International Journal of Modern Physics B

HECKE GROUPS, POLYNOMIAL MAPS AND MATINGS

SHAUN BULLETT and MARIANNE FREIBERGER *

School of Mathematical Sciences,
Queen Mary, University of London, London E1 4NS, UK

26 September 2002

We show that there is a topological conjugacy between the action of the Hecke group G_n on the completed positive real axis and the action of the shift on base $n-1$ expressions of real numbers in the unit interval. This conjugacy is shown to occur in *holomorphic* dynamical systems: we construct $(n-1 : n-1)$ holomorphic correspondences (multi-valued self-maps of the Riemann sphere) which are *matings* between G_n acting on the complex upper half plane, and polynomials of degree $n-1$ acting on their filled Julia sets. Certain of these correspondences split into unions of $(2:2)$ and $(1:1)$ correspondences: we present combinatorial descriptions of limit sets and the connectivity locus in the case $n = 4$.

AMS(2000) classification: 37F05

1. Hecke Groups and Shifts

For each integer value of $n \geq 3$, the *Hecke group* G_n is the subgroup of $PSL(2,\mathbb{R})$ generated by the matrices:

$$\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad and \quad \rho = \begin{pmatrix} 0 & 1 \\ -1 & -2\cos(\pi/n) \end{pmatrix}.$$

For each $n \geq 3$ this is a discrete and faithful representation of the free product of cyclic groups C_2 and C_n, generated by σ and ρ respectively. The Hecke group G_n acts via fractional linear transformations on the complex upper half-plane, with limit set $\hat{\mathbb{R}} = \mathbb{R} \cup \infty$. See figure 1, where G_4 and a fundamental domain for its action are illustrated. The Hecke group G_3 is the *modular group* $PSL(2,\mathbb{Z})$.

In mathematics there are many facts which are rediscovered in different contexts by generation after generation of researchers. One such is that the action of the modular group on $\mathbb{R}_+ = \mathbb{R}^{\geq 0} \cup \infty$ is conjugate to the action of the shift on binary expressions for real numbers in the closed unit interval $I = [0,1]$. Explicitly, if we define

$$h : \mathbb{R}_+ \to I$$

*s.r.bullett@qmul.ac.uk m.freiberger@qmul.ac.uk

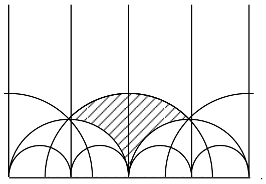

Fig. 1. The Hecke group G_4: the hatched region is a fundamental domain

by sending

$$a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \ldots}} \to .11\ldots100\ldots011\ldots1...$$

where the blocks of ones and zeros have lengths $a_0, a_1, a_2 \ldots$, then h is a homeomorphism and conjugates the pair of maps $x \to x+1,\ x \to x/(x+1)$ on $\mathbb{R}_+$ to the pair of maps 'right shift with insertion of zero or one' on I.

Less well known is that there is an analogous conjugacy for all $n \geq 3$:

Proposition 1: *For each $n \geq 3$ there exists a homeomorphism $h : \mathbb{R}_+ \to I$ conjugating the actions of the elements $\sigma\rho, \sigma\rho^2, \ldots \sigma\rho^{n-1}$ of the Hecke group G_n on $\mathbb{R}_+$ to the actions of 'right shift with insertion of $n-2, n-3, \ldots, 0$' on base $n-1$ expansions of real numbers in I.*

Proof: For $1 \leq j \leq n-1$ the maps

$$\sigma\rho^j : \mathbb{R}_+ \to [\sigma\rho^j(0), \sigma\rho^j(\infty)]$$

are order-preserving homeomorphisms. But each of these maps is a contraction in the spherical metric on $\mathbb{R}_+ \subset \hat{\mathbb{C}} = \mathbb{C} \cup \infty$, and their images partition $\mathbb{R}_+$ since $\sigma\rho(\infty) = \infty$, $\sigma\rho^{j+1}(\infty) = \sigma\rho^j(0)$ for all j, and $\sigma\rho^{n-1}(0) = 0$. The result follows. □

2. Polynomial Maps and the Böttcher Co-ordinate

A natural setting in holomorphic dynamics in which we see the action of the shift on base m expressions of real numbers in I, is on the boundary of the filled Julia set of a polynomial P of degree m. Recall that the filled Julia set of P is defined to be

$$K(P) = \{z : P^n(z) \nrightarrow \infty\}$$

and that if $K(P)$ is connected the Riemann mapping $\mathbb{D} \to \hat{\mathbb{C}} - K(P)$ (the 'Böttcher co-ordinate') defines a conjugacy from the action of $z \to z^m$ on the unit disc to the action of P on $\hat{\mathbb{C}} - K(P)$, or equivalently from the action of the m-valued map $z \to z^{1/m}$ on the unit disc to the action of the m-valued map P^{-1} on $\hat{\mathbb{C}} - K(P)$. If $\partial K(P)$ is locally connected this conjugacy extends to a semi-conjugacy on boundaries, from 'right shift with insertion of $0, 1, ..., m-1$ on base m expressions of real numbers in $\mathbb{R}/\mathbb{Z}$' to the action of P^{-1} on $\partial K(P)$. Combining this with Proposition 1 we obtain a semi-conjugacy from the action of G_n on $\mathbb{R}_+$ to the action of P^{-1} on the boundary of the filled Julia set of a polynomial P of degree $n-1$. The miracle is that this semi-conjugacy arises in practice, in a *holomorphic* dynamical system.

3. Correspondences and Matings

An $(m:n)$ *holomorphic correspondence* on the Riemann sphere $\hat{\mathbb{C}}$ is a multivalued map $z \to w$ defined by a relation $p(z,w)=0$, where $p(z,w)$ is a polynomial of degree m in z and degree n in w. The notion of a holomorphic correspondence is a generalization of that of a rational map (an $(m:1)$ correspondence) and also of that of a finitely generated group G of fractional linear maps. We may think of the latter as the correspondence $(w-g_1(z))\dots(w-g_m(z))=0$, where the Möbius transformations $g_1,\dots g_m$ generate G: this correspondence has the same grand orbits as G.

The dynamical behaviour of a holomorphic correspondence under iteration can exhibit similarities to that of a rational map or a Kleinian group (a discrete group of fractional linear maps). A construction [1] was presented for $(2:2)$ correspondences which are *matings* of the modular group with quadratic maps, in a sense which we now generalize. We say that an $(n-1:n-1)$ holomorphic correspondence $\mathcal{G}$ on the Riemann sphere, equipped with an involution J conjugating $\mathcal{G}$ to $\mathcal{G}^{-1}$, is a *mating* between G_n and a polynomial map $z \to P(z)$ of degree $n-1$ if the Riemann sphere is partitioned into simply-connected completely invariant sets Ω (open) and Λ (closed), such that:

• on Ω the action of $\mathcal{G}$ is conjugate to the action of $\{\sigma\rho, \sigma\rho^2, \dots, \sigma\rho^{n-1}\}$ on the complex upper half-plane (where σ and ρ are the generators of G_n defined above), and this conjugacy carries the action of J to that of σ;
• Λ contains a subset Λ_+ such that restricted to domain and range Λ_+ the correspondence $\mathcal{G}^{-1}$ is a holomorphic map conjugate to the polynomial P acting on its filled Julia set, the conjugacy being conformal on interiors;
• $\Lambda = \Lambda_- \cup \Lambda_+$, where $\Lambda_- = J(\Lambda_+)$ and Λ_- meets Λ_+ in a single point.

See figure 2 for a computer plot of an example. We can think of such a mating as being obtained from the complex upper half plane by gluing the points zero and infinity together and then gluing to the two halves $\mathbb{R}_+$ and $\mathbb{R}_-$ of its boundary two copies of the filled Julia set for P, using the matching of boundaries provided by the homeomorphism h of Proposition 1.

Before showing that there exist correspondences realising matings, we first consider the simplest kind of $(n:n)$ correspondence. If $Q:\hat{\mathbb{C}} \to \mathbb{C}$ is any polynomial of degree n, we define the *covering correspondence* of Q to be the $(n:n)$ correspondence Cov^Q given by

$$Cov^Q : z \to w \Leftrightarrow Q(z) - Q(w) = 0$$

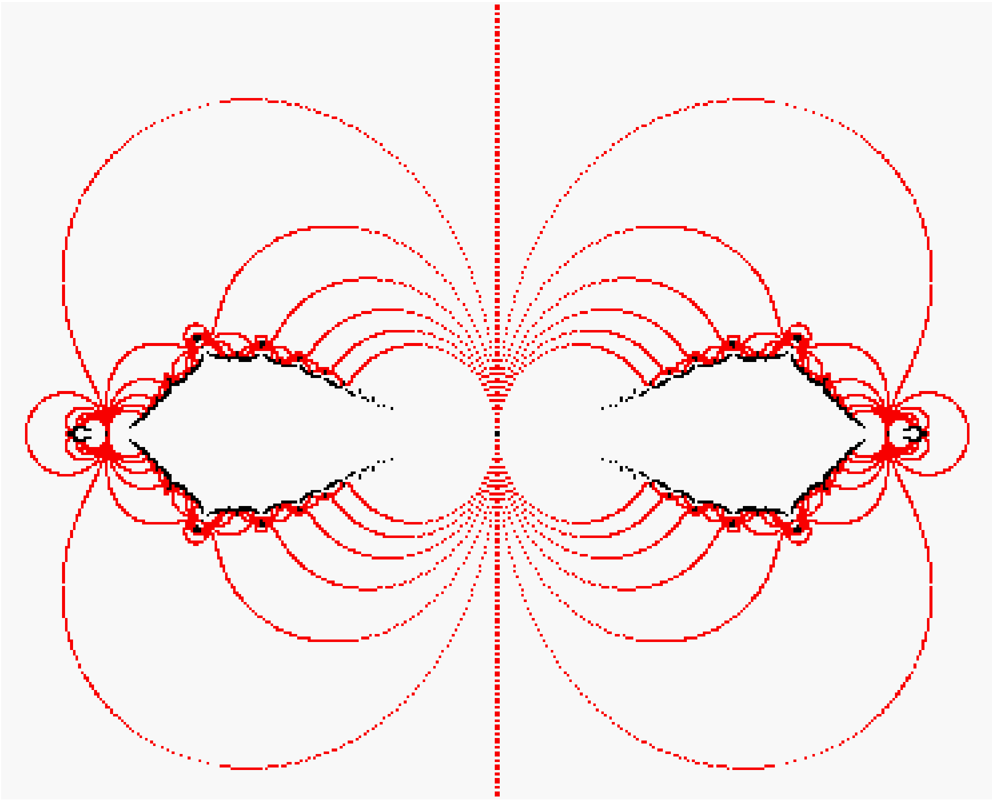

Fig. 2. Computer plot of an example of a mating between the Hecke group G_4 and a cubic polynomial. Λ_+ and Λ_- are regions to the right and left of the center of the picture, intersecting in a single point (in a neighbourhood of which the computer plot is incomplete, for numerical reasons). The lines drawn on the outer region, Ω, are the images of the vertical axis (the boundary of D_J) under iteration of $\mathcal{G}$ and $\mathcal{G}^{-1}$, and correspond to the images of the imaginary axis under the Hecke group G_4 on the complex upper half-plane.

By 'striking out the diagonal' we obtain the $(n-1 : n-1)$ correspondence

$$Cov^Q - I : z \to w \Leftrightarrow \frac{Q(z) - Q(w)}{z - w} = 0$$

Note that a *critical point* of Q (a branch point) is a fixed point of $Cov^Q - I$ and vice versa. Neither Cov^Q nor $Cov^Q - I$ has much interest from the point of view of dynamics, since grand orbits for Cov^Q are simply the fibers of Q. However we shall obtain our matings as compositions of correspondences $Cov^Q - I$ followed by the simplest kind of map, an involution J of the Riemann sphere.

A *transversal* for Q is a maximal set D_Q on which Q is injective. One should think of it as a 'fundamental domain' for Cov^Q. For simplicity we shall assume all our transversals and fundamental domains are simply-connected and bounded by Jordan curves. We say that a pair (D_Q, D_J) consisting of a transversal for a polynomial map Q and a fundamental domain for an involution J satisfy the *Klein combination condition* if the interiors of D_Q and D_J together cover $\hat{\mathbb{C}}$, and they satisfy the *contact condition* if the interiors of D_Q and D_J cover all of $\hat{\mathbb{C}}$ except for a single point which is a fixed point of both $Cov^Q - I$ and J. The Klein combination (or 'ping-pong') principle [2] ensures that in either case the composite $\mathcal{G} = J \circ (Cov^Q - I)$ acts freely on

$$\Omega = \hat{\mathbb{C}} - (\Lambda_+ \cup \Lambda_-)$$

where

$$\Lambda_+ = \bigcap_{m>0} \mathcal{G}^m(D_Q) \qquad \Lambda_- = \bigcap_{m<0} \mathcal{G}^m(D_J).$$

The intersection $D_Q \cap D_J$ is a fundamental domain for this action on Ω, and the restriction of $\mathcal{G}$ to Λ_+ as both domain and range is a $(1 : n-1)$ correspondence, i.e. the inverse of a holomorphic map. The following can be proved [3]:

Theorem 2: Let Q be a polynomial of degree n. If (D_Q, D_J) satisfies the contact condition, and all the critical points of Q other than ∞ lie in Λ, then
(i) Λ_+ and Λ_- are each simply-connected, and
(ii) there is a conformal homeomorphism ϕ from the upper half plane $\mathbb{H}$ to Ω carrying the action of the elements $\sigma\rho, ..., \sigma\rho^{n-1}$ of the Hecke group G_n to the action of the $(n-1 : n-1)$ correspondence $\mathcal{G} = J \circ (Cov^Q - I)$.

A consequence of this theorem is that when the restriction $\mathcal{G}^{-1} : \Lambda_+ \to \Lambda_+$ (which is a holomorphic map of degree $n-1$) is conjugate to a genuine polynomial P of degree $n-1$ acting on its filled Julia set, we have a *mating* of G_n with P in the sense of our definition above. In many cases we can show that $\mathcal{G}^{-1}$ is indeed conjugate to a polynomial, by using the theory of *polynomial-like mappings* developed by Douady and Hubbard [4]. We conjecture that examples of matings exist for *all* polynomials P of degree $n-1$ which have connected Julia sets.

4. Covering Correspondences of Chebyshev Polynomials

The nth *Chebyshev polynomial* Q_n expresses $\cos(n\theta)$ in terms of $\cos(\theta)$. For example $Q_3(z) = 4z^3 - 3z$ and $Q_4(z) = 8z^4 - 8z^2 + 1$.

When we take $Q_n(z)$ for the polynomial $Q(z)$ in our $(n-1 : n-1)$ correspondence $\mathcal{G} = J \circ (Cov^Q - I)$ we find that the correspondence $\mathcal{G}$ splits into a union of $(2 : 2)$ correspondences if n is odd and a union of $(2 : 2)$ correspondences together with a $(1 : 1)$ correspondence if n is even. This follows from the fact that for n odd:

$$Q_n(w) - Q_n(z) = 2^{n-1}(w - z) \prod_{j=1}^{(n-1)/2} (w^2 - 2zw\cos(2j\pi/n) + z^2 - \sin^2(2j\pi/n))$$

and for n even:

$$Q_n(w)-Q_n(z) = 2^{n-1}(w-z)(w+z)\prod_{j=1}^{n/2-1}(w^2-2zw\cos(2j\pi/n)+z^2-\sin^2(2j\pi/n))$$

We write g for the $(2:2)$ correspondence

$$z \to w \Leftrightarrow w^2 - 2zw\cos(2\pi/n) + z^2 - \sin^2(2\pi/n) = 0$$

and remark that the other $(2:2)$ (or $(1:1)$) correspondences in the decomposition of Cov^{Q_n} can be obtained by iterating g. It follows that the grand orbits of the $(n:n)$ correspondence $\mathcal{G} = J \circ Cov^{Q_n}$ are the same as those of the $(2:2)$ correspondence $J \circ g$.

The covering correspondence of the Chebyshev polynomial Q_3, followed by an involution J, gives rise to the matings of the modular group with quadratic maps [5, 1]. In the remaining part of this note we shall consider the dynamics of $J \circ (Cov^Q - I)$ for $Q = Q_4$.

5. Combinatorics of Limit Sets and Connectivity Loci

When $n = 4$ our correspondence $\mathcal{G} = J \circ (Cov^{Q_n} - I)$ has the form

$$z \to w \Leftrightarrow ((Jw)^2 + z^2 - 1)(Jw + z) = 0$$

Thus $\mathcal{G} = f \cup A$ where f is the $(2:2)$ correspondence

$$f = J \circ g : z \to w \Leftrightarrow (Jw)^2 + z^2 - 1 = 0$$

and A is the Möbius transformation $z \to w = J(-z)$.

The Chebyshev polynomial $Q = Q_4$ has critical points at $\infty, 0, \pm\frac{1}{\sqrt{2}}$, and the singular fibers of Q (those containing less that four points) are $\{\infty\}, \{\pm\frac{1}{\sqrt{2}}\}, \{-1, 0, 1\}$. For our transversal for Q we may take the set

$$D_Q = \{(x,y)|x \geq \frac{1}{\sqrt{2}}, |y| < \sqrt{\frac{2x^2-1}{2}}\}$$

We consider involutions J which have $\frac{1}{\sqrt{2}}$ as fixed point, and hence as contact point. We have a one complex parameter family of $(3:3)$ correspondences $\mathcal{G} = f \cup A$ to consider, parametrized by the other fixed point of J. We investigate the combinatorics of Λ_+ for correspondences in this family: when Λ_+ is connected we shall see that it is conformally homeomorphic to the filled Julia set of a cubic polynomial, and so satisfies our definition of a mating. Moreover, it (and therefore the cubic filled Julia set) is the closure of a countable union of copies of a single filled *quadratic* Julia set, itself associated to some *branch* of iteration of f and A. Thus it makes sense to investigate the structure of Λ_+ in terms of this quadratic filled Julia set. Suppose that D_J is such that (D_Q, D_J) satisfy the contact condition

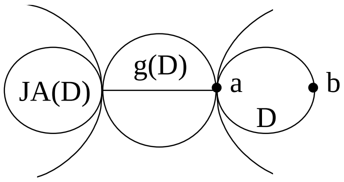

Fig. 3. The sets $D, g(D)$ and $JA(D)$; the fixed points of J are marked $a(=\frac{1}{\sqrt{2}})$ and b.

(see figure 3), and let $D = \hat{\mathbb{C}} - D_J$. Then

$$\Lambda_+ = \bigcap_0^\infty \mathcal{G}^n(D_Q) = \bigcap_0^\infty \mathcal{G}^n(D)$$

Denote the co-critical point one by c. It is easily proved [3] that Λ_+ is connected if and only if $c \in \mathcal{G}^n(D)$ for all n. We consider the possible locations for c which satisfy this condition. Figure 4 illustrates $\mathcal{G}(D) = f(D) \cup A(D)$, and figure 5 illustrates the two possible configurations for $\mathcal{G}^2(D)$, depending on whether (i) $c \in f(D)$ or (ii) $c \in A(D)$. Figure 6 shows how Figure 5 (i) and (ii) arises via the conjugacy ϕ in the respective cases from the action of G_4 on the upper half-plane $\mathbb{H}$.

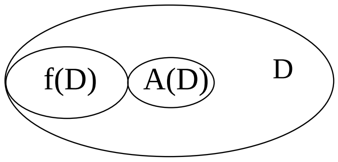

Fig. 4. $\mathcal{G}(D) = f(D) \cup A(D)$.

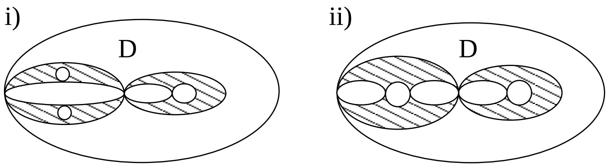

Fig. 5. The shaded area is $\mathcal{G}(D) - \mathcal{G}^2(D)$.

Keeping track of the configuration of $\mathcal{G}^n(D)$ becomes increasingly difficult as n becomes larger. For example there are eight possible pictures for $\mathcal{G}^3(D)$, depending on the position of c. To simplify our analysis we specialize to the case that c is fixed under some branch h of $\mathcal{G}^m$ commencing with f (i.e. $c \in \mathcal{G}^{m-1}f(c)$) and we suppose that m is minimal in the sense that h cannot be written as k^j for some j

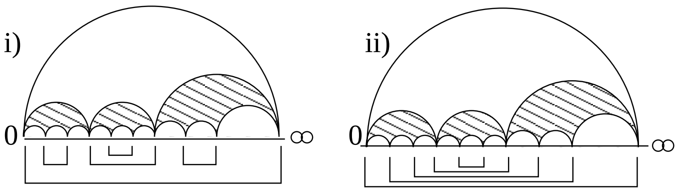

Fig. 6. The pairings of points of $\mathbb{R}_+$ indicate the identifications made in the surjection $\mathbb{R}_+ \to \partial\Lambda_+$ in Figure 5(i) and (ii) respectively.

dividing m and branch k of $\mathcal{G}^{m/j}$. Let C denote the component C of the interior of Λ_+ containing c. Under these hypotheses it can be shown that the restriction of h to C, as both domain and range, is a $(1:2)$ correspondence which is conformally conjugate to the correspondence $z \to \pm\sqrt{z}$ on the unit disc. It can be shown [3] that when c is fixed by h as above:

Proposition 3:
(i) $\phi : \mathbb{H} \to \Omega$ extends to a continuous surjection $\mathbb{R}_+ \to \partial\Lambda_+$. Thus each point of $\partial\Lambda_+$ has a well-defined set of 'external angles' (inverse images under ϕ: the terminology comes from the theory of iterated rational maps [6]). These angles can be written in their base three expansions as in Proposition 1.
(ii) ∂C is homeomorphic to a circle and $h : \partial C \to \partial C$ is conjugate to 'right shift with insertion of zero or one' on binary expressions in $\mathbb{R}/\mathbb{Z}$: the external angles of the fixed point of h which lies in ∂C have the form $\bar{X}$ and $\bar{Y}$ for certain words X and Y in $0, 1, 2$ of length m (where $\bar{X}$ denotes the infinite word $XX\ldots$). The points of ∂C have external angles words in X and Y, and the action of h on C is that of 'right shift with insertion of X or Y'.

See figure 7. The interval $[\bar{X}, \bar{Y}]$ in $\mathbb{R}_+$ is referred to as the *critical interval* for the correspondence under consideration.

5.1. *Algorithm for a model for* Λ_+

Define the *dual* of $0, 1, 2$ by $\tilde{0} = 2, \tilde{1} = 1, \tilde{2} = 0$. Now let $\sim$ be the closure of the relation defined inductively by
- $\bar{0} \sim \bar{2}$
- If $x \sim y$ are words in $0, 1, 2$ and $w \in \{0, 1, 2\}$ then:
  $wx \sim \tilde{w}y$ if $[x, y]$ contains the critical interval (or is equal to it), and
  $wx \sim wy$ otherwise.

A model for Λ_+ is obtained from the unit disc by contracting geodesics joining pairs of boundary points related by $\sim$. It can be shown [3] that in the situation considered in the Proposition above, Λ_+ consists of the closure of a union of a countable infinity of copies of C (the images of C under mixed iteration of $\mathcal{G}$ and $\mathcal{G}^{-1}$) arranged together in the form of a tree.

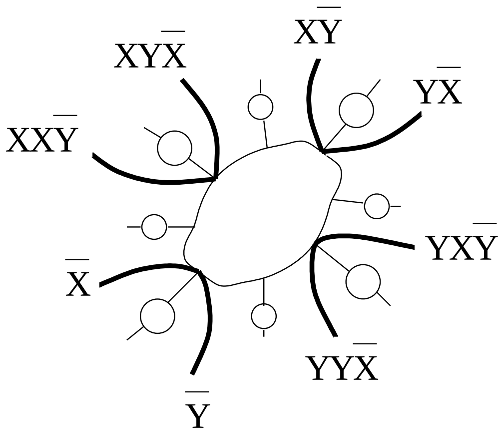

Fig. 7. The critical interval is $[\bar{X}, \bar{Y}]$.

The case considered so far is where the branch h of $\mathcal{G}^m$ which takes the critical component C back to itself is conjugate to $z \to \pm\sqrt{z}$. By varying the fixed point of J it is possible to create examples where h is conjugate to $z \to \pm\sqrt{z-a}$ for other values of a in the interior of the quadratic Mandelbrot set M: the algorithm for the combinatorial model for Λ_+ in these cases proceeds as before but we add extra relations for pairs of words in X and Y which correspond to pairs of words in zero and one identified together in the combinatorial model for the filled Julia set K_a of $z \to z^2 + a$. Now Λ_+ is the closure of a union of countably many copies of K_a, in place of K_0, arranged in the same tree.

5.2. *Algorithm for a model for the connectivity locus*

Let $\mathcal{N}$ denote the *connectivity locus* in parameter space — that is to say the set of all positions of the second fixed point of J such that the limit set Λ of our $3:3$ correspondence $\mathcal{G}$ is connected. This is the analogue of the *Mandelbrot set* M for quadratic maps. Recall that there is a combinatorial model for M obtained from the closed unit disc by contracting geodesics joining certain pairs of boundary points [6] (usually represented by their binary expansions: e.g. $\overline{01}$ is joined to $\overline{10}$): if, as conjectured, M is locally connected, then M is known to be homeomorphic to the combinatorial model [6]. We construct a combinatorial model for $\mathcal{N}$ from the unit disc as follows [3]. Represent points on the boundary circle $\mathbb{R}/\mathbb{Z}$ by their base three expansions. Define the first step in the construction of a relation $\approx$ by

- $\bar{0} \approx \bar{2}$ (this is just the identification of $0 \in \mathbb{R}$ with $1 \in \mathbb{R}$)
- $0\bar{2} \approx 2\bar{0}$
- $W0\bar{2} \approx W2\bar{0}$ for W any word in $0, 2$
- $W \approx W'$ for any words W, W' in $0, 2$ such that the corresponding words in $0, 1$ (which have a 1 or 0 where W has a 2 or 0 respectively) are joined in the combinatorial model for the Mandelbrot set M.

For the second step in the construction of $\approx$ repeat the first step but with 10 in place of 0 and 12 in place of 2. For each subsequent step take a shortest word X which ends in 0 or 2 which has not already been considered, and associate to X the unique word Y such that $[\bar{X}, \bar{Y}]$ is a critical interval of the form considered in Proposition 2. Construct a copy of the Mandelbrot set M for words in X and Y, just as in the first two steps. We remark that we believe Y to be uniquely determined by the properties that Y has the same length as X, has one in the positions where X has one, has either zero or two in the positions where X has either zero or two, ends in the dual to the last letter of X, and has the property that $\bar{Y}$ can be joined to $\bar{X}$ without crossing any of the geodesics between related pairs already constructed in earlier steps. The first few steps in the construction are illustrated in figure 8.

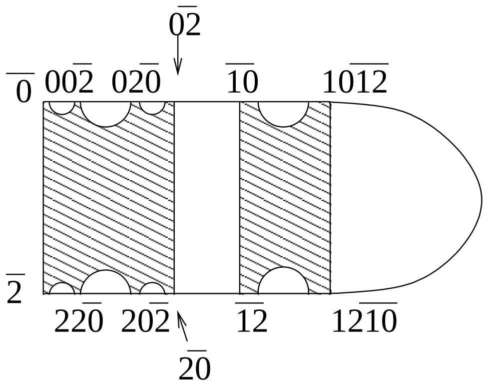

Fig. 8. Copies of M are glued into the hatched regions, after the arcs making up the boundaries of these regions have been contracted to points.

In the limit we obtain a combinatorial model for $\mathcal{N}$ consisting of the closure of a union of a countable infinity of copies of the Mandelbrot set M, one for each critical interval $[\bar{X}, \bar{Y}]$, arranged in the form of a tree.

Acknowledgement

We thank the UK EPSRC for their support through grant no. GR/R73232/01.

References

1. S Bullett and C Penrose, Inventiones Mathematicae **115** (1994), 483-511
2. S Bullett, Conformal Geometry and Dynamics 4 (2000) 75-96
3. S Bullett and M Freiberger, *Holomorphic correspondences mating quadratic maps with Hecke groups*, preprint, Queen Mary (University of London), 2002
4. A Douady and J H Hubbard, An. de l'École Norm. Sup. 18 (1985) 287-343
5. S Bullett and C Penrose, Fund. Math. 167 (2001) 111-171
6. A Douady and J H Hubbard, C.R. Acad. Sci. Paris 294 (1982) 123-126