V. Analytic Continuation and Riemann Surfaces

This section will introduce various ideas that are important for more advanced work in complex analysis.

1. Continuation along paths

A holomorphic function f defined on some open disc containing z_0, has a Taylor series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$$

We can use this series to *define* f at any points inside the disc $D(z_0, r_0)$ of convergence of the series where f has not already been defined. Now choosing a new point z_1 near the edge of $D(z_0, r_0)$ we can repeat the same procedure for the Taylor series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(z_1)}{n!}(z-z_1)^n$$

and this will enable us to define f in a new disc $D(z_1, r_1)$, overlapping the first disc. This process is known as *analytic continuation* of f. An *analytic* function is one that can be written as the sum of a power series: note that we can only carry out the *continuation* process because of Taylor's Theorem, which says that a holomorphic function is an analytic function.

The most interesting examples of this process occur when f is one branch of a multi-valued function, and we follow the analytic continuation of this branch along some path, or even round a loop back to where we started. This is best illustrated with an example.

Let $f(z) = \sqrt{z}$. Around the point $z = 1$, one branch of this function can be defined by the power series:

$$(1+(z-1))^{1/2} = 1 + \frac{(1/2)}{1!}(z-1) + \frac{(1/2)(-1/2)}{2!}(z-1)^2 + \frac{(1/2)(-1/2)(-3/2)}{3!}(z-1)^3 + \dots$$

which converges for $|z-1| < 1$. Using this as our definition of a function g in the open disc $D(1, 1)$, we can choose any point within this disc, for example $z_1 = 1 + i/2$, compute the derivatives of g there, and hence construct a Taylor series

$$\sum_{n=0}^{\infty} \frac{g^{(n)}(z_1)}{n!}(z-z_1)^n$$

which is valid in some disc with centre z_1. By taking disc centres $1, z_1, z_2, z_3, \dots$, which lie on some path which goes once around the origin we can follow the branch $g(z)$ of $\sqrt{z}$ which takes the value 1 at

$z = 1$. When we get back to 1 we will find our function is the other branch of $\sqrt{z}$, the branch which takes the value -1 at $z = 1$. This is because we have followed $\sqrt{z}$ continuously round from $arg(z) = 0$ to $arg(z) = 2\pi$, and $\sqrt{}$ halves arguments. Of course if we follow either branch of $\sqrt{}$ *twice* around the origin we get back to the same branch that we started with.

2. Riemann Surfaces

Riemann surfaces provide a way to deal with analytic continuation of multivalued functions. Rather than try to give a general definition we consider two examples.

2.1 The Riemann surface for $z \to \sqrt{z}$

Take two copies of $\hat{\mathbb{C}}$ and slit each of them along the positive real axis, from 0 to ∞. Denote these slit copies of $\hat{\mathbb{C}}$ by S_1 and S_2. Now glue the upper edge of the slit on S_1 to the lower edge of the slit on S_2, and the lower edge of the slit on S_1 to the upper edge of the slit on S_2, in each case gluing each point $x \in [0, \infty]$ to the corresponding point x on the other copy.

The resulting surface S has a projection $p : S \to \hat{\mathbb{C}}$, which sends each point in either of S_1 or S_2 to the corresponding point of $\hat{\mathbb{C}}$. The surface S tells us what happens as we follow the two branches of $\sqrt{}$ around $\hat{\mathbb{C}}$.

The points 0 and ∞ in this example are called the *branch points* (or sometimes called the *ramification points*) of the Riemann surface S. Although they are special in the sense that they are the points z where $\sqrt{z}$ has only one value, rather than two, in another sense they are not so special: S is just a copy of $\hat{\mathbb{C}}$, and the projection $p : S \to \hat{\mathbb{C}}$ is just the map $z \to z^2$.

2.2 The Riemann surface for $z \to \log z$

Recall that for any complex number $z \neq 0$, the multivalued function $\log z$ is defined by $\log z = \ln r + \arg\theta + 2k\pi i$ (where $z = re^{i\theta}$ with $0 \leq \theta < 2\pi$, and k takes any integer value, positive or negative), and that if we follow a branch of log once around the origin in an anticlockwise direction the value of $\log z$ increases by $2\pi i$. The Riemann surface S for log is constructed from infinitely many copies of the punctured complex plane $\mathbb{C} \setminus \{0\}$, each slit along the positive real axis. We denote the copies by S_k, one for each $k \in \mathbb{Z}$. We construct the Riemann surface S for log by gluing the lower edge of the slit in S_k to the upper edge of the slit in S_{k+1} and the upper edge of the slit in S_k to the lower edge of the slit in S_{k-1}, for each $n \in \mathbb{Z}$. Observe that S looks like an infinitely high 'multi-storey car-park': as you drive round and round the origin in a positive direction you pass to higher and higher floors.

This time the Riemann surface S is a copy of the whole complex plane $\mathbb{C}$ and the projection map $p : S \to \mathbb{C} \setminus \{0\}$ is the map $z \to e^z$. Notice that there are no branch points in this example as we cannot extend p to a map which has 0 or ∞ in its image.

3. Schwarz's Reflection Principle

This is a method of analytic continuation which can be very useful when we have a holomorphic function which we know up to and including some smooth boundary, for instance on the upper half plane and its boundary the real axis, and we want to continue the function across the boundary.

3.1 Proposition (Schwarz reflection principle) *Let U be a connected open set in $\mathbb{C}$ which is symmetric about the real axis, that is, $z \in U \Leftrightarrow \bar{z} \in U$. Let $U^+ = \{z \in U : Im(z) > 0\}$, $U^- = \{z \in U : Im(z) < 0\}$ and $U^0 = \{z \in U : Im(z) = 0\}$. Suppose $f : U^+ \cup U^0 \to \mathbb{C}$ is continuous, that $f(z)$ is real for all $z \in U^0$, and that f is holomorphic on U^+. Then f can be extended analytically to U^- by setting $f(z) = \overline{f(\bar{z})}$ for each $z \in U^-$.*

Proof *(Because of lack of time this proof was not be covered in lectures, and is therefore non-examinable.)* First we observe that for $z \in U^0$ the two definitions agree, that is $\overline{f(\bar{z})} = f(z)$. So if we write $F(z)$ for $f(z)$ when $z \in U^+ \cup U^0$ and for $\overline{f(\bar{z})}$ when $z \in U^-$, the function F is continuous on the whole of U and holomorphic in $U^+ \cup U^-$.

Next we note that for any simple closed path γ in U the integral of F around γ is zero. This is clear by Cauchy's Theorem for closed paths entirely inside U^+ or entirely inside U^-. For a path that contains points of both (and of U^0) we have to work a little harder: we divide the region $I(\gamma^*)$ inside the path into the parts above and below the real axis and we consider the two closed paths γ^+ an γ^- running round the edge of the boundary of each. Clearly it will suffice to show that the integral of F around each is zero, since the parts that run along the real axis cancel out. But the integral of F around γ^+ is zero by Cauchy's Theorem because we can approximate γ^+ by a nearby path inside U^+ and F is holomorphic there, and similarly the integral of F around γ^- is zero.

Finally the fact that F is continuous on U and has zero integral around every simple closed path implies that F has an antiderivative G on U. But a holomorphic function can be differentiated twice (in fact arbitrarily many times) and thus F (which is G') is holomorphic on U. $\square$

4. Elliptic Functions

Recall that a meromorphic function f on $\mathbb{C}$ is a function which is holomorphic except for isolated singularities, all of which are poles. We can define the value of f at a pole to be ∞ and f may have infinitely many such singularities (in which case the singularity of f at $z = \infty$ is not isolated).

Definition A meromorphic function f on $\mathbb{C}$ is said to be *doubly periodic* if there exist complex numbers ω_1 and ω_2 which are linearly independent over $\mathbb{R}$ (in other words $\omega_2/\omega_1 \notin \mathbb{R}$), such that

$$f(z+\omega_1) = f(z) = f(z+\omega_2) \ \forall z \in \mathbb{C}$$

We shall see that every doubly-periodic holomorphic function is bounded on $\mathbb{C}$, and hence constant (by Liouville's Theorem). This is why we have to consider meromorphic functions to find non-constant examples. First we observe that it follows at once from the definition that if f is doubly-periodic then

$$f(z+m\omega_1+n\omega_2) = f(z) \ \forall m,n \in \mathbb{Z}$$

.

Definition Let ω_1 and ω_2 be non-zero complex numbers which are linearly indepent over $\mathbb{R}$. The *lattice* generated by ω_1 and ω_2 is the set

$$\Omega = \{m\omega_1 + n\omega_2 : m,n \in \mathbb{Z}\} \subset \mathbb{C}$$

We say that a meromorphic function f is *an elliptic function with repect to the lattice* Ω if

$$f(z+\omega) = f(z) \ \forall \omega \in \Omega \text{ and } \forall z \in \mathbb{C}$$

Note that a given lattice can have many different generating sets. For example ω_1 and $\omega_1+\omega_2$ generate the same lattice as ω_1 and ω_2. But once we having a chosen a generating pair ω_1 and ω_2 we can define a *fundamental domain* P for the doubly periodic function f by

$$P = \{s\omega_1 + t\omega_2 : s,t \in \mathbb{R}, 0 \le s \le 1, 0 \le t \le 1\}$$

It is easy to see that copies $m\omega_1 + n\omega_2 + P$ of the parallelogram P, one for each $m,n \in \mathbb{Z}$, fill up the whole plane meeting neatly at their edges without overlapping. Thus if f is elliptic with respect to Ω then for every $z \in \mathbb{C}$ there exist a $z' \in P$ with $f(z) = f(z')$ (indeed there are two such z' if z is on the edge of a copy of P, and four such z' if it is at a vertex). Notice also that instead of P we can choose any $t \in \mathbb{C}$ and then:

$$P' = P + t = \{z + t : z \in P\}$$

is also a fundamental domain for the lattice Ω. If, for example, f has a pole on one of the sides of P, then replacing P by $P' = P + t$ for some small $t \in \mathbb{C}$ gives us a fundamental domain such that f no longer has a pole on the boundary.

4.1 Lemma *Every double-periodic holomorphic function is constant.*

Proof The fundamental domain P is closed and bounded. So $f(P)$ is bounded. Hence f is bounded on $\mathbb{C}$, since for any $z \in \mathbb{C}$, $f(z) = f(z')$ for some $z' \in P$. Hence, by Liouville's Theorem, f is constant. □

So we look at meromorphic functions. We can think of a doubly-periodic meromorphic function $f : \mathbb{C} \to \hat{\mathbb{C}}$ as a function defined on the torus obtained from the parallelogram P by identifying pairs of opposite edges. Thus f can be thought of as a complex differentiable function from a torus to the Riemann sphere (much as we thought of a rational map as a complex differentiable map from the Riemann sphere to itself). Thinking of f this way, and assuming we have chosen a fundamental domain P for f which doesn't have any poles or zeros of f lying on its boundary ∂P, we define the *order* of f to be $ord(f) =$ number of solutions of $f(z) = 0$ in P (counted with multiplicity). It can be shown that this is the same as the number of *poles* of f in P (counted with multiplicity), or indeed the number of solutions of $f(z) = c$ in P for any fixed constant $c \in \mathbb{C}$

4.2 Proposition

(i) Every elliptic function of order 0 *is constant.*

(ii) There are no elliptic functions of order 1.

Proof

We assume the fundamental parallelogram P chosen so as not to have any poles on its edges. Part (i) is just Lemma 4.1 (since if f has no poles then it is holomorphic). For part(ii) observe that if we integrate f once around the boundary of P we get zero, since the integrals along opposite sides cancel. So by the Residue Theorem the sum of the residues of f inside P is zero. Hence there must be at least two simple poles, or at least one double pole or pole of higher order. □

It is not too hard to construct examples of elliptic functions:

4.3 Proposition *For each* $N \geq 3$

$$F_N(z) = \sum_{\omega \in \Omega} \frac{1}{(z-\omega)^N}$$

is elliptic of order N *with respect to* Ω

Proof First we have to show that the sum is convergent so that it defines a genuine function F_N. We omit the details of this part of the proof, but once we know F_N is a well defined (meromorphic) function it is obvious from the expression for F_N that it is doubly periodic and of order N. □

What about elliptic functions of order 2? The sum for F_2 does not converge, but fortuantely it can be made to converge by a slight modification.

Definition The *Weierstrass p-function* is defined to be

$$\mathcal{P}(z) = \frac{1}{z^2} + \sum_{\omega\in\Omega}{}' \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)$$

where $\sum'$ denotes the sum taken over all non-zero ω in Ω.

Note that the derivative $\mathcal{P}'$ of $\mathcal{P}$ is elliptic with respect to Ω, since $\mathcal{P}' = -2F_3$. This fact can be used to prove that $\mathcal{P}$ is itself elliptic with respect to Ω. For details see, for example, Chapter 3 of *Jones and Singerman 'Complex functions:an algebraic and geometric viewpoint', Cambridge University Press 1987.*

It can be shown that every elliptic (with respect to Ω) function f can be expressed in the form

$$f = R_1(\mathcal{P}) + \mathcal{P}' R_2(\mathcal{P})$$

where R_1 and R_2 are rational functions, and that these elliptic functions form a *field*. This is the just the start of a whole area of complex analysis. For more details, and also for the connection with real 'elliptic integrals', see Jones and Singerman or any of the classic text books on complex functions written in the first half of the 20th century, for example: *Whittaker and Watson: 'Modern Analysis', Cambridge University Press 1920.* 'Elliptic integrals' are real integrals of the type that one finds when evaluating the perimeter length of an ellipse, or the period of a pendulum. For example the perimeter length of the ellipse $(x^2/a^2) + (y^2/b^2) = 1$ is of the form

$$\int \sqrt{\left(\frac{a^2 - ex^2}{a^2 - x^2} \right)} dx$$

where $e = 1 - (b^2/a^2)$. The Riemann surface for the multi-valued function

$$g(z) = \sqrt{\left(\frac{a^2 - ez^2}{a^2 - z^2} \right)}$$

is a torus $\mathbb{T}$, and the inverse function to g is an elliptic function

$$\mathbb{C} \to \mathbb{C}/\Omega = \mathbb{T} \to \hat{\mathbb{C}}$$

in the sense that we have described in this section.

5. Picard's Theorem

5.1 Picard's Theorem *If $f : \mathbb{C} \to \mathbb{C}$ is an entire function such that there exist at least two points in $\mathbb{C}$ not in the image of f, then f is constant.*

Comment This is much stronger than Liouville's Theorem! It is generally known as 'Picard's Little Theorem'. A related result (which we shall not prove), known as 'Picard's Big Theorem', states that in every neighbourhood of an isolated essential singularity a function takes *every* value in $\mathbb{C}$, except possibly one, infinitely many times.

Proof of Picard's (Little) Theorem *(This proof is not for examination.)*

Without loss of generality we can assume the missing values are 0 and 1 (since we can compose f with any automorphism $z \to az + b$ of $\mathbb{C}$).

Consider the region Δ of the complex upper half-plane H bounded by the vertical lines $Re(z) = 0$ and $Re(z) = 1$ and the semicircle which has centre on the real axis at $z = 1/2$ and radius $1/2$. By the Riemann mapping theorem there exists a conformal bijection $\lambda : \Delta \to H$. This bijection will send the boundary of Δ continuously to $\mathbb{R} \cup \{\infty\}$ and we can choose it to have $\lambda(\infty) = 0$, $\lambda(0) = 1$ and $\lambda(1) = \infty$ (here we are using a form of the Riemann Mapping Theorem that says that if a simply connected region U has boundary a Jordan curve, then the Riemann mapping extends to a continuous map on the boundary). By the Schwarz reflection principle we can extend λ across each of its three boundaries and map the three adjacent 'tiles' (copies of Δ) to the lower half plane. Continuing to reflect we can extend λ to a map sending the whole of H (see pictures of 'tiling' in lecture) onto $\mathbb{C} - \{0, 1\}$. Let ν denote the inverse of λ mapping the upper half plane back to the original tile Δ.

Now consider our entire map $f : \mathbb{C} \to \mathbb{C} - \{0, 1\}$. We can assume the image of f meets the upper half plane (if not, we can compose f with a map which switches the half-planes and interchanges 0 and 1). Thus ν is defined at some point $f(z_0)$ and we can analytically continue $\nu \circ f$ in all directions from z_0 until we have $\nu \circ f$ defined on the whole of $\mathbb{C}$. This is possible because $\mathbb{C}$ is simply-connected, so whatever paths we use to define the analytic continuation will give the same values. But now $\nu \circ f$ is an entire map with image contained in H, and by composing with a Möbius transformation ϕ we can send H to the unit disc D. Now $\phi \circ \nu \circ f$ is bounded, hence (by Liouville) constant. So $\nu \circ f$ is constant. So f is constant $\Box$.

Comment In fact there is a formula for λ. Given any point τ in the upper half plane, let Ω be the lattice generated by $\omega_1 = 1$ and $\omega_2 = \tau$. The value of the Weierstrass p-function $\mathcal{P}$ (with respect to Ω) evaluated at 0 is ∞. Let e_1, e_2 and e_3 be the values of $\mathcal{P}$ evaluated at $\omega_1/2$,$\omega_2/2$ and $(\omega_1+\omega_2)/2$ respectively. Then

$$\lambda(\tau) = [\infty, e_1; e_2, e_3] = \frac{(\infty - e_2)(e_1 - e_3)}{(\infty - e_3)(e_1 - e_2)} = \frac{e_1 - e_3}{e_1 - e_2}$$

Even more explicitly, it turns out that

$$\lambda(\tau) = 16q \prod_{n=1}^{\infty} \left(\frac{1 + q^{2n}}{1 + q^{2n-1}} \right)^8 \text{ where } q = e^{2\pi i \tau}$$

This too, is the beginning of a whole fascinating area of mathematics. The function λ, and its close relation the *modular function* J, have applications in many different areas of recent research, from the proof of Fermat's Theorem to the study of large finite simple groups, and to theoretical physics.