LTCC

Holomorphic Dynamics and Hyperbolic Geometry (Feb-March 2013)

Week 4 Exercises - SOLUTION TO QUESTION 1

These exercises are more open-ended. An answer to any one of questions 2,3 or 4 is an acceptable alternative to the set of assessment exercises (an answer to question 1 is too easy to find in a textbook).

1. Prove that the area of a hyperbolic triangle with angles α, β, γ is $\pi - (\alpha + \beta + \gamma)$. Deduce a formula for the area of a hyperbolic polygon with a finite number of sides.

[HINT: In the half-plane model the area of a triangle A is $\int\int_A \frac{dxdy}{y^2}$. Start by calculating the area of a triangle which has an ideal vertex, in which case you can assume that this vertex is at ∞. If it makes the calculation easier assume the triangle has angle $\pi/2$ at one vertex. A general triangle can be expressed as the difference of two triangles which have an ideal vertex.]

SOLUTION

Case 1: T a triangle with angles $0, \pi/2, \alpha$.
By applying a suitable Möbius transformation we may assume T to be the following triangle in the upper half-plane:

$$T = \{z \in \mathcal{H}^2_+ : \ 0 \leq Re(z) \leq \cos(\alpha), \ |z| \geq 1\}$$

So

$$Area(T) = \int\int_T \frac{dx.dy}{y^2} = \int_0^{\cos(\alpha)} \left(\int_{\sqrt{1-x^2}}^{\infty} \frac{dy}{y^2} \right) dx = \int_0^{\cos(\alpha)} \left[\frac{-1}{y} \right]_{\sqrt{1-x^2}}^{\infty} dx = \int_0^{\cos(\alpha)} \frac{1}{\sqrt{1-x^2}} dx$$

which, by substituting $x = \cos\theta, \ dx = -\sin\theta d\theta$, gives

$$Area(T) = \int_{\pi/2}^{\alpha} (-1).d\theta = \pi/2 - \alpha$$

Case 2: T with angles $0, \alpha, \beta$. Drop a perpendicular from the ideal vertex to the opposite side, apply Case 1 to the two resulting triangles, and add, to get $Area(T) = \pi - (\alpha + \beta)$.

Case 3: Given a triangle with non-zero angles α, β, γ at vertices A, B, C, arrange it in the upper half-plane with the side AB running along the imaginary axis. Then draw a line from C to $i\infty$ parallel to the imaginary axis. Let D denote the point $i\infty$. Now the triangle ABC is the difference between the triangles CAD and CBD, each of which has an ideal vertex, so we know their areas by Case 2, and the formula $\pi - (\alpha + \beta + \gamma)$ follows.

Given a polygon with n sides, we can subdivide it into n triangles T_j by adding a new interior vertex v_o. Suppose T_j has angle α_j at v_0 and angles β_j, γ_j at the other two vertices. Then the area of the polygon is:

$$\sum_{j=1}^{n} \pi - (\alpha_j + \beta_j + \gamma_j) = n\pi - 2\pi - \sum_{j=1}^{n} (\beta_j + \gamma_j) = (n-2)\pi - (\text{sum of interior angles})$$

$$= (\text{sum of exterior angles}) - 2\pi.$$

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