Holomorphic Dynamics and Hyperbolic Geometry (February-March 2013)

Week 2 Exercises

1. If f is a rational function with a fixed point at ∞ show that the multiplier λ at ∞ is equal to $\lim_{z\to\infty} 1/f'(z)$. Deduce that the fixed point at ∞ is a superattractor if and only if $\lim_{z\to\infty} f'(z) = \infty$. (Hint: consider the power series expansion around $\zeta = 0$ of $\sigma f \sigma$, where $\sigma(\zeta) = 1/\zeta$).

2. Picard's Theorem states that if a holomorphic function $f : \mathbb{C} \to \mathbb{C}$ (i.e. an *entire* function) has the property that there are at least two points of $\mathbb{C}$ that are not in the image of f, then f is constant. Deduce Picard's Theorem from Liouville's Theorem and the fact that $\mathbb{D}$ is the universal cover of the thrice-punctured Riemann sphere $\hat{\mathbb{C}}$. Write down a non-constant entire function the image of which omits just one point of $\mathbb{C}$.

3. Let f be a rational map. Using the 'normal families' definition of the Fatou set, prove that the Fatou set of f^2 (i.e. f composed with f) is the same set as the Fatou set $F(f)$ of F. Now consider $f(z) = z^2 - 1$. Show that $0, -1$ and ∞ are attracting fixed points of f^2 (i.e. f composed with f) and deduce that they are in different components of the Fatou set $F(f)$ of f. Deduce that $F(f)$ contains infinitely many components. Let F_0 denote the component containing 0. Sketch the position of the components of $f^{-n}(F_0)$ for $n = 1, 2, 3$, indicating how they map to each other under f.

4. A non-identity element $\alpha \in PSL(2, \mathbb{R})$ is said to be:
elliptic if it has just one fixed point in the open upper half plane;
hyperbolic if it has two distinct fixed points on the extended real line $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$;
parabolic if it has just one fixed point in $\hat{\mathbb{C}}$ (necessarily on $\hat{\mathbb{R}}$).
(i) Regarding α as a 2×2 real matrix of determinant 1, show that α is elliptic, hyperbolic, parabolic $\Leftrightarrow$ $|tr(\alpha)| < 2,\ > 2,\ = 2$ respectively (where the trace of a matrix is the sum of the entries on the main diagonal).
(ii) Show that if α is hyperbolic then it is conjugate in $PSL(2, \mathbb{R})$ to $z \to \lambda z$ for some non-zero $\lambda \in \mathbb{R}$, and in fact that we may require λ to be > 0.
(iii) Show that if α is parabolic then it is conjugate in $PSL(2, \mathbb{R})$ to $z \to z + 1$ or to $z \to z - 1$.
(iv) Show that in the Poincaré disc model of the hyperbolic plane the elliptic isometries fixing the origin are the Euclidean rotations.

5. On the hyperbolic plane a *reflection* is an orientation-reversing isometry β which fixes some geodesic pointwise.
(i) Show that every reflection β is an *involution* (i.e. $\beta^2 = I$);
(ii) Show that for every reflection β there is an element of $PSL(2, \mathbb{R})$ which conjugates β to 'reflection in the imaginary axis', i.e. the map $z \to -\bar{z}$.
(iii) Show that every orientation-preserving isometry of the hyperbolic plane can be written as the composition of a pair of reflections (by the previous question it will suffice to consider $z \to \lambda z$ and $z \to z \pm 1$ on $\mathcal{H}_+$, and $z \to e^{i\theta} z$ on $\mathbb{D}$). Deduce that every orientation-reversing isometry can be written as a composition of at most three reflections.
(iv) Show that the orientation-reversing isometries of the hyperbolic plane are precisely the maps

$$z \to \frac{a\bar{z} + b}{c\bar{z} + d} \qquad a, b, c, d \in \mathbb{R},\ ad - bc = -1$$

(Hint: if an isometry reverses orientation then composing it with a reflection will preserve orientation.)