LTCC

Holomorphic Dynamics and Hyperbolic Geometry (February-March 2013)

Assessment Exercises - SOLUTIONS

1. Show that for any $a \in \mathbb{D}$ the map:

$$\phi_a(z) = \frac{z-a}{1-\overline{a}z}$$

carries the unit circle to itself, and the origin to a point of $\mathbb{D}$, and hence carries the unit disc $\mathbb{D}$ isomorphically to itself. *[HINT: Observe that dividing the numerator and denominator of $\phi_a(e^{i\theta})$ by $e^{i\theta/2}$ gives an expression of the form $\zeta/\overline{\zeta}$.]*

SOLUTION.
We know that $z \in S^1$ if and only if $z = e^{i\theta}$ for some $\theta \in \mathbb{R}$. But

$$\phi_a(e^{i\theta}) = \frac{e^{i\theta}-a}{1-\overline{a}e^{i\theta}} = \frac{e^{i\theta/2}-ae^{-i\theta/2}}{e^{-i\theta/2}-\overline{a}e^{i\theta/2}} = \frac{\zeta}{\overline{\zeta}}$$

where $\zeta = e^{i\theta/2} - ae^{-i\theta/2}$. However $|\zeta/\overline{\zeta}| \in S^1$ since $|\zeta| = |\overline{\zeta}|$.
The map ϕ_a is a Möbius transformation, so it is a bijection, and since it sends S^1 to S^1 it must send the connected component $\mathbb{D}$ of $\hat{\mathbb{C}} \setminus S^1$ bijectively either to itself or to $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$. As $\phi_a(0) = -a \in \mathbb{D}$ we deduce that ϕ_a is a bijection from $\mathbb{D}$ to itself.

2. A finite product of the form

$$(B) \qquad f(z) = e^{i\theta}\phi_{a_1}(z)\phi_{a_2}(z)\ldots\phi_{a_n}(z)$$

with $a_1, \ldots, a_n \in \mathbb{D}$ is called a *Blaschke product* of degree n.

Show that f is a rational map which carries $\mathbb{D}$ onto $\mathbb{D}$ and $\hat{\mathbb{C}} \setminus \mathbb{D}$ onto $\hat{\mathbb{C}} \setminus \mathbb{D}$. Deduce that the unit circle S^1 is completely invariant and hence that the Julia set $J(f) \subseteq S^1$.

SOLUTION.
f is a rational map since it has the form of a polynomial divided by a polynomial. In Question 1 we have just seen that if $z \in \mathbb{D}$ then $\phi_a(z) \in \mathbb{D}$, and since a product of complex numbers having modulus < 1 also has modulus < 1 it follows that

$$(a) \qquad f(\mathbb{D}) \subseteq \mathbb{D}.$$

From our solution to Question 1 it is also true that if $z \in \hat{\mathbb{C}} \setminus \mathbb{D}$ then $\phi_a(z) \in \hat{\mathbb{C}} \setminus \mathbb{D}$, and since a product of complex numbers having modulus ≥ 1 has modulus ≥ 1 it follows that

$$(b) \qquad f(\hat{\mathbb{C}} \setminus \mathbb{D}) \subseteq \hat{\mathbb{C}} \setminus \mathbb{D}.$$

Similarly if $|z| = 1$ it follows that $|f(z)| = 1$, that is

$$(c) \qquad f(S^1) \subseteq S^1.$$

$f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is surjective (by the Fundamental Theorem of Algebra), so it follows from (b) that $f : \mathbb{D} \to \mathbb{D}$ is surjective, and from (a) that $f : \hat{\mathbb{C}} \setminus \mathbb{D} \to \hat{\mathbb{C}} \setminus \mathbb{D}$ is surjective.

To prove that S^1 is completely invariant we must show that $f^{-1}(S^1) \subseteq S^1$ (since we already know that $f(S^1) \subseteq S^1$ by (c) above). However if $f(z) \in S^1$ then $z \in S^1$, since a product of complex numbers of modulus < 1 has modulus < 1 and a product of complex numbers of modulus > 1 has modulus > 1.

Since S^1 is closed and completely invariant, and the Julia set $J(f)$ can be characterised as the minimal closed completely invariant set, we deduce that $J(f) \subseteq S^1$. Alternatively one can prove that $J(f) \subseteq S^1$ by applying Montel's Theorem to the iterates of f on $\mathbb{D} \cup (\hat{\mathbb{C}} \setminus \mathbb{D})$, to show that $\mathbb{D} \cup (\hat{\mathbb{C}} \setminus \mathbb{D})$ is contained in the Fatou set, $F(f)$.

3. If $g(z) = 1/f(z)$, where f is a Blaschke product, show that $J(g)$ is also contained in the unit circle.

SOLUTION.
If $g(z) = 1/f(z)$ then g carries $\mathbb{D}$ onto $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ and $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ onto $\mathbb{D}$. By a similar argument to that in the solution of Question 2 it follows that S^1 is completely invariant under the map g, and hence that $J(g) \subseteq S^1$.

4. If f is a Blaschke product of the form (B) with $n \geq 2$ and one of its factors is $\phi_0(z) = z$, show that:
(i) f has an attracting fixed point at 0.
(ii) $1/f(1/z)$ is also a Blaschke product with one of its factors $\phi_0(z) = z$, so f has an attracting fixed point at ∞ as well as at 0.
(iii) Deduce that $J(f)$ is the entire circle. (You may assume without proof that for any attracting fixed point z_0 all points in the component of $F(f)$ containing z_0 have forward orbits which converge to z_0.)

SOLUTION.
(i) If $n > 2$ and $f(z) = z.h(z)$ where $h(z) = e^{i\theta}\phi_{a_2}(z)\dots\phi_{a_n}(z)$, then $f(0) = 0.h(0)$. But $h(0) = e^{i\theta}\prod_{j=2}^{n}(-a_j) \neq \infty$ so $f(0) = 0$. Thus 0 is a fixed point.

From the rule for differentiating a product, $f'(z) = h(z) + z.h'(z)$ so $f'(0) = h(0)$ and therefore $|f'(0)| < 1$ (since $h(0) \in \mathbb{D}$). So 0 is an attracting fixed point.

(ii) The statement about $1/f(1/z)$ follows at once from:

$$\frac{1}{\phi_a(1/z)} = \frac{1 - \overline{a}/z}{1/z - a} = \frac{z - \overline{a}}{1 - az}.$$

But $z \to 1/f(1/z)$ is just f conjugated by $z \to 1/z$. So it now follows from (i) that ∞ is an attracting fixed point of f.

(iii) It follows at once from (i) and (ii) that 0 and ∞ are both attracting fixed points of f. Since the immediate basins of 0 and ∞ are distinct components of the Fatou set $F(f) = \hat{\mathbb{C}} \setminus J(f)$ we know $J(f)$ cannot be simply-connected (as the complement of any simply-connected subset of $\hat{\mathbb{C}}$ is connected). But $J(f) \subseteq S^1$ (by Question 2). So $J(f) = S^1$.

5. Let
$$f(z) = z\left(\frac{z-a}{1-az}\right)$$
with $a \in \mathbb{R}$ and $|a| < 1$ (so f satisfies the hypotheses of Question 4). Let $\psi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ denote the map $\psi(z) = z + 1/z$. Show that there is a unique rational map F such that $F\psi = \psi f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. In this way construct a 1-real-parameter family of non-conjugate quadratic rational maps with Julia set the real interval $[-2, +2]$, each with a fixed point at ∞. (You may assume that $J(F) = \psi(J(f))$, or you can prove this.)

SOLUTION.
$$\begin{aligned}\psi(f(z)) &= \frac{z(z-a)}{1-az} + \frac{1-az}{z(z-a)} = \frac{z-a}{1/z-a} + \frac{1/z-a}{z-a} \\ &= \frac{(z-a)^2 + (1/z-a)^2}{(z-a)(1/z-a)} \\ &= \frac{(z+1/z)^2 - 2 - 2a(z+1/z) + 2a^2}{1 + a^2 - a(z+1/z)} \\ &= \frac{\zeta^2 - 2a\zeta + 2(a^2-1)}{(a^2+1) - a\zeta}\end{aligned}$$
where $\zeta = z + 1/z$. Thus $\psi f = F\psi$ where
$$F(\zeta) = \frac{\zeta^2 - 2a\zeta + 2(a^2-1)}{(a^2+1) - a\zeta}.$$

The fact that F is unique follows at once from the requirement that $\psi f = F\psi$, since this tells us that for every $\zeta \in \hat{\mathbb{C}}$ the value of $F(\zeta)$ must be equal to $\psi f(z)$ for both of the two values of $z \in \psi^{-1}(\zeta)$.

Since $J(f) = S^1$ by Question 4, and assuming that $J(F) = \psi(J(f)$ (which can be proved) we have that $J(F) = [-2, +2] \subset \mathbb{R}$.

The functions F have a fixed point at $\psi(0) = \infty$. To show that the functions F are not conjugate for different values of a, it will suffice to compute the multiplier at the fixed point ∞, that is to say the multiplier of $G(\zeta) = 1/F(1/\zeta)$ at $\zeta = 0$.

$$G(\zeta) = \frac{1}{F(1/\zeta)} = \zeta . \frac{\zeta(a^2+1) - a}{2(a^2-1)\zeta^2 - 2a\zeta + 1}, \text{ so } G'(0) = -a.$$

An alternative proof of non-conjugacy follows the fact that the fixed points of F are ∞, 2 and $a-1$. These are distinguishable by the properties that ∞ is outside $[-2, +2]$, that 2 is at one end, and that $a-1$ is in $[-2, 2]$ but not at an end. While there exists a Möbius transformation sending $\infty, 2, a-1$ to $\infty, 2, b-1$, this transformation will not send the point -2 (the other end of $[-2, +2]$) to itself unless $a = b$.

Comments:

1. There are many ways to prove that $J(F) = \psi(J(f))$: one can use any of the characterisations of $J(f)$ as the closure of the set of periodic points, the accumulation set of any non-exceptional orbit, or the complement of the equicontinuity set or of the normality set, and show that $\psi^{-1}(J(f))$ has the same property for F.

2. The quadratic rational map f in this question has fixed points at 0 and ∞, with multiplier equal to $-a$ at each, and its Julia set is the unit circle. If we set $c = -a/2 - a^2/4$, so that $q_c : z \to z^2 + c$ has multiplier at its attracting fixed point equal to $-a$, the map f can be regarded as a 'mating of q_c with q_c' in the sense of the final paragraph of the final section of the lecture notes.

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