (rational maps that can be expressed as matings in more than one way), questions of *continuity* and *non-continuity* of matings, and questions about the local and global structure of moduli spaces of rational maps. See, for example, [34, 35, 16, 25].

5 Matings between polynomial maps and Fuchsian groups

The first hint that it might be possible to mate polynomials of degree 2 with the modular group is the existence of a homeomorphism

$$h : \hat{\mathbb{R}}_{\geq 0} = [0, \infty] \to [0, 1]$$

conjugating the pair of maps on $[0, \infty]$:

$$\{x \to x + 1,\ x \to x/(x + 1)\}$$

(which generate $PSL_2(\mathbb{Z})$) to the pair of maps on $[0, 1]$:

$$\{t \to t/2,\ t \to (t + 1)/2\}$$

(the inverse binary shift). This conjugacy h (known as the 'Minkowski Question Mark Function') sends $x \in \mathbb{R}$ represented by the continued fraction $[x_0; x_1, x_2, \ldots]$ to $t \in I$ represented by the binary expression consisting of x_0 copies of 1, followed by x_1 copies of 0, followed by x_2 copies of 1, and so on.

If the Julia set $J(q_c)$ of $q_c : z \to z^2 + c$ is connected and locally connected then the Böttcher coordinate $\phi_c^{-1} : \mathbb{D} \to \hat{\mathbb{C}} \setminus K(q_c)$ extends to a continuous surjection $S^1 \to J(q_c)$, which semi-conjugates the map $z \to z^2$ on S^1 (the binary shift) to the map q_c on $J(q_c)$. We deduce that we may use the homeomorphism h described above to glue the action of q_c^{-1} on $J(q_c)$ to that of $\{z \to z + 1,\ z \to z/z + 1\}$ on $[0, \infty]/_{0 \sim \infty}$.

Now consider how the pair of maps $\tau_1 : x \to x + 1$ and $\tau_2 : x \to x/(x + 1)$ behave on the rest of of the real axis. Restricted to $[-\infty, 0]$ as domain and codomain they define a two-to-one map, conjugate to the binary shift on $[0, 1]$, and restricted to $[-\infty, 0]$ as domain and $[0, \infty]$ as codomain they define a bijection.

Thus if we identify together the points $-\infty, 0$ and ∞ of $[-\infty, \infty]$, to give a 'figure of eight', the pair of maps τ_1, τ_2 induce a $(2 : 2)$ correspondence made up of three parts: on the circle $[-\infty, 0]/_\sim$ a $(2 : 1)$ correspondence conjugate to the binary shift, on the circle $[0, \infty]/_\sim$ a $(1 : 2)$ correspondence conjugate to the inverse shift, and from the circle $[-\infty, 0]/_\sim$ to the circle $[0, \infty]/_\sim$ a $(1 : 1)$ correspondence.

We now take two copies K_1 and K_2 of $K(q_c)$ and glue them together at the boundary point of external angle 0 on each, to form a space $K_1 \vee K_2$. Each

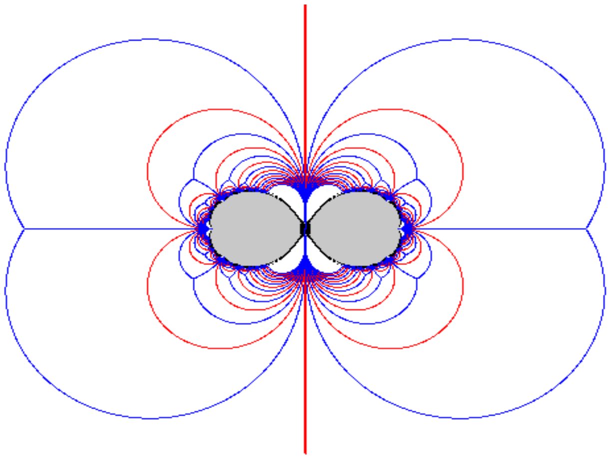

Figure 3: The mating between $q_{1/4} : z \to z^2 + 1/4$ and $PSL_2(\mathbb{Z})$.

point $z \in K(q_c)$ has a 'conjugate' point z' defined by $q_c(z') = q_c(z)$ (where $z' \neq z$ unless z is the critical point 0 of q_c). Consider the $(2:2)$ correspondence defined on $K_1 \vee K_2$ by sending

- $z \in K_1$ to $q_c(z) \in K_1$ and to $z' \in K_2$;
- $z \in K_2$ to the pair of points $q_c^{-1}(z) \in K_2$.

It is an elementary exercise to check that this correspondence on $K_1 \vee K_2$ can be glued to that defined by $z \to z+1$ and $z \to z/(z+1)$ on the complex upper half-plane using the surjections $[-\infty, 0]/_{-\infty \sim 0} \to \partial K_1$ and $[0, \infty]/_{0 \sim \infty} \to \partial K_2$ defined above. Thus we have a topological mating between the action of the modular group on the upper half-plane and our $(2:2)$ correspondence on $K_1 \vee K_2$. The question is then whether this topological mating can be realised by a *holomorphic correspondence* on $\hat{\mathbb{C}}$, that is to say a multi-valued map $f : z \to w$ defined by a polynomial equation $P(z, w) = 0$, where in this case P is of bidegree $(2:2)$ (that is to say of degree 2 in z and also in w).

Definition *Let $q_c : z \to z^2 + c$ be a quadratic map with connected filled Julia set $K(q_c)$. A holomorphic correspondence $f : z \to w$ of bidegree $(2:2)$ is called a mating between q_c and the modular group $PSL_2(\mathbb{Z})$ if:*

(a) there exists a completely invariant open simply-connected region $\Omega \subset \hat{\mathbb{C}}$ and a conformal bijection h from Ω to the upper half-plane conjugating the two branches of $f|_\Omega$ to the pair of generators $z \to z+1,\ z \to z/(z+1)$ of $PSL_2(\mathbb{Z})$;

(b) the complement of Ω is the union of two closed sets Λ_- and Λ_+, which intersect in a single point and are equipped with homeomorphisms $h_\pm : \Lambda_\pm \to K(q_c)$, conformal on interiors, respectively conjugating f restricted to Λ_- as domain and codomain to q_c on $K(q_c)$, and conjugating f restricted to Λ_+ as domain and codomain to q_c^{-1} on $K(q_c)$.

In [6] the following one (complex) parameter family of correspondences $z \to w$ was introduced:

$$\left(\frac{az+1}{z+1}\right)^2 + \left(\frac{az+1}{z+1}\right)\left(\frac{aw-1}{w-1}\right) + \left(\frac{aw-1}{w-1}\right)^2 = 3 \tag{1}$$

It was proved in [6] that the parameter value $a = 4$ gives a correspondence which is a mating between $PSL_2(\mathbb{Z})$ and the quadratic polynomial $q_{-2} : z \to z^2 - 2$, and the following conjecture was made.

Conjecture 1 *The family (1) of* $(2:2)$ *correspondences contains matings between* $PSL_2(\mathbb{Z})$ *and every quadratic polynomial having a connected Julia set, that is to say every* $z \to z^2 + c$ *with* $c \in M$*, the Mandelbrot set.*

The correspondence in the family (1) given by the parameter value $a = 7$ is plotted in figure 3. In [6] this correspondence was conjectured to be a mating between $q_{1/4}$ and $PSL_2(\mathbb{Z})$: it follows from recent results of Bullett and Haïsinsky [4] (see below) that this is indeed the case. In the figure the 'filled Julia sets' Λ_- and Λ_+ are the two regions shown in grey: on these the correspondence is conjugate to $z \to \sqrt{z - 1/4}$ and $z \to z^2 + 1/4$ respectively. On their complement, $\Omega = \hat{\mathbb{C}} \setminus (\Lambda_- \cup \Lambda_+)$, the action of the correspondence is conjugate to the standard action of $PSL_2(\mathbb{Z})$ on the open upper half plane $\mathcal{U}$: the tiling of Ω illustrated in figure 3 corresponds to the tiling of $\mathcal{U}$ by copies of the fundamental domain $\{z : 0 \leq Re(z) \leq 1/2, Im(z) > 0, |z-1| \geq 1\}$. As one varies the parameter a, the sets Λ_- and Λ_+ change shape, appearing to run through all the topological types of filled Julia sets $K(q_c)$ for $c \in M$, the Mandelbrot set. While Λ_- and Λ_+ remain connected, the action of $PSL_2(\mathbb{Z})$ on their complement Ω appears to remains conformally equivalent to the standard action of $PSL_2(\mathbb{Z})$ on the upper half-plane (in [6] this is proved to be the case when a is *real*). These observations, and computer plots of parameter space suggesting a resemblance between the Mandelbrot set and the set of correspondences in the family (1) for which Λ_- and Λ_+ are connected, provided strong evidence to support Conjecture 1.

However, difficulties in adapting the theory of *polynomial-like maps* [13] to one of 'pinched polynomial-like maps' prevented a general proof. A different question turned out to be easier to answer. The modular group may be considered as a representation of the free product $C_2 * C_3$ of cyclic groups, of orders two and three, in $PSL_2(\mathbb{C})$. Up to conjugacy there is a one (complex) parameter family of such representations, and in the parameter space there is a set $\mathcal{D}$, homeomorphic to a once-punctured closed disc, for which the representation is discrete and faithful. The modular group corresponds to a particular *boundary* point of $\mathcal{D}$. Let r be any representation of $C_2 * C_3$ corresponding to a parameter value in the *interior* $\mathcal{D}^\circ$ of $\mathcal{D}$. The ordinary set $\Omega(r)$ of the Kleinian group defined by such a representation r is connected and the limit set $\Lambda(r)$ is a Cantor set. In [5] the notion of a mating between such a representation r of $C_2 * C_3$ and a quadratic polynomial $q_c : z \to z^2 + c$ was introduced: Λ_- and Λ_+ are now disjoint, and their complement Ω is associated to $\Omega(r)$ by the property that the

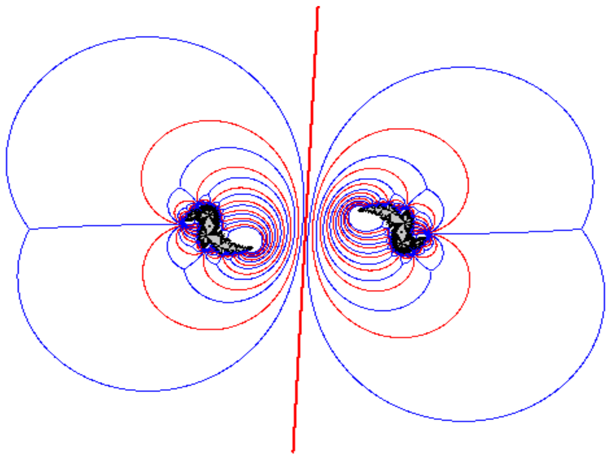

Figure 4: A mating between a generic faithful discrete representation of $C_2 * C_3$ in $PSL_2(\mathbb{C})$ and a quadratic map with an attracting orbit of period 6. This example is a correspondence in the family (2) referred to in the text.

grand orbit space of Ω under the correspondence is conformally isomorphic to the orbit space of $\Omega(r)$ under the Kleinian group generated by $C_2 * C_3$ and the unique involution χ which fixes C_2 and sends each element of C_3 to its inverse (see [5]: in the case of the representation $PSL_2(\mathbb{Z})$ of $C_2 * C_3$ the involution χ is $z \to 1/z$). The correspondence plotted in figure 4 is an example of such a mating.

Theorem 2 *For every quadratic map $q_c : z \to z^2 + c$ with $c \in M$, the Mandelbrot set, and every faithful discrete representation r of $C_2 * C_3$ in $PSL(2, \mathbf{C})$ having connected ordinary set, there exists a polynomial relation $p(z, w) = 0$ defining a $(2:2)$ correspondence which is a mating between q_c and r.*

The proof [5] of this theorem is similar in spirit to the proof of Theorem 1 outlined in the previous section: we associate an annulus A and boundary identification data to each $q_c : z \to z^2 + c$, and we associate another annulus B with similar boundary identification data to each representation r of $C_2 * C_3$ in $PSL_2(\mathbb{C})$ which has a connected ordinary set. We then show that there is a quasiconformal homeomorphism carrying one annulus and data to the other and use this to paste together the different parts of a topological mating. Finally we apply the Measurable Riemann Mapping Theorem to guarantee the existence of an invariant complex structure (see [5] for details).

It was also shown in [5] that the matings f constructed in the proof of Theorem 2 are conjugate to members of the following two-parameter generalisation of the family (1):

$$(2) \qquad \left(\frac{az+1}{z+1}\right)^2 + \left(\frac{az+1}{z+1}\right)\left(\frac{aw-1}{w-1}\right) + \left(\frac{aw-1}{w-1}\right)^2 = 3k$$

The reason for this is that the method of construction produces not just a correspondence $f : z \to w$, but also an extension of the involution σ on Ω to an involution J on $\hat{\mathbb{C}}$, exchanging Λ_+ with Λ_-, indeed conjugating f to f^{-1} on the whole of $\hat{\mathbb{C}}$, and with the crucial property that $z \to \{z, J \circ f(z)\}$ is an equivalence relation (where '$\circ$' denotes composition). It follows that $J \circ f$ is the *deleted covering correspondence* $Cov_0^Q : z \to w$ of some rational map Q, that is to say $J \circ f$ is defined by

$$\frac{Q(w) - Q(z)}{w - z} = 0.$$

Thus

$$f = J \circ Cov_0^Q$$

for some involution J and rational map Q. We can choose coordinates so that either Q or J (but not both) is in a standard form. Counting singular points of the mating f tells us that Q has one double critical point, corresponding to the fixed point of the rotation ρ on Ω, and two single critical points, corresponding to the critical points of q_c on Λ_+ and Λ_-. Thus, up to pre- and post-composition by Möbius transformations, Q is equivalent to the polynomial $Q(z) = z^3 - 3z$, and by taking coordinates in which Q has this expression we may write the relation defining f in the form

$$(Jw)^2 + (Jw)z + z^2 = 3.$$

Finally if we now put J into the standard form $J(z) = -z$ (necessarily pre-composing our normalised expression for Q by a Möbius transformation) we obtain the expression (2).

As we remarked earlier in this Section, the moduli space $\mathcal{D}$ of faithful discrete representations r of $C_2 * C_3$ is homeomorphic to a once-punctured disc. It is closely related to the moduli space of once-punctured torus groups considered in Section 3. Indeed the orbifold of the ordinary set of any representation r corresponding to an interior point of $\mathcal{D}$ is commensurable with a hyperbolic torus having a single cone point of angle $2\pi/3$ rather than a puncture point, and to each $p/q \in \mathbb{Q} \cup \{\infty\}$ one can associate a simple closed geodesic on this orbifold [4], just as in the case of a punctured torus. Contracting the p/q geodesic corresponds to deforming r to a representation corresponding to a boundary cusp of $\mathcal{D}$. For example contracting the geodesic associated to 0 takes one to the representation of $C_2 * C_3$ in $PSL_2(\mathbb{C})$ defined by $PSL_2(\mathbb{Z})$, the modular group, which has limit set $\hat{\mathbb{R}}$. Contracting the geodesic associated to p/q takes one to a representation $r_{p/q}$ of $C_2 * C_3$ in $PSL_2(\mathbb{C})$ which has limit set a *circle-packing*.

In [4], HaÏssinsky's geodesic-pinching technology is applied to the matings constructed in Theorem 2. Applied to the geodesic of rotation number 0 in a mating between a generic faithful representation r of $C_2 * C_3$ and q_c, it yields a proof of Conjecture 1 for a large class of quadratic polynomials q_c, including all hyperbolic quadratic polynomials and all quadratic polynomials with parabolic points, for example $q_{1/4}$, as illustrated in figure 3 (see [4] for the precise definition of this class).

The same method can be applied to pinch a geodesic of any rational rotation number p/q in a mating between r and q_c, at least for certain values of c, but the technicalities become more complicated and a complete proof is only given in [4] for the following case:

Theorem 3 *For all $p/q \in \mathbb{Q}$ the family (2) contains a mating between the circle-packing representation $r_{p/q}$ of $C_2 * C_3$ in $PSL_2(\mathbb{C})$ and $q_0 : z \to z^2$.*

We remark that there is a topological obstruction to the existence of a mating between a circle-packing representation $r_{p/q}$ and any q_c for which c is in the $(1-p/q)$-limb of the Mandelbrot set. This is for precisely the same reason that matings between quadratic polynomials q_c and $q_{c'}$ with c and c' in conjugate limbs of M are excluded, namely that the pinching process that would realise such a mating would pinch certain simple closed curves on the sphere to points.

5.1 Open Questions

1. Is q_0 matable with a *totally degenerate* representation of $C_2 * C_3$ (a representation r_ν with ν irrational)? If such a mating exists and has locally connected limit set, then the combinatorial structure of the mating would be that of two copies of the unit disc carrying the action of $z \to z^2$, glued together along Cantor sets corresponding to *Sturmian orbits* [8] of rotation number ν in opposite directions on their boundaries, with the remaining parts of the boundaries of these discs 'folded in' to form dendrites.

2. Can one mate a preperiodic q_c (which has limit set a dendrite) with a *totally degenerate* representation of $C_2 * C_3$? This would be a correspondence analogue of a doubly degenerate Kleinian group or a rational map which is a mating of two preperiodic quadratic polynomials.

3. More generally is it true that every discrete faithful representation r of $C_2 * C_3$ in $PSL_2(\mathbb{C})$ is matable with every q_c having $c \in M$, provided only that if c lies in the ρ-limb of M, or (if ρ is irrational) that c lies on the boundary of the cardioid at internal angle ρ, then r is not the circle-packing (or totally degenerate) representation of $C_2 * C_3$ corresponding to the boundary point of moduli space indexed by $1-\rho$?

Examples of matings between higher degree polynomials and Hecke groups (representations of $C_2 * C_n$ analogous to the modular group) are considered in [3]. For the general theory of iterated holomorphic correspondences see [7].

6 Deforming the modular group: quasifuchsian correspondences

A finitely generated Fuchsian group of G, generated by (say)

$$z \to \frac{a_j z + b_j}{c_j z + d_j} \quad (1 \leq j \leq n)$$

has the same grand orbits as those of the $(n : n)$ holomorphic correspondence defined by the polynomial relation $P(z, w) = 0$ where

$$P(z, w) = \prod_{j=1}^{n} (w(c_j z + d) - (a_j z + b))$$

Any deformation of G as a group necessarily retains the reducibility of $P(z, w) = 0$ into factors which are linear in z and w, but we have more freedom when we deform G as a correspondence. As an example, we shall examine a family of correspondence deformations of the modular group $PSL_2(\mathbb{Z})$, with chosen generators $z \to z+1$ and $z \to z/(z+1)$. The deformations will be such that the limit set of the correspondence remains a topological circle (indeed a *quasicircle*), on which the action of the correspondence continues to be topologically conjugate to the action of $PSL_2(\mathbb{Z})$ on $\hat{\mathbb{R}}$ throughout. Note that as a *quasifuchsian group* (a Kleinian group with limit set a topological circle) $PSL_2(\mathbb{Z})$ is *rigid*, and no such deformations are possible. A detailed discussion of this family of examples will be presented in [10].

As already observed, the modular group is a representation of the free product $C_2 * C_3$ of cyclic groups of orders 2 and 3. Writing σ for the involution $z \to -1/z$ and ρ for the order three element $z \to -1/(z+1)$, our chosen generators $z \to z+1$ and $z \to z/(z+1)$ for $PSL_2(\mathbb{Z})$ are $\sigma\rho$ and $\sigma\rho^{-1}$. By a suitable change of coordinates, we may view the triple $1, \rho, \rho^{-1}$ as the *covering correspondence* Cov^{Q_0} of the function $Q_0(z) = z^3$, that is to say the correspondence $z \to w$ defined by

$$Q_0(w) - Q_0(z) = 0$$

and so we may view the pair ρ, ρ^{-1} as the *deleted covering correspondence* $Cov_0^{Q_0}$ of $Q_0(z) = z^3$, that is to say the correspondence defined by

$$\frac{Q_0(w) - Q_0(z)}{w - z} = 0.$$

Thus, up to Möbius conjugacy, we may regard $PSL_2(\mathbb{Z})$ as a $(2 : 2)$ holomorphic correspondence f_0 defined by a composition

$$f_0 = J \circ Cov_0^{Q_0}$$

where $Q_0(z) = z^3$ and $J(= \sigma)$ is an involution which has the property that the fixed points of f_0 are parabolic, with the derivative of f equal to $+1$ at each of these fixed points. While there are many choices for J, all give the same correspondence f_0 up to Möbius conjugacy. This is just a translation into the language of correspondences of the fact that up to conjugacy $PSL_2(\mathbb{Z})$ is the unique representation of $C_2 * C_3$ for which the fixed points of $\sigma\rho$ and $\sigma\rho^{-1}$ are parabolic, each with derivative equal to $+1$.

We deform the correspondence $f_0 = J \circ Cov_0^{Q_0}$ by perturbing $Q_0(z) = z^3$ to $Q_\epsilon(z) = z^3 - 3\epsilon z$, at the same time perturbing the involution J in such a way that the fixed points of $f_\epsilon = J \circ Cov_0^{Q_\epsilon}$ remain parabolic. We now look for a normal form for this family of deformations. Note that provided $\epsilon \neq 0$ the rational function Q_ϵ is determined up to equivalence (that is, up to pre- and post-multiplication by Möbius transformations) by the property that it has one double and two single critical points. Therefore Q_ϵ is equivalent to $Q_1(z) = z^3 - 3z$ (provided $\epsilon \neq 0$), and thus every correspondence of the form $J \circ Cov_0^{Q_\epsilon}$ (where $\epsilon \neq 0$) is conjugate to one of the form $J \circ Cov_0^{Q_1}$. But in the previous Section, in a different context, we have already seen that correspondences $z \to w$ of the form $J \circ Cov_0^{Q_1}$ are conjugate to members of the two parameter family:

$$\left(\frac{az+1}{z+1}\right)^2 + \left(\frac{az+1}{z+1}\right)\left(\frac{aw-1}{w-1}\right) + \left(\frac{aw-1}{w-1}\right)^2 = 3k \tag{2}$$

It remains to identify the restrictions imposed on a and k by the requirement that the fixed points of the correspondence (2) be parabolic. But the fixed points are parabolic if and only they are double (as fixed points) and a straightforward calculation [10] shows that the correspondence (2) has two double fixed points (rather than the generic four single ones) if and only if:

$$k = 1 - \frac{(a-7)^2}{48.} \tag{3}$$

In this normalization, the group $PSL_2(\mathbb{Z})$ corresponds to the parameter values $k = 0$, $a = 7 + 4\sqrt{3}$. It can be proved (see [36]) that for sufficiently small perturbations of a from this value, with k required to continue to satisfy (3), the limit set Λ of the correspondence (2) remains a quasicircle, the action of the correspondence on Λ remains topologically conjugate to that of the modular group on $\hat{\mathbb{R}}$, and the action of the correspondence on one component, Ω_1, of $\hat{\mathbb{C}} \setminus \Lambda$, remains conformally conjugate to that of the modular group on the complex upper half-plane. However while the action of the correspondence on the other component, Ω_2, of $\hat{\mathbb{C}} \setminus \Lambda$, remains 'discrete' in the sense that the grand orbit space is Hausdorff, this action is no longer conjugate to that of a group.

To see why one might expect such behaviour, we first consider the geometry of the standard action of the modular group interpreted as the action of the unperturbed correspondence $J \circ Cov_0^{Q_0}$. As $Q_0(z) = z^3$ has fixed points at 0 and ∞, the subset of $\hat{\mathbb{C}}$ defined by $\{z : -\pi/3 \leq Arg(z) \leq \pi/3\}$ is a transversal D_{Q_0} for Q_0, that is to say a fundamental domain for ρ, and the parabolic condition on the fixed points of $J \circ Cov_0^{Q_0}$ may be achieved as follows. Draw any circle which has centre on the positive real axis and which touches the two lines bounding D_{Q_0}. Take J to be the involution which has its fixed points α and β the points where this circle cuts the real axis. Writing D_J for the region exterior to this circle, D_J is a fundamental domain for J, and the pair $\{D_{Q_0}, D_J\}$ satisfy the conditions of the Klein Combination Theorem (their union covers $\hat{\mathbb{C}}$) so their intersection is a fundamental domain for a faithful discrete action of $C_2 * C_3$ on $\hat{\mathbb{C}}$. The limit set of this action is the closure of the orbit of the points P_1 and P_2 where the circle ∂D_J touches the boundary ∂D_{Q_0}. This limit set is a round circle, and what we see is an action of $PSL_2(\mathbb{Z})$ which is Möbius conjugate to the standard action on the upper half plane (completed at ∞). Note that although we have a choice of transversal D_{Q_0}, and also a choice of fundamental domain D_J, all choices give the same action of $PSL_2(\mathbb{Z})$ up to Möbius conjugacy.

Now replace $Q_0(z) = z^3$ by $Q_1(z) = z^3 - 3z$. The double critical point $z = 0$ of Q_0 is split into two single critical points $z = \pm 1$ of Q_1, but the double critical point $z = \infty$ of Q_0 remains unchanged. The subset $[-\infty, -2]$ of the negative real axis is mapped by $Cov_0^{Q_1}$ to a smooth curve asymptotic to the lines $Arg(z) = \pm\pi/3$ and crossing the positive real axis orthogonally at $z = 1$. The region to the right of this curve is easily seen to be a transversal D_{Q_1} for the function Q_1 (a maximal set on which Q_1 is injective) and thus the analogue of a fundamental domain for the correspondence $Cov_0^{Q_1}$. We choose an involution J in the same way as before, that is to say we choose a circle which has centre on the real axis and which is tangent to ∂D_{Q_1}, and we let J be the involution which has as its fixed points the two points where this circle intersects the real axis. As before we let D_J denote the region exterior to the circle. Now D_J and D_{Q_1} again satisfy the conditions of the Klein Combination Theorem (which works just as well for correspondences as for groups [9]), so our correspondence has a 'faithful discrete action' on $\hat{\mathbb{C}}$, with fundamental domain $D = D_J \cap D_{Q_1}$, and limit set the closure of the grand orbit of the pair of points P_1 and P_2 where ∂D_J touches ∂D_{Q_1}. It can be shown [36] that in these circumstances the limit set Λ is a quasicircle, and that the action of the correspondence on the component Ω_1 of $\hat{\mathbb{C}} \setminus \Lambda$ containing $z = \infty$ is conformally conjugate to that of the modular group on the upper half plane. This is because in a neighbourhood of $z = \infty$ the covering correspondence of Q_1 is conjugate to a pair of rotations $\{\rho, \rho^{-1}\}$ through $\pm 2\pi/3$ since Q_1, being a polynomial of degree 3, has a double critical point at $z = \infty$.

Although the construction described in the previous paragraph works for any involution J which has fixed points on the real axis and fundamental domain a round disc whose boundary touches ∂D_{Q_1}, this is only part of the story, as it only gives those members of the family (2) which correspond to *real* values of a. We call these *real* correspondence perturbations of the modular group. To get the full one complex parameter family of deformations we must consider transversals D_{Q_1} for Q_1 which are not symmetric with respect to the real axis, and involutions J whose fixed points are no longer on the real axis, but which still have a fundamental domain D_J with boundary ∂D_J (no longer necessarily a round circle), touching ∂D_{Q_1} at a pair of parabolic fixed points of $J \circ Cov_0^{Q_1}$. The full case is treated in [10], where it is shown that there is a open region $\mathcal{Q}$ in moduli space where Λ remains a quasicircle and the correspondence remains a mating between the modular group on one component of $\hat{\mathbb{C}} \setminus \Lambda$ and a 'quasi-fuchsian correspondence' on the other. In [10] we call this region $\mathcal{Q}$, which we conjecture to be simply-connected, a *quasifuchsian slice* of moduli space.

To understand the action of the correspondence $f = J \circ Cov_0^{Q_1}$ on the component Ω_2 of $\hat{\mathbb{C}} \setminus \Lambda$, we must consider the behaviour of the forward and backward *singular* points of f: these are the points z which have a single image under f or f^{-1} rather than two images (the case for generic z). The forward singular points of f are the *co-critical* points p_1 and p_2 of Q_1, and the backward singular points are $q_1 = J(p_1)$ and $q_2 = J(p_2)$. In the normalization where Q_1 is the function $Q_1(z) = z^3 - 3z$ the point p_1 is $z = 2$ and p_2 is $z = -2$.

Note that J is a branch of $f \circ f^{-1} \circ f$, and also a branch of $f^{-1} \circ f \circ f^{-1}$, so for each of $i = 1, 2$ the singular points p_i and q_i are always on the same grand orbit of f. We say that f satisfies a *critical relation* if there is any other grand orbit relationship between p_1, p_2, q_1 and q_2. To be more precise, we say that f satisfies a *critical relation* if either p_1 and p_2 lie on the same grand orbit of f (in which case q_1 and q_2 will also lie on this grand orbit) or for at least one of $i = 1$ and $i = 2$ the points p_i and q_i are related by some branch of mixed iteration of f and f^{-1} other than J. A sufficient (but not necessary) condition for the second possibility is that the grand orbit of p_i contains a fixed point of J.

Theorem 4 *[10] There exists a countable set $\mathcal{C}$ of parameter values in $\mathcal{Q}$ at which the correspondence f satisfies a critical relation. In the normalization* (2) *these form a subset $\{a_n : n \geq 0\}$, of the positive real a-axis. At the parameter value a_n the backward singular point q_1 is mapped by a branch of f^n to the forward singular point p_1. The correspondences f given by all parameter values in $\mathcal{Q}$ other than these a_n are quasiconformally conjugate to one another.*

The points a_n accumulate on the boundary of $\mathcal{Q}$ at $a = 7$, where the correspondence f is the mating of $q_{1/4} : z \to z^2 + 1/4$ with $PSL_2(\mathbb{Z})$ described in [6] and pictured in figure 3 in Section 5 above.

We outline some of the ingredients of the proof of this theorem. First we examine how the correspondence f acts on Ω_2. In the normalization in which f is written $J \circ Cov_0^{Q_1}$ (with Q_1 the function $Q_1(z) = z^3 - 3z$), consider a triangular 'tile' Δ which has one side the segment P_1P_2 of ∂D_J running between the fixed points P_1 and P_2 of f, and as its other two sides the images of P_1P_2 under $Cov_0^{Q_1}$, which are a pair of curves P_1P_3 and P_2P_3. See figure 5: if the fixed points of J are on the real axis then ∂D_J is a round circle, as illustrated, and the edges of Δ are the arc P_1P_2 of that circle, and the smooth curves shown as dashed lines P_1P_3 and P_2P_3 in the figure, but in general the shape of ∂D_J, and hence of the 'triangle' Δ, may be much distorted. The $(2:2)$ correspondence $Cov_0^{Q_1}$ maps Δ to itself. On the boundary of Δ its action is topologically that of a rotation ρ of order 3 and its inverse ρ^{-1}. But on the interior of Δ the action

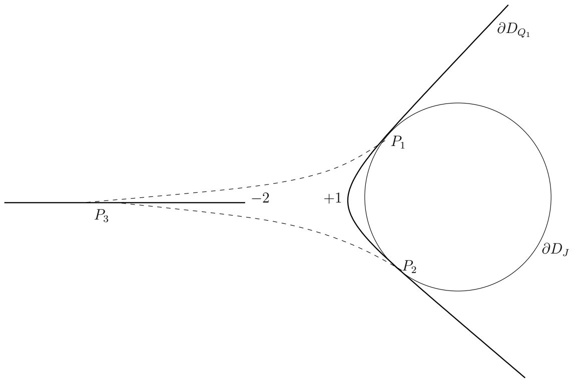

Figure 5: The tile Δ with vertices P_1, P_2, P_3.

of the $(2:2)$ correspondence $Cov_0^{Q_1}$ is best described as opening up a slit from $p_1(=-2)$ to $p_2(=+2)$ and then closing down the resulting circle to the same line segment in a different way. The correspondence $f = J \circ Cov_0^{Q_1}$ acts on Δ by this action followed by a rotation J through an angle π around the mid point of one edge of Δ to send it to the neighbouring tile $J(\Delta)$. The inverse correspondence f^{-1} sends $J(\Delta)$ back to Δ, also opening up a slit and closing it down again. Each of the other tiles in the global orbit making up Ω_2 is an image of $J(\Delta)$ under a unique branch of f^n for some $n > 0$ or an image of Δ under a unique branch of f^{-n} for some $n > 0$: the action of the correspondence f permuting the tiles on Ω_2 is combinatorially identical to the action of the generators $\sigma\rho$ and $\sigma\rho^{-1}$ of the modular group $PSL_2(\mathbb{Z})$, permuting copies of the ideal triangle $\Delta = \{z : |Re(z) + 1/2| \leq 1/2, |z + 1/2| \geq 1/2\}$ on the upper half plane.

Choose a (generic) base point $\underline{a}$ in $\mathcal{Q}$ and denote by $f_{\underline{a}}$ the correspondence at this parameter value. For convenience in drawing pictures we suppose $\underline{a}$ to be close to (but not equal to) the parameter value $a = a_0$ at which our correspondence f_a is the modular group. The orbifold $\mathcal{O}$ of grand orbits of $f_{\underline{a}}$ on Ω_2 is a punctured sphere with three cone points, each of cone angle π, corresponding to the fixed point of J and the singular points p_1 and p_2. If we mark $\mathcal{O}$ with a path from the cusp to the cone point corresponding to the fixed point of J, and make sure that this path avoids the other two cone points, then we can recover a triangular tile Δ (as in figure 5) as a triple branched cover of $\mathcal{O}$, with each edge of Δ a double cover of the marked path on $\mathcal{O}$. Once we have Δ we can recover the whole of Ω_2, and the action of $f_{\underline{a}}$ on it, by analytic continuation (laying out adjacent tiles inductively). Thus $f_{\underline{a}}$ is determined by the complex structure of $\mathcal{O}$, together with the marking. The parameter values in $\mathcal{Q}$ at which there are critical relations form a set $\mathcal{C}$ of isolated points, so given any value $a \in \mathcal{C}$ we can find a path from $\underline{a}$ to a in $\mathcal{Q} \setminus \mathcal{C}$ along which the complex structure on $\mathcal{O}$ is continuously deformed. But if we mark one more curve on $\mathcal{O}$, a path from the cusp to one of the cone points corresponding to p_1 and p_2, and we

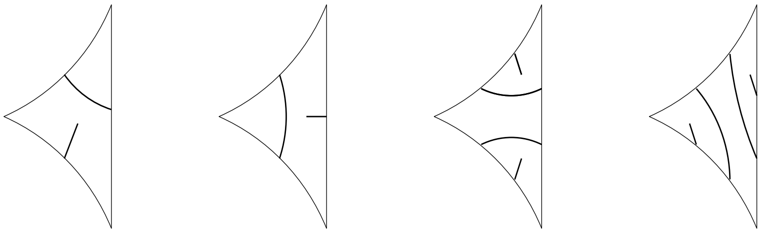

Figure 6: Patterns $P_{1,1}, \rho(P_{1,1}), P_{1,2}$ and $\rho(P_{1,2})$: the vertical edge is E_1.

double cover the marked orbifold, taking as ramification points the cone points and the puncture point, we obtain the orbifold of a (marked) punctured torus group, as considered in Subsection 3.1. The deformation theory of these groups is well-understood, and we can deduce that every quasiconformal deformation of $f_{\underline{a}}$ can be achieved in such a way as to preserve a simple closed geodesic $\gamma_{p/q}$ of rational slope p/q, or a geodesic lamination γ_ν of slope ν, on the punctured torus.

Under the procedure of quotienting by the elliptic involution on the punctured torus to obtain $\mathcal{O}$, and then passing to a triple cover and cutting along a marked curve to obtain Δ, a simple closed geodesic on the punctured torus becomes a *pattern* $\mathcal{P}$ on Δ with the following properties:

(a) $\mathcal{P}$ is invariant under Cov^{Q_1};
(b) every arc in $\mathcal{P}$ meets the interval $[p_1, p_2]$ exactly once;
(c) every arc in $\mathcal{P}$ which ends in the interior of Δ, does so at either p_1 or p_2.
(d) the end points of arcs on the edge of Δ containing the fixed point of the involution J form a set invariant under J.

We call such a $\mathcal{P}$ an *invariant pattern.* Write E_1 for the edge of Δ containing the fixed point of J, and write E_2, E_3 for the other two edges (in anticlockwise order from E_1). In [10] a standard set of invariant patterns $\{P_{m,n}\}$ is defined, for pairs (m, n) with $0 \leq m \leq n$. These patterns have the following properties, which characterize $P_{m,n}$ up to isotopies of Δ which keep its vertices fixed, but which are allowed to move the singular points p_1 and p_2:

(i) Each edge of Δ meets n arcs of $P_{m,n}$;
(ii) m of the arcs starting on E_1 continue to E_2, and the remaining $n - m$ continue to E_3.

See figure 6 for examples. It is shown in [10] that every $\gamma_{p/q}$ gives rise on the tile Δ to one of these patterns $P_{m,n}$ or to a rotated version $\rho^{\pm 1}(P_{m,n})$ (where ρ is a rotation through $2\pi/3$). If we transfer the pattern to each of the other tiles making up Ω_2 by applying the appropriate branch of f^n to $J(\Delta)$ or of f^{-n} to Δ, we obtain a *global lamination* on Ω_2. Such laminations are of two types: those where each leaf meets only a finite number of tiles, and those where each leaf meets infinitely many tiles (see figure 7 for an example of each type). It is a straightforward exercise to deform the complex structure of $f_{\underline{a}}$ to pinch

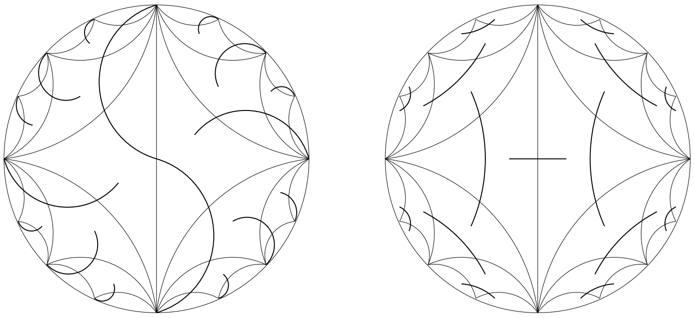

Figure 7: The global laminations corresponding to $P_{1,1}$ and $\rho(P_{1,1})$.

each component of a global lamination of the first kind to a point, and obtain a critical relation corresponding to a point of $\mathcal{C}$, that is to say in the interior of $\mathcal{Q}$. Moreover our knowledge of the deformation space of punctured tori can be applied to show that every critical relation corresponding to a point of $\mathcal{C}$ can be obtained in this way. Conjecturally, laminations of the second kind correspond to a dense set of points on the boundary of $\mathcal{Q}$, but it seems quite a difficult technical problem to apply Haïssinsky's technology to pinch leaves which meet the boundary of Ω_2.

To complete the classification of the points of $\mathcal{C}$, we must examine which patterns give rise to the same critical relation. This is a matter of keeping track of markings: as we move once around any loop in $\mathcal{Q} \setminus \mathcal{C}$ the global lamination is isotoped *globally*, the boundary of Δ is isotoped to a new position (but with the vertices unchanged), and the intersection of the new lamination with the original Δ will in general be a new pattern.

Proposition 1 *Under the action of $\pi_1(\mathcal{Q}\setminus\mathcal{C})$ every pattern is equivalent to one of the following:*
(i) $P_{0,0}$, $P_{0,1}$, $P_{1,1} = \rho(P_{0,1})$, $\rho^{-1}(P_{0,1}) = \rho(P_{1,1})$;
(ii) $P_{m,n}$ for some coprime pair of non-negative integers (m,n) with $1 \leq m < n$;
(iii) $\rho(P_{1,n})$ or $\rho^{-1}(P_{n-1,n})$ for some $n > 1$.

For a proof of this proposition see [10]. It is also shown there that the global laminations associated to those patterns which have leaves which each meet only a *finite* number of copies of Δ are $P_{0,0}$, $\rho^{-1}(P_{0,1})$ and those of type (iii), so these are the only patterns associated to critical relations in the interior of $\mathcal{Q}$. Moreover it is not hard to see that the patterns $\rho(P_{1,n})$ and $\rho^{-1}(P_{n-1,n})$ give rise to the same critical relation. The proof of Theorem 4 is completed by showing that these give the *real* critical relations described in the statement. The pattern $P_{0,0}$ corresponds to an arc from p_1 to p_2 inside Δ and gives the group case, corresponding to the parameter value a_0. Figure 8 illustrates the dynamics of the correspondence at the parameter value a_1, where the next critical relation occurs. Here the normalization is chosen to put the fixed points of J at the origin

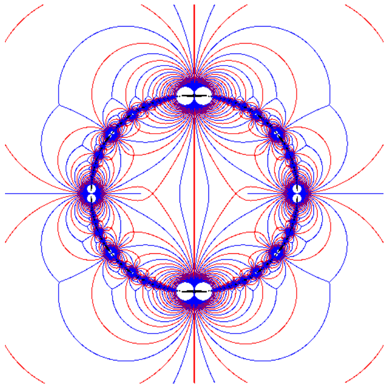

Figure 8: Dynamics of the correspondence at the parameter value a_1.

and infinity. We see the action of $PSL_2(\mathbb{Z})$ on the outer component, Ω_1, of $\hat{\mathbb{C}}\setminus\Lambda$: the action of the correspondence on the inner component, Ω_2, is that obtained from the $PSL_2(\mathbb{Z})$ action by pinching to points the connected components of the global lamination corresponding to $\rho(P_{1,1})$ (illustrated on the right in figure 7). After pinching, the tile boundaries in the right-hand picture in figure 7 become the vertical diameter and its images under the correspondence in figure 8: these are alternate lines in the figure (in the coloured version they appear in red). We remark that the limit set in figure 8 is not a *round* circle - indeed by the Schwarz reflection principle it cannot be a smooth curve.

The global laminations corresponding to the (non-rotated) $P_{m,n}$ with $(m,n) \neq (0,0)$ have leaves which all meet infinitely many tiles and therefore all have at least one end on the limit set Λ. We remark that pinching either of the laminations $P_{0,1}$ or $P_{1,1}$ (the latter is illustrated on the left in figure 7) deforms the generic correspondence to the mating of the quadratic map $q_{1/4}$ with $PSL_2(\mathbb{Z})$ illustrated in figure 3.

Finally we remark that if instead of splitting just one of the double critical points of $Q_0(z) = z^3$, we split both, and replace Q_0 by a generic rational map Q of degree 3, the construction we have outlined in this section yields quasifuchsian correspondences that do not act as a group on either of the components of $\hat{\mathbb{C}}\setminus\Lambda$. A generic such Q has four single critical points, and by choosing Q appropriately we can create a real one-parameter family of Fuchsian correspondence deformations of the modular group which all have limit set Λ a *round* circle and where the actions on the two components of its complement are reflections of one another in this round circle.

6.1 Open Questions

1. Is it possible to realise holomorphic examples of all the pinched correspondences given by pinching the laminations $P_{m,n}$ described above? Is the only

restriction to realising two such pinched correspondences simultaneously on the two sides of the limit set the condition that patterns are not of the form $P_{m,n}$ on one side and $P_{n-m,n}$ on the other? Are the conjectured degenerate representations corresponding to irrational laminations all realisable, either on just one component of $\hat{\mathbb{C}} \setminus \Lambda$ or on both? The latter would be a type of *doubly degenerate* correspondence.

2. Given a mating of $PSL_2(\mathbb{Z})$ with a quadratic polynomial q_c, as in Section 5, we can deform the representation of $PSL_2(\mathbb{Z})$ as a correspondence. Can we deform it to a degenerate correspondence? When $K(q_c)$ is a dendrite, this would yield yet another type of *doubly degenerate* correspondence.

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