LTCC

Holomorphic Dynamics and Hyperbolic Geometry

Solutions 1

1. For the angle-doubling map $t \to 2t \bmod 1$ on the circle $\mathbb{R}/\mathbb{Z}$ prove that the periodic points are the points $t \in [0,1)$ of the form $t = m/(2^n - 1)$ (where $0 \le m < 2^n - 1$ with $m, n \in \mathbb{N}$).

Solution: $t \in 0,1)$ has period dividing $n \Leftrightarrow 2^n t \equiv t \bmod 1 \Leftrightarrow 2^n t - t = m$ for some integer m in the range $0 \le m < 2^n - 1$.

2. Show that $h : z \to z + 1/z$ is a semiconjugacy from $f : z \to z^2$ to $g : z \to z^2 - 2$ (that is, h is a surjection satisfying $hf = gh$) and that h sends the Julia set of f (the unit circle) onto the real interval $[-2, +2]$.

Solution: $hf(z) = z^2 + 1/z^2 = (z + 1/z)^2 - 2 = gh(z)$.
For $z = e^{i\theta}$ with θ real, $h(z) = e^{i\theta} + e^{-i\theta} = 2\cos(\theta)$. Hence $h(S^1) = [-2, +2]$.
[Comment: more generally the same function h semi-conjugates $z \to z^n$ to $z \to 2T_n(z/2)$, where $T_n(z)$ is the nth Chebyshev polynomial, i.e. the polynomial that expresses $\cos(n\theta)$ in terms of $\cos(\theta)$.]

3. Find a Möbius transformation which sends the upper half plane $\mathcal{H}_+$ bijectively onto the unit disc $\mathbb{D}$. Assuming the structure of $Aut(\mathbb{D})$ (Prop 2.9) prove that $Aut(\mathcal{H}_+) = PSL(2, \mathbb{R})$ (Cor 2.10).

Solution: The Möbius transformation α which sends $-1 \to -1$, $0 \to -i$ and $1 \to 1$ will do the trick, since it sends the real axis, oriented in the positive direction, to the unit circle, oriented anticlockwise, so it sends the upper half-plane to the unit disc. This transformation is:

$$\alpha = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \quad \text{so} \quad \alpha^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$$

From Prop 2.9 every element of $Aut(\mathbb{D})$ has the form:

$$\beta(z) = e^{i\theta} \frac{z - a}{1 - \bar{a}z} = \frac{e^{i\theta/2}(z - a)}{e^{-i\theta/2}(1 - \bar{a}z)}$$

where $a \in \mathbb{D}$, and therefore every element of $Aut(\mathcal{H}_+)$ has the form:

$$\alpha^{-1}\beta\alpha = \begin{pmatrix} e^{i\theta/2}(1 + ai) + e^{-i\theta/2}(1 - i\bar{a}) & e^{i\theta/2}(-i - a) + e^{-i\theta/2}(i - \bar{a}) \\ e^{i\theta/2}(-a + i) + e^{-i\theta/2}(-i - \bar{a}) & e^{i\theta/2}(1 - ai) + e^{-i\theta/2}(1 + \bar{a}i) \end{pmatrix}$$

As these matrices have real entries, they are all elements of $PSL(2, \mathbb{R})$. It is not quite obvious that every element of $PSL(2, \mathbb{R})$ is one of these matrices, but to prove this it is sufficient to show that $z \to z \pm 1$, $z \to kz$ ($k \in \mathbb{R}^{>0}$) and $z \to -1/z$ are all represented by matrices expressible in this form, since every element of $PSL(2, \mathbb{R})$ is a composition of such maps (we omit the details here).

4. Let $w = e^{i\theta}(z-a)/(1-\bar{a}z)$ with $\theta \in \mathbb{R}$ and a in the open unit disc $\mathbb{D}$. Show that $\left|\frac{dw}{dz}\right| = \frac{1-|w|^2}{1-|z|^2}$ and hence $\frac{2|dz|}{1-|z|^2} = \frac{2|dw|}{1-|w|^2}$. Deduce that the infinitesimal metric $d\rho = \frac{2|dz|}{1-|z|^2}$ is invariant under $Aut(\mathbb{D})$.
(To verify that $d\rho$ is what we get when we transfer the Poincaré metric from the upper half-plane to $\mathbb{D}$, it now suffices to check that integrating $d\rho$ gives the distance between 0 and $t \in \mathbb{D} \cap \mathbb{R}$ to be $\ln|(0, t; -1, +1)|$.)

Solution: By the usual rule for differentiating a quotient

$$\frac{dw}{dz} = e^{i\theta}\frac{1-a\bar{a}}{(1-\bar{a}z)^2}$$

$$\text{and therefore } \left|\frac{dw}{dz}\right| = \frac{1-a\bar{a}}{(1-\bar{a}z)(1-a\bar{z})}.$$

$$\text{But } 1-|w|^2 = 1-\left(\frac{z-a}{1-\bar{a}z}\right)\left(\frac{\bar{z}-\bar{a}}{1-a\bar{z}}\right) = \frac{1-a\bar{a}}{(1-\bar{a}z)(1-a\bar{z})}.\left(1-|z|^2\right).$$

$$\text{Hence } \frac{2|dz|}{1-|z|^2} = \frac{2|dw|}{1-|w|^2}$$

and the result concerning the infinitesimal metric $d\rho$ follows.
To verify that $d\rho$ gives the distance between 0 and $t \in \mathbb{D} \cap \mathbb{R}$ to be $\ln|(0, t; -1, +1)|$ it suffice to integrate $2dy/(1-y^2)$ from 0 to t. But

$$\int_0^t \frac{2dy}{1-y^2} = \int_0^t \left(\frac{1}{1+y} - \frac{1}{y-1}\right) dy = \ln\left(\frac{t+1}{t-1}\right).$$

5. Show that a rational map f of degree > 1 is conjugate to a polynomial of the form $z \to z^n$ (some $n > 1$) if and only if there exist distinct points $z_0, z_1 \in \hat{\mathbb{C}}$ such that $f^{-1}(z_0) = \{z_0\}$ and $f^{-1}(z_1) = \{z_1\}$.

Solution: For $g(z) = z^n$, the points 0 and ∞ satisfy the specified condition. So if $f = h^{-1}gh$ with h Möbius, the points $z_0 = h(0)$ and $z_1 = h(\infty)$ satisfy the condition.
Conversely if there exist z_0 and z_1 satisfying the condition, set h to be any Möbius transformation such that $h(z_0) = 0$ and $h(z_1) = \infty$. Now hfh^{-1} is a rational map for which the only inverse image of 0 is 0 and the only inverse image of ∞ is ∞, i.e. its only zero is at 0 and its only pole is at ∞. The map hfh^{-1} therefore has the form $z \to \lambda z^n$ for some $0 \neq \lambda \in \mathbb{C}$. It remains to show that if $n > 1$ then $z \to \lambda z^n$ is conjugate to $z \to z^n$. But $z \to \mu z$ will give the desired conjugacy provided $\mu(\lambda z^n) = (\mu z)^n$ for all z i.e. provided $\mu^{n-1} = \lambda$. As $n > 1$ there exists such a $\mu \in \mathbb{C}$.

6. Show that every degree 2 polynomial $z \to \alpha z^2 + \beta z + \gamma$ ($\alpha \neq 0$) is conjugate to a (unique) one of the form $z \to z^2 + c$.

Solution: A Möbius conjugacy sending a polynomial to another polynomial must have the form $z \to \lambda z + \mu$ (since it must send ∞ to ∞). This is the desired conjugacy if and only if

$$\lambda(\alpha z^2 + \beta z + \gamma) + \mu = (\lambda z + \mu)^2 + c$$

for all $z \in \mathbb{C}$, that is to say (equating coefficients):

$$\lambda\alpha = \lambda^2; \quad \lambda\beta = 2\lambda\mu : \quad \lambda\gamma + \mu = \mu^2 + c.$$

So, given $\alpha \neq 0$, β and γ, our $z \to \lambda z + \mu$ is a conjugacy if and only if $\lambda = \alpha$, $\mu = \beta/2$, and $c = \lambda\gamma + \mu - \mu^2 = \alpha\gamma + \beta/2 - (\beta/2)^2$.

7. Let f be the rational map

$$z \to \frac{-2z - 1}{z^2 + 4z + 2}$$

Find the critical points of f and their orbits. Deduce that f is conjugate to $z \to z^2 - 1$.

Solution:

$$f'(z) = 0 \Leftrightarrow (z^2 + 4z + 2)(-2) - (-2z - 1)(2z + 4) = 0$$

$$\Leftrightarrow 2z^2 + 2z = 0 \Leftrightarrow z = 0 \text{ or } z = -1.$$

The point -1 is fixed by f. The point 0 maps to $-1/2$ which maps back to 0. But $g : z \to z^2 - 1$ has critical points at ∞ and 0, and these are a fixed point and a point of a period 2 cycle $0 \to -1 \to 0 \to -1 \ldots$ respectively. So if there is a conjugacy h such that $f = hgh^{-1}$ then we must have $h(\infty) = -1$, $h(0) = 0$ and $h(-1) = -1/2$. Thus h, being a Möbius transformation, must be:

$$h(z) = \frac{z}{1 - z}.$$

It now only remains to check that this function h does indeed satisfy the equation

$$f(z) = hgh^{-1}(z).$$

SB 25/2/13