E.J. Beggs and S. Majid

Quantum Riemannian Geometry

– Monograph –

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Preface

There are many ways to extend the ideas of classical differential geometry to a noncommutative world. Our view is that there is no clear answer as to which of these is correct, given that many of them have their own rich pure mathematical theory. However, if we were to think about what *should* ideally be noncommutative differential geometry, we might identify the following considerations. There should be a broad collection of examples of interest across different branches of mathematics. Noncommutative geometry should reduce to classical geometry as a special case, though some aspects of the theory may become trivial in the classical case. Most constructions in classical differential geometry should have noncommutative geometry analogues. Last but not least, as geometry originated as a practical subject, there should be applications, which historically has meant applications in physics and applied mathematics. With this in mind, it has been one of the principles of this book to include both the pure mathematical background and applications, from categories to cosmology and from modules to Minkowski space. We shall try to explain both aspects from a relatively elementary starting point. Much of the work will be taken from our own research papers, which have been inspired by the above point of view, particularly from our experience with quantum groups (a 'quantum groups approach to noncommutative geometry'), although not limited to this. In short, we will provide one particularly constructive and computable style of noncommutative geometry, but we will also include links to other approaches, where possible.

Noncommutative geometry, one way or another, arose from experience with quantum theory. By the 1920s, Dirac has already speculated about geometry with noncommuting x, p coordinates, and the great theorems of Gel'fand and Naimark for C^*-algebras and the GNS construction of Hilbert space representations of noncommutative C^*-algebras were driven by mathematical physics in the context of quantum theory. K-theory, universal differentials, vector bundles (as projective modules) and connections for noncommutative algebras, as well as Hochschild and cyclic cohomology, were a natural progression in this direction, culminating in Connes' famous notion of a spectral triple as an abstract 'Dirac operator' in the early 1980s. Meanwhile, and quite separately in the mid 1980s, large classes of noncommutative algebras appeared as part of the 'quantum groups revolution'. These objects arose on the one hand in the context of generalised symmetries in quantum integrable systems (the Drinfeld–Jimbo quantum groups $U_q(\mathfrak{g})$) and on the other hand from ideas of quantum Born reciprocity or observable-state duality in quantum gravity (the bicrossproduct quantum groups in the PhD thesis of one of the authors). These remain two main classes of quantum groups, with the first class of direct interest in many branches of mathematics, including knot theory and category theory, and the second class of particular interest as Poincaré quantum symmetries of noncommutative or 'quantum' spacetime. In either case, such quantum groups provide key examples of noncommutative algebras with a clear geometric significance which therefore *should* be foundational examples of noncommutative geometry, just as classical Lie groups were of classical differential geometry.

An outline of the book is as follows. In Chapter 1, we cover the basic theory of algebras equipped with differential structure expressed as exterior algebras of differential forms. The chapter also introduces the notion of a quantum metric and, in the case of an 'inner' calculus, an induced quantum Laplacian, as elementary layers of the theory that depend only on the differential structure. We also introduce many of the basic examples which will be further developed in subsequent chapters as we build up the different layers of noncommutative geometry in our approach. The last sections provide some applications so that, all together, Chapter 1 could be read as a self-contained first introduction to noncommutative differential geometry as we see it.

Chapter 2 provides a condensed introduction to Hopf algebras or 'quantum groups' and their representations as monoidal or (in the (co)quasitriangular case) braided monoidal categories. Much more can be found in several textbooks on quantum groups, including a text by one of the present authors, from which we borrow our notation. The chapter then develops the theory of differential structures and exterior algebras on Hopf algebras, including a braided antisymmetric algebra approach to the Woronowicz construction, and quantum/braided Lie algebras on quantum groups. The last section of Chapter 2 introduces the notion of a bar category, which is needed to formulate complex conjugation and $*$-operations in a more categorical way.

Chapter 3 introduces the basic notions of a vector bundle (as a projective module of sections) over an algebra and of a connection on such a vector bundle over a differential algebra. We also cover elements of cyclic cohomology and K-theory, including the famous Chern–Connes pairing between the two. Particularly important for in this chapter is the idea that 1-forms in noncommutative geometry are bimodules (one can multiply by the algebra from either side) with the result that one should ask for a connection to obey a Leibniz type rule from both sides. If the Leibniz rule on one side has a standard form then the other side will need to refer to a 'generalised braiding' σ and when this exists we say that we have a 'bimodule connection'. Chapters 1–3 constitute the basic foundation of the book.

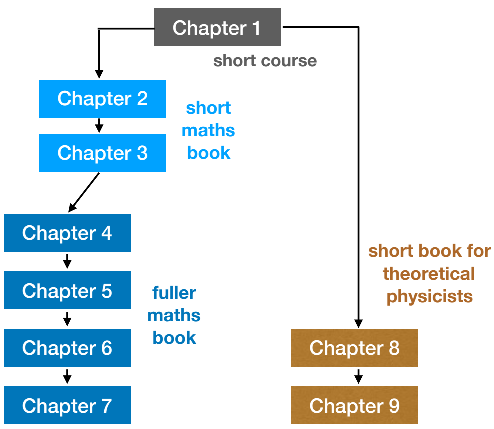
Chapter 4 proceeds to harder results about the curvature of connections such as the Bianchi identities and characteristic classes. The chapter also includes a study of the category of modules equipped with flat connections, which could be seen as playing the role of sheaves over the algebra. Various constructions with sheaves and cohomology are given which are analogous to classical constructions. Some applications of spectral sequences are given, including the Leray–Serre spectral sequence of a fibration, for a differential definition of noncommutative fibration. We also look at positive maps and Hilbert C^*-modules, and extend the idea of bimodules as generalised morphisms between algebras to a differential setting using bimodule connections on B-A bimodules for different algebras A, B.

Chapter 5 looks at quantum principal bundles, where the fibre is now a Hopf algebra or quantum group. This is a theory that goes back to Brzeziński and one of the authors, while in the case of the universal calculus it is also known in algebra as a Hopf–Galois extension. We explain the link with Galois theory and provide the general theory of associated bundles and induced bimodule connections on them when the principal bundle has a connection-form. We also study differential fibrations more generally. The last section of the chapter is an application to quantum framed spaces, i.e., differential algebras appearing as the base of a quantum principal bundle and data such that the space Ω^1 of differential 1-forms is an associated bundle. The chapter includes bundles and q-monopole bimodule connections over the standard q-deformed sphere, among other examples, which combines with our quantum framing theory to provide our first encounter with a 'quantum Levi-Civita connection' as a torsion free metric compatible bimodule connection on Ω^1. The framing approach also allows us to solve for a connection on the quantum group $\mathbb{C}_q[SU_2]$ with its 4D calculus, which turns out to be 'weak quantum Levi-Civita' in the sense of torsion free and cotorsion free.

Chapter 6 develops the theory of vector fields and the algebra of differential operators $\mathcal{D}_A$ associated to an algebra with differential calculus. As in the classical theory, modules over the algebra $\mathcal{D}_A$ are the same as A-modules with flat connection. The algebra $\mathcal{D}_A$ has extra algebraic structure, which can be expressed by saying that it is a braided-commutative algebra in the centre of the monoidal category of bimodules with bimodule connections. $\mathcal{D}_A$ for A the algebra of 2×2 matrices with a natural differential structure turns out to be generated by A and a single quantised fermion. We also introduce $T\mathfrak{X}_\bullet$ with modules the same as A-modules with connection, not necessarily flat. Chapter 7 introduces complex structures in the same manner as classical complex manifold theory. This involves a bigrading of the exterior algebra to give a double complex and allows the definition of holomorphic modules along with implications for cohomology theories. These are shorter chapters and complete the more advanced mathematical content of the book. A mathematically-minded reader should be able to use the book as Chapters 1–3, or Chapters 1–7, depending on how far one wishes to travel.

Chapter 8 brings together previously encountered notions of Riemannian and other structures from Chapters 3,4,5 into a self-contained account of noncommutative Riemannian geometry over an algebra equipped with differential structure and choice of metric. Finding an associated torsion free and metric compatible bimodule connection (or quantum Levi-Civita connection) on Ω^1 here is a well-posed non-linear problem and the chapter shows how it can be solved directly in a variety of models. The chapter also includes a section on Connes' spectral triples and how they can sometimes arise in a weakened form in our constructive approach, as a Dirac operator built along geometric lines from a connection and a Clifford structure. Examples include the q-sphere and the algebra of 2×2 matrices. Other topics include a wave-operator approach to quantum Riemannian geometry that short-cuts the layer-by-layer treatment by directly formulating the quantum Laplacian as a partial derivative of an extended calculus. The chapter also includes a slightly different theory of hermitian-metric compatible connections and Chern connections in noncommutative geometry.

Chapter 9 concludes the book with applications specifically to quantum spacetime. Unlike quantum phase space in quantum mechanics, there is as yet no physical evidence that the coordinates of spacetime themselves form a noncommutative differential algebra. Indeed this effect, if observed, would be a discovery on a par with the discovery of gravity itself (one can call it 'co-gravity' as it is in some sense dual to gravity). By now, this striking possibility is widely accepted in quantum gravity circles as a plausible better-than-classical model of spacetime that takes into account Planck scale or quantum gravity corrections. At the same time, physicists and applied mathematician readers should consider that just as geometry has many roles beyond gravity, so does quantum or noncommutative geometry, for example to the geometry of discrete systems as well as potentially to actual quantum-mechanics. We have written the book in such a way that such readers should be able to focus on Chapters 1,8,9, with the intervening chapters dipped into as needed for further details of the underlying mathematics.



There are inevitably other topics which we have not had room to include, and many of these are of great importance and could form the basis of a future volume. We included some of these topics, at an introductory level only, in the final sections of the book. Thus, we briefly treat the semiclassical behaviour within deformation theory or 'Poisson–Riemannian geometry', a paradigm which includes first-order quantum gravity effects but bears the same relation to quantum gravity as does classical mechanics to quantum mechanics. We have also treated only briefly the construction of examples by 'functorial twisting', which is particularly interesting in the cochain case where the data on the symmetry quantum group is not a cocycle and the exterior algebra becomes nonassociative. This is a big topic in its own right and also closely tied to the semiclassical theory, where it corresponds to curvature of the Poisson-compatible connection. The reader will also notice that we have only briefly presented some areas from the C^*-algebras side of noncommutative geometry and in this case there are more comprehensive works elsewhere.

Outline of notations and examples

Here we outline some of the notational conventions for reference and orientation. Most constructions work over a general field $\Bbbk$ but most of the time one can keep in mind $\mathbb{R}$ or $\mathbb{C}$. We use a standard convention for expressing a tensor product of matrices as a single matrix: for the example of a 3×3 matrix A,

$$A \otimes B = \begin{pmatrix} A_{11}\,B & A_{12}\,B & A_{13}\,B \\ A_{21}\,B & A_{22}\,B & A_{23}\,B \\ A_{31}\,B & A_{32}\,B & A_{33}\,B \end{pmatrix}$$

which is equivalent to $A^i{}_j B^k{}_l$ written as a matrix with rows ik and columns jl taken in order $11, 12, \cdots, 33$. A dot is typically used to emphasise a product, often to separate different types of elements. For example $\mathrm{d}a.b$ may be used to indicate $(\mathrm{d}a)b$ (rather than $\mathrm{d}(ab)$) or $e.a$ for the product of a right module by an element of the algebra. Left and right actions that are different from default module or bimodule structures will typically be denoted by $\triangleright, \triangleleft$ respectively. Composition of maps may for emphasis be written as $f \circ g$, meaning to apply g first.

The symbol δ_x is used for the function taking the value 1 at x and zero elsewhere. We also use the Kronecker delta $\delta_{x,y}$ with value 1 if $x = y$ and zero otherwise. We use $C(X)$ for the continuous complex-valued (unless otherwise specified) functions on a topological space X, $C^\infty(X)$ for the smooth (differentiable arbitrarily many times) continuous and typically real-valued functions on a smooth manifold X, $\Bbbk[X]$ for the coordinate algebra of an algebraic variety X and $\Bbbk_q[X]$ its q-deformation. For $\mathbb{C}_q[S^1]$, the q refers to the differential calculus as the algebra itself in this case is not deformed. (We also write $\mathbb{C}_q\mathbb{Z}$ for the same algebra but in the different context of a group Hopf algebra with coquasitriangular structure involving q.) We similarly write $\mathbb{C}_\theta[\mathbb{T}^2]$ for the algebraic noncommutative torus rather than the more common notation $\mathbb{T}^2_\theta$. The polynomial algebra in n variables is $\Bbbk[x_1, \cdots, x_n]$ and the free associative algebra in n variables is $\Bbbk\langle x_1, \cdots, x_n\rangle$. Angular brackets also denote an ideal generated in an algebra and in other contexts a duality pairing or a hermitian inner product.

We use $[x, y] = xy - yx$ for the commutator of two elements in an algebra, $\{x, y\} = xy + yx$ for the anticommutator and $[x, y\} = xy - (-1)^{|x|\,|y|}yx$ for the graded commutator when this applies. Where there is no confusion we also use $[\ ,\ ]$ for Lie brackets and quantum and braided Lie brackets. In other contexts, $\{\ ,\ \}$ could denote a Poisson bracket or indicate a list. We use distinct notations

$$[n]_q = \frac{1-q^n}{1-q}, \quad (n)_q = \frac{q^n - q^{-n}}{q - q^{-1}}$$

for two sorts of q-integer for any determinate or indeterminate q.

We most often use Ω^n for the n-forms on an algebra $A = \Omega^0$, with the entire collection of forms simply denoted by Ω and forming a differential graded algebra (or DGA). When we have different algebras of interest, we may write Ω_A, Ω_B etc., to avoid confusion. A roman d denotes the differential. Clearly, each Ω^n is a bimodule over A as we can multiply by degree 0 from either side. In general, for modules over an algebra A, we use ${}_A\mathrm{Hom}(E, F)$ for left A-linear maps from the left module E to the left module F. Correspondingly, $\mathrm{Hom}_A(E, F)$ is the right A-linear maps from the right module E to the right module F, and ${}_A\mathrm{Hom}_A(E, F)$ is the A-bilinear maps between bimodules. We write 'fgp' for 'finitely generated projective' as a module over an algebra. We use id for the identity map.

As noncommutative geometry often relates to complex-valued objects, the $*$ operation plays a prominent role and follows the usual rules for a complex-linear involution when extended to other contexts. It should not be confused with the Hodge operation, for which we reserve $\circledast$. Sometimes we will want to be extremely clear and use a more formal language (of bar categories). Thus, the conjugate $\overline{E}$ of a bimodule E over a $*$-algebra A has elements $\overline{e} \in \overline{E}$ denoting the same element as $e \in E$ but viewed in the conjugate module. The bimodule actions on the conjugate are $a.\overline{e} = \overline{e.a^*}$ and $\overline{e}.a = \overline{a^*.e}$ for $a \in A$. For a second bimodule F, there is a map $\Upsilon : \overline{E \otimes_A F} \to \overline{F} \otimes_A \overline{E}$ given by $\Upsilon(\overline{e \otimes f}) = \overline{f} \otimes \overline{e}$. The notation $e \mapsto e^*$ is used for the usual conjugate linear $*$ map, but we also use the more categorical notation $\star : E \to \overline{E}$ as the linear map (typically also a bimodule map) defined by $\star(e) = \overline{e^*}$. Round brackets $(\ ,\ )$ are often used for inner products linear in both factors in contrast to angular brackets $\langle\ ,\ \rangle$ for hermitian inner products. The latter could be conjugate linear on the right, or in other cases conjugate linear on the left. Often an explicit conjugate is used, for example $\langle u, \overline{v}\rangle$.

Elsewhere in algebra, the map 'flip' swaps tensor factors. A counit or 'augmentation' on an algebra means a character $A \to \Bbbk$ and when this is fixed, A^+ denotes its kernel or augmentation ideal. Left and right coactions are denoted by

$$\Delta_L v = v_{(\bar{1})} \otimes v_{(\bar{\infty})}, \quad \Delta_R v = v_{(\bar{0})} \otimes v_{(\bar{1})}.$$

Categories of modules over an algebra are denoted by ${}_A\mathcal{M}$ for left modules, $\mathcal{M}_A$ for right modules and ${}_A\mathcal{M}_A$ for bimodules. When H is a Hopf algebra, or at least a coalgebra, we use ${}^H\mathcal{M}, \mathcal{M}^H, {}^H\mathcal{M}^H$ to denote left, right and bicomodules. We also have crossed versions where the different structures do not commute, for example $\mathcal{M}_H^H$ denotes right crossed H-modules where H both acts and coacts and there is a compatibility between these which is a linearised 'Hopf' version of the crossed G-set condition of J.C. Whitehead in algebraic topology (these objects are also called right Radford–Drinfeld–Yetter modules in the literature).

We will typically use G for a group, $\mathfrak{g}$ for a Lie algebra, not to be confused with g for a metric. We typically use ∇ for left-covariant derivatives or connections (we use these terms interchangeably) and $\tilde{\nabla}$ for right-covariant derivatives. These are frequently given subscripts such as ∇_E to indicate which module they act on. In addition, D is often used for a connection arising from the framing construction and $\heartsuit$ for a background connection on Ω^1 or on vector fields $\mathfrak{X}$ in the theory of differential operators. We use ω for principal or 'spin' connection forms on a quantum principal bundle and ϖ for the Maurer–Cartan form on a group or quantum group (which can be seen as a flat connection form). $Z(A)$ is the centre of an algebra A and $\mathcal{Z}(\mathcal{C})$ of a monoidal category $\mathcal{C}$ (as introduced independently by Drinfeld and by one of the authors as a dual $\mathcal{C}^\circ$).

We have adopted a particular compromise for the notation of indices on differential forms. To be consistent with the geometry literature, local coordinates $\{x^\mu\}$ have an upper index hence so do differential forms $\mathrm{d}x^\mu$. In Chapters 8 and 9 an 'n-bein' basis of differential forms $\{e^i\}$, when it exists, generally then gets an upper index for consistency. We also adopt a geometric normalisation of the exterior derivative d so as to have a classical limit as a deformation parameter $q \to 1$ or $\lambda \to 0$. However, in the less specialised earlier chapters we do *not* adopt such strict conventions and typically keep a basis of 1-forms as $\{e_i\}$ and keep the canonical normalisations for d intrinsic to the relevant noncommutative constructions, notably in Chapter 2.

In Table 0.1, we give the occurrences of the most common examples used to illustrate various different aspects of the theory. The references are to example or proposition number, or by chapter and section number as indicated by §.

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Mumbles, Swansea, *Edwin James Beggs*
Hampstead, London, *Shahn Majid*
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Algebra	Ω^1, d	metric	Laplacian	Ω max	de Rham H_{dR}	Ω canonical	integral	projection	connection	bimod conn.	$*$ conn.	line module	regularity	coaction	fibration	principal bun.	vector fields	complex str.	Levi-Civita	spectral triple
$C^\infty(M)$			1.23	1.35											4.60		6.7			
$M_2(\mathbb{C})$	1.8	1.20	1.20	1.37	1.38				4.89	4.89			4.89				§6.5.3	7.6	8.13	8.46
$\mathbb{C}_\lambda[\mathbb{R}]$	1.10			1.34	1.34															
$\mathbb{C}_q[S^1]$	1.11	8.5		1.34	1.34			4.22				4.22	4.22				§6.5.2		8.5	
$\Bbbk_\Theta[V]$	1.14	1.21																		
$\Bbbk(X)$	§1.4	1.28	1.28	1.40												5.44	6.3			
$\mathbb{C}_\theta[\mathbb{T}^2]$	1.36			1.36	1.36			3.17	3.30								6.13	7.11	8.16	
$\mathbb{C}_{q,\theta}[\mathbb{T}^2]$	E1.5						E3.6							E4.3		E5.10				
$U(\mathfrak{g})$	§1.6.1	1.43																		
$U(su_2)$	1.45	1.45	1.45																8.15	8.50
$u_q(b_+)$	E2.4				E4.5	E4.5	E2.3		E4.9											
$\mathbb{C}_\lambda[S^2]$	1.46							1.46												
$\mathbb{C}S_3$	1.48	1.48	1.50		1.50				4.48					5.43		5.43	E6.2			8.49
$\Bbbk(G)$	§1.7	1.59	1.59	1.53		2.29	2.20			3.75	3.87		4.21			5.49			8.17	E8.10
$\Bbbk(S_3)$	1.60	1.60	1.60		1.60	1.60				3.76	3.88		4.18		5.64		6.30			
$U_q(sl_2)$	2.11																			
$\mathbb{C}_q[\mathbb{C}^2]$	2.79					2.79								4.33				§7.4.2		
$\mathbb{C}_q^{3D}[SU_2]$	2.32				4.68		2.21			3.77	3.89				5.63	5.51	6.4			
$\mathbb{C}_q^{4D}[SU_2]$	2.13	2.60	2.62		2.77	2.59													5.85	8.51
$\mathbb{C}_q[S^2]$	2.35	2.36			4.34		4.36	3.15	3.27	4.24		3.99			5.63		6.14	7.12	5.80	8.47
$\mathbb{C}_q[D]$	3.40			3.40	4.37		4.37	3.100				3.100		4.31		5.24	E6.3	8.57	E8.7	8.48
$\mathbb{C}(\mathbb{Z}_N)$	3.86									3.86	3.86					5.49		E7.1		
$\mathbb{C}PSL_2(\mathbb{Z})$	4.19					4.19							4.19							
$\mathbb{C}_q[SO_3]$	4.23								4.23	4.23		4.23	4.23							
$\mathbb{C}$Hg	4.62			4.62	4.67										E5.5	5.12				
$\mathbb{C}$[Klein]	5.45				5.45							5.45		5.11	5.45	5.11				

Table 0.1 Some examples by statement number, chapter-section number § or exercise number E

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