MTH4100 Calculus I

Lecture notes for Week 12

Thomas' Calculus, Sections 8.1 to 8.3, 8.8 and 10.5

Rainer Klages

School of Mathematical Sciences
Queen Mary, University of London

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Techniques of integration

- Basic properties (Thomas' Calculus, Chapter 5)
- Rules (substitution, integration by parts - see today)
- Basic formulas, integration tables (Thomas' Calculus, pages T1-T6)
- Procedures to simplify integrals (bag of tricks, methods)

This needs practice, practice, practice, ...:

$\boxed{\text{Last exercise class and } \textit{voluntary online exercises}}$

TABLE 8.1 Basic integration formulas

1. $\int du = u + C$
2. $\int k\,du = ku + C$ (any number k)
3. $\int (du + dv) = \int du + \int dv$
4. $\int u^n\,du = \frac{u^{n+1}}{n+1} + C$ $(n \neq -1)$
5. $\int \frac{du}{u} = \ln|u| + C$
6. $\int \sin u\,du = -\cos u + C$
7. $\int \cos u\,du = \sin u + C$
8. $\int \sec^2 u\,du = \tan u + C$
9. $\int \csc^2 u\,du = -\cot u + C$
10. $\int \sec u \tan u\,du = \sec u + C$
11. $\int \csc u \cot u\,du = -\csc u + C$
12. $\int \tan u\,du = -\ln|\cos u| + C = \ln|\sec u| + C$
13. $\int \cot u\,du = \ln|\sin u| + C = -\ln|\csc u| + C$
14. $\int e^u\,du = e^u + C$
15. $\int a^u\,du = \frac{a^u}{\ln a} + C$ $(a > 0, a \neq 1)$
16. $\int \sinh u\,du = \cosh u + C$
17. $\int \cosh u\,du = \sinh u + C$
18. $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$
19. $\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$
20. $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C$
21. $\int \frac{du}{\sqrt{a^2 + u^2}} = \sinh^{-1}\left(\frac{u}{a}\right) + C$ $(a > 0)$
22. $\int \frac{du}{\sqrt{u^2 - a^2}} = \cosh^{-1}\left(\frac{u}{a}\right) + C$ $(u > a > 0)$

Integration tricks:

Procedures for Matching Integrals to Basic Formulas

PROCEDURE	EXAMPLE
Making a simplifying substitution	$\frac{2x-9}{\sqrt{x^2-9x+1}}\,dx = \frac{du}{\sqrt{u}}$
Completing the square	$\sqrt{8x-x^2} = \sqrt{16-(x-4)^2}$
Using a trigonometric identity	$(\sec x + \tan x)^2 = \sec^2 x + 2\sec x \tan x + \tan^2 x$ $= \sec^2 x + 2\sec x\tan x + (\sec^2 x - 1)$ $= 2\sec^2 x + 2\sec x \tan x - 1$
Eliminating a square root	$\sqrt{1+\cos 4x} = \sqrt{2\cos^2 2x} = \sqrt{2}\,\lvert\cos 2x\rvert$
Reducing an improper fraction	$\frac{3x^2-7x}{3x+2} = x - 3 + \frac{6}{3x+2}$
Separating a fraction	$\frac{3x+2}{\sqrt{1-x^2}} = \frac{3x}{\sqrt{1-x^2}} + \frac{2}{\sqrt{1-x^2}}$
Multiplying by a form of 1	$\sec x = \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x}$ $= \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}$

see book p.554 to p.557 and exercise sheet 10 for further examples

Integration by parts

differentiation $\longleftrightarrow$ integration:

- chain rule $\longleftrightarrow$ substitution

$$\int f(g(x))g'(x)dx = \int f(u)du\,, \quad u = g(x)$$

- product rule $\longleftrightarrow$?

$$\frac{d}{dx}\left(f(x)g(x)\right) = f'(x)g(x) + f(x)g'(x)$$

Integrate:

$$\int \frac{d}{dx}\left(f(x)g(x)\right)dx = \int \left(f'(x)g(x) + f(x)g'(x)\right)dx$$

Therefore,

$$f(x)g(x) = \int f'(x)g(x)dx + \int f(x)g'(x)dx$$

(in this case we neglect the integration constant - it is implicitly contained on the rhs)

leading to

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx \tag{1}$$

abbreviated:

Integration by Parts Formula

$$\int u\,dv = uv - \int v\,du \tag{2}$$

Integration by Parts Formula for Definite Integrals

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big]_a^b - \int_a^b f'(x)g(x)\,dx \tag{3}$$

example: Evaluate

$$\int x\cos x\,dx \ :$$

Choose

$$u = x\ , \quad dv = \cos x\,dx\ ,$$

then

$$du = dx\ , \quad v = \sin x \text{ neglect any constant}$$

gives, according to formula,

$$\begin{aligned}\int x\cos x\,dx &= x\sin x - \int \sin x\,dx \\ &= x\sin x + \cos x + C\end{aligned}$$

(do *not* forget the constant here!)

Explore four choices of u and dv for

$$\int x\cos x\,dx \ :$$

1. $u = 1$, $dv = x\cos x\,dx$:
   We don't know of how to compute $\int dv$: no good!

2. $u = x$ and $dv = \cos x\,dx$:
   Done above, works!

3. $u = \cos x$, $dv = x\,dx$:
   Now $du = -\sin x\,dx$ and $v = x^2/2$ so that

$$\int x \cos x\,dx = \frac{1}{2}x^2 \cos x + \int \frac{1}{2}x^2 \sin x\,dx$$

   This makes the situation worse!

4. $u = x \cos x$ and $dv = dx$:
   Now $du = (\cos x - x \sin x)dx$ and $v = x$ so that

$$\int x \cos x\,dx = x^2 \cos x - \int x(\cos x - x \sin x)dx$$

   This again is worse!

General advice:

- Choose u such that du "simplifies".
- Choose dv such that vdu is easy to integrate
- If your result looks more complicated after doing integration by parts, it's most likely not right. Try something else.
- Remember: generally

$$\int f(x)g(x)dx \neq \int f(x)dx \int g(x)dx\,!$$

Read Thomas' Calculus:
p.563 to 565, examples 3 to 5:
Three further examples of integration by parts...
...and practice by doing voluntary online exercises!

The method of partial fractions

example: If you know that

$$\frac{5x-3}{x^2-2x-3} = \frac{2}{x+1} + \frac{3}{x-3}$$

you can integrate easily

$$\begin{aligned} \int \frac{5x-3}{x^2-2x-3}dx &= \int \frac{2}{x+1}dx + \int \frac{3}{x-3}dx \\ &= 2\ln|x+1| + 3\ln|x-3| + C \end{aligned}$$

To obtain such simplifications, we use the **method of partial fractions**.
Let $f(x)/g(x)$ be a rational function, for **example**,

$$\frac{f(x)}{g(x)} = \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3}$$

If $\deg(f) \geq \deg(g)$, we first use polynomial division:

$$\frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} = 2x + \frac{5x - 3}{x^2 - 2x - 3}$$

and consider the remainder term. We also have to know the factors of $g(x)$:

$$x^2 - 2x - 3 = (x+1)(x-3)$$

Now we can write

$$\frac{5x - 3}{x^2 - 2x - 3} = \frac{A}{x+1} + \frac{B}{x-3}$$

and obtain from

$$5x - 3 = A(x-3) + B(x+1) = (A+B)x + (-3A+B)$$

that $A = 2$ and $B = 3$, see above.
note: Alternatively, determine the coefficients by setting $x = -1$ and $x = 3$ in the above equation. However, you need to know about *complex numbers* (taught later) in order to apply this method to more complicated fractions.

Method of Partial Fractions ($f(x)/g(x)$ Proper)

1. Let $x - r$ be a linear factor of $g(x)$. Suppose that $(x-r)^m$ is the highest power of $x - r$ that divides $g(x)$. Then, to this factor, assign the sum of the m partial fractions:

$$\frac{A_1}{x-r} + \frac{A_2}{(x-r)^2} + \cdots + \frac{A_m}{(x-r)^m}.$$

Do this for each distinct linear factor of $g(x)$.

2. Let $x^2 + px + q$ be a quadratic factor of $g(x)$. Suppose that $(x^2 + px + q)^n$ is the highest power of this factor that divides $g(x)$. Then, to this factor, assign the sum of the n partial fractions:

$$\frac{B_1x + C_1}{x^2 + px + q} + \frac{B_2x + C_2}{(x^2 + px + q)^2} + \cdots + \frac{B_nx + C_n}{(x^2 + px + q)^n}.$$

Do this for each distinct quadratic factor of $g(x)$ that cannot be factored into linear factors with real coefficients.

3. Set the original fraction $f(x)/g(x)$ equal to the sum of all these partial fractions. Clear the resulting equation of fractions and arrange the terms in decreasing powers of x.

4. Equate the coefficients of corresponding powers of x and solve the resulting equations for the undetermined coefficients.

example for a repeated linear factor: Find

$$\int \frac{6x+7}{(x+2)^2} dx \, .$$

- Write
$$\frac{6x+7}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2} \, .$$

- Multiply by $(x+2)^2$ to get
$$6x+7 = A(x+2)+B = Ax + (2A+B) \, .$$

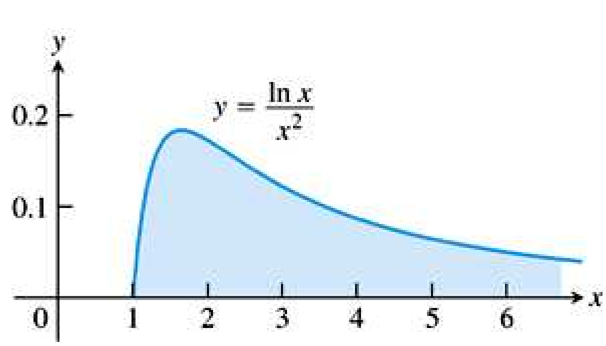
- Equate coefficients of equal powers of x and solve:
$$A = 6 \text{ and } 2A+B = 12+B = 7 \Rightarrow B = -5 \, .$$

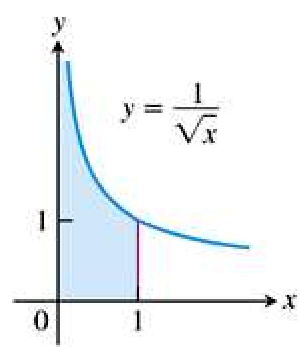
- Integrate:
$$\int \frac{6x+7}{(x+2)^2} dx = 6 \int \frac{dx}{x+2} - 5 \int \frac{dx}{(x+2)^2} = 6 \ln|x+2| + 5(x+2)^{-1} + C \, .$$

Read Thomas' Calculus:
p.572 to 575, examples 1, 4 and 5:
Three more advanced examples...
...and practice by doing voluntary online exercises!

Improper integrals

Can we compute areas under *infinitely extended curves*?
Two **examples** of **improper integrals**:

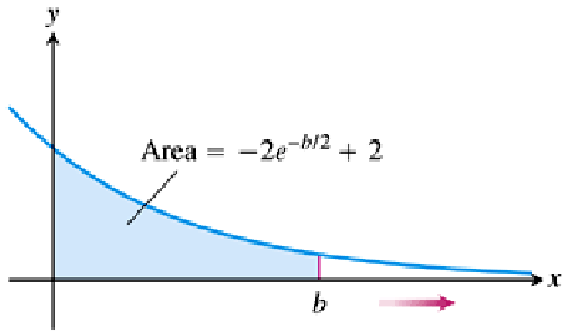




Type 1: area extends from $x = 1$ to $x = \infty$.
Type 2: area extends from $x = 0$ to $x = 1$ *but* $f(x)$ diverges at $x = 0$.

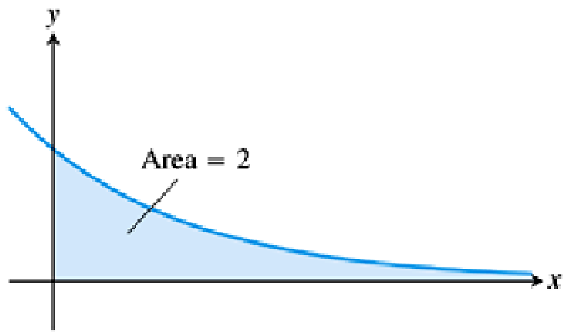
Calculation of type I improper integrals in two steps.

example: $y = e^{-x/2}$ on $[0, \infty)$
1. Calculate bounded area:



$$A(b) = \int_0^b e^{-x/2} dx = -2e^{-x/2}\Big|_0^b = -2e^{-b/2} + 2$$

2. Take the limit:



$$\lim_{b\to\infty} A(b) = \lim_{b\to\infty} (-2e^{-b/2} + 2) = 2$$
$$\Rightarrow \int_0^\infty e^{-x/2} dx = 2$$

DEFINITION Type I Improper Integrals

Integrals with infinite limits of integration are **improper integrals of Type I**.

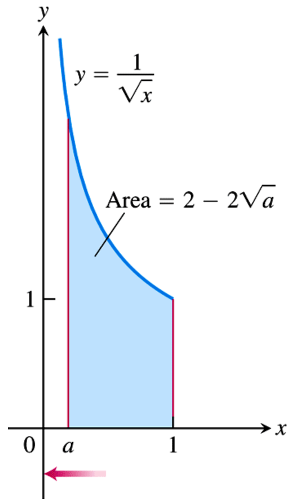
1. If $f(x)$ is continuous on $[a, \infty)$, then
$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx.$$

2. If $f(x)$ is continuous on $(-\infty, b]$, then
$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx.$$

3. If $f(x)$ is continuous on $(-\infty, \infty)$, then
$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx,$$
where c is any real number.

In each case, if the limit is finite we say that the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit fails to exist, the improper integral **diverges**.

Calculation of type II improper integrals in two steps.
example: $y = 1/\sqrt{x}$ on $(0, 1]$



1. Calculate bounded area:

$$A(a) = \int_a^1 \frac{dx}{\sqrt{x}} = 2\sqrt{x}\Big|_a^1 = 2 - 2\sqrt{a}$$

2. Take the limit:

$$\lim_{a \to 0^+} A(a) = \lim_{a \to 0^+} (2 - 2\sqrt{a}) = 2$$

$$\Rightarrow \int_0^1 \frac{dx}{\sqrt{x}} = 2$$

DEFINITION **Type II Improper Integrals**

Integrals of functions that become infinite at a point within the interval of integration are **improper integrals of Type II**.

1. If $f(x)$ is continuous on $(a, b]$ and is discontinuous at a then

$$\int_a^b f(x)\,dx = \lim_{c \to a^+} \int_c^b f(x)\,dx.$$

2. If $f(x)$ is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx.$$

3. If $f(x)$ is discontinuous at c, where $a < c < b$, and continuous on $[a, c) \cup (c, b]$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

In each case, if the limit is finite we say the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit does not exist, the integral **diverges**.

Remarks:

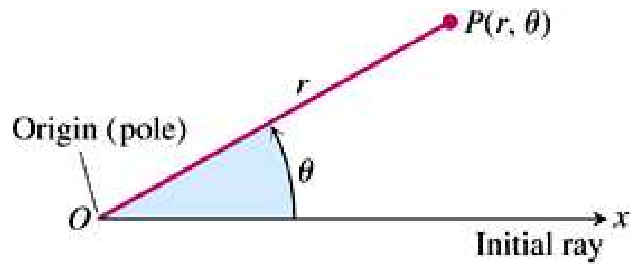
- If you need more examples, please read through Section 8.8, p.619 to p.626.
- **Voluntary reading assignment:** *Tests for convergence and divergence*, see 2nd part of Section 8.8, p.627 to 629; states two conditions under which improper integrals converge or diverge.

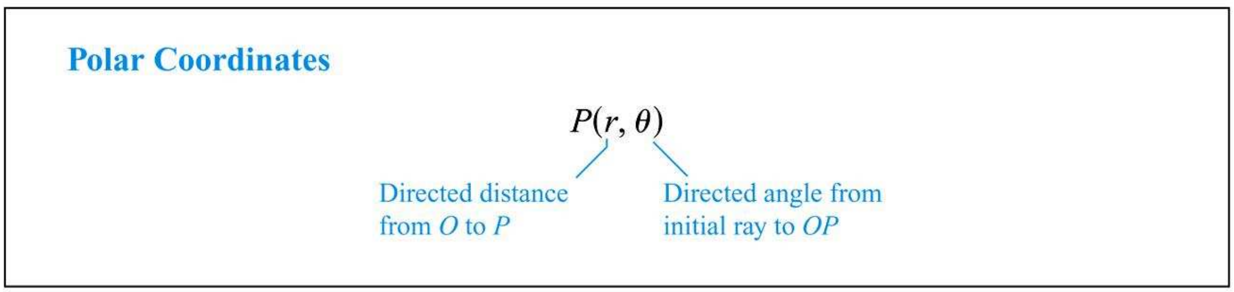
Reading assignment for all students (GL11, GG14)
***not* taking MTH4102 Differential Equations:**
Read Thomas' Calculus Sections 9.1 and 9.2 about differential equations

Polar coordinates

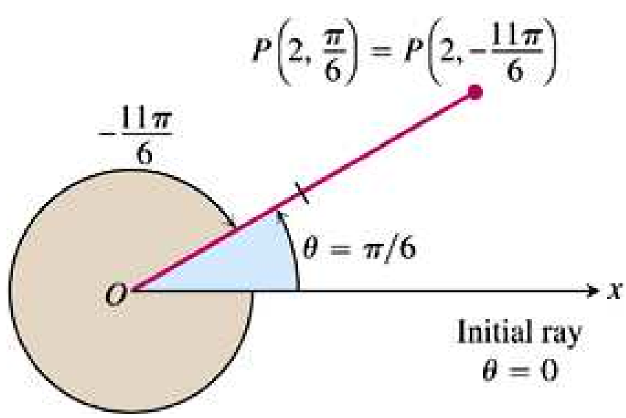
How can we describe a point P in the plane?

- by **Cartesian coordinates** $P(x, y)$
- by **polar coordinates**:



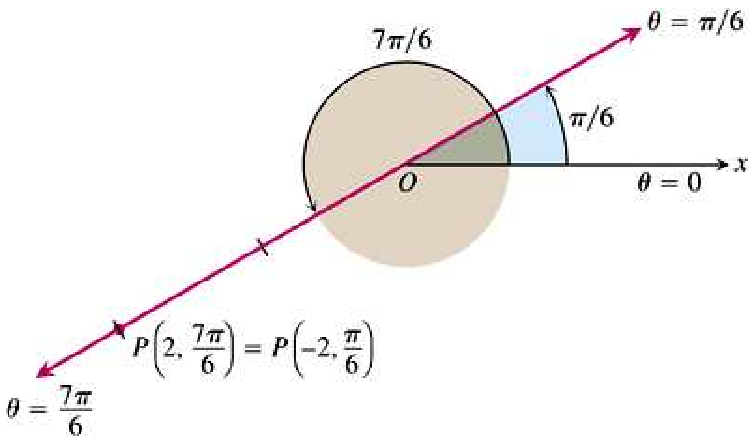


While Cartesian coordinates are unique, polar coordinates are *not*!
example:



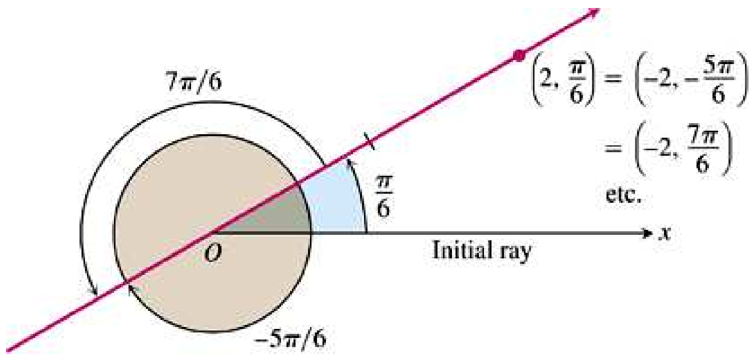
$$(r, \theta) = (r, \theta - 2\pi)$$

Apart from negative angles, we also allow negative values for r:



$$(r, \theta) = (-r, \theta + \pi)$$

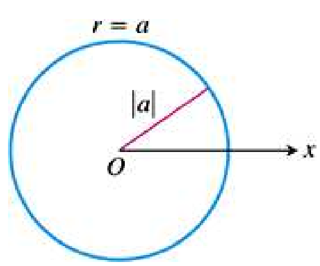
example: Find all polar coordinates of the point $(2, \pi/6)$.



- $r = 2$: $\theta = \pi/6,\ \pi/6 \pm 2\pi,\ \pi/6 \pm 4\pi,\ \pi/6 \pm 6\pi,\ \ldots$
- $r = -2$: $\theta = 7\pi/6,\ 7\pi/6 \pm 2\pi,\ 7\pi/6 \pm 4\pi,\ 7\pi/6 \pm 6\pi,\ \ldots$

Some graphs have simple equations in polar coordinates.
examples:

1. A circle about the origin.



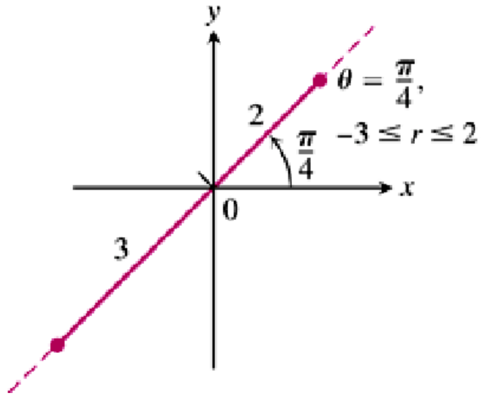
equation: $r = a \neq 0$ (by varying θ over any interval of length 2π)

note: $r = a$ and $r = -a$ both describe the *same* circle of radius $|a|$.

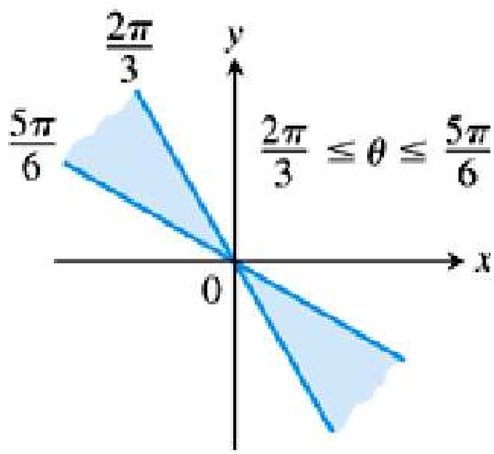
2. A line through the origin.
equation: $\theta = \theta_0$ (by varying r between $-\infty$ and ∞)

examples: Find the graphs of

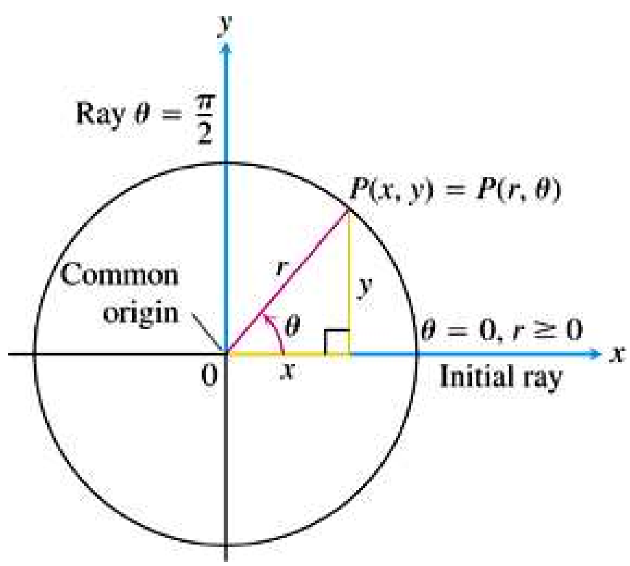
1. $-3 \leq r \leq 2$ and $\theta = \pi/4$



2. $2\pi/3 \leq \theta \leq 5\pi/6$:



Polar and Cartesian coordinates can be converted into each other:



- polar $\rightarrow$ Cartesian coordinates:

$$x = r\cos\theta\,, \quad y = r\sin\theta$$

Given (r, θ), we can uniquely compute (x, y).

- Cartesian $\longrightarrow$ polar coordinates:

$$r^2 = x^2 + y^2 \; , \quad \tan\theta = y/x$$

Given (x, y), we have to choose one of many polar coordinates.

Often as **convention** (particularly in physics): $r \geq 0$ ("distance") and $0 \leq \theta < 2\pi$. (if $r = 0$, choose also $\theta = 0$ for uniqueness)

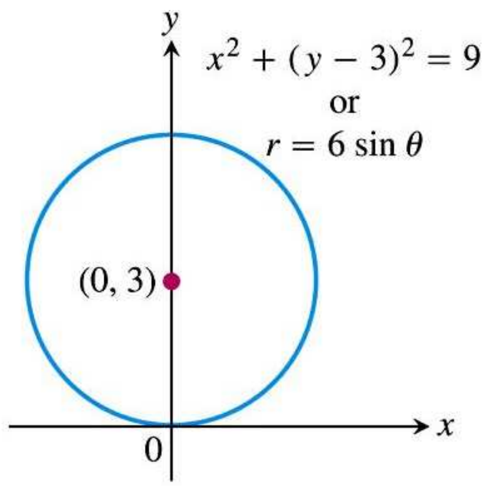
examples: equivalent equations

Cartesian	polar
$x = 2$	$r\cos\theta = 2$
$xy = 4$	$r^2\cos\theta\sin\theta = 4$
$x^2 - y^2 = 1$	$r^2(\cos^2\theta - \sin^2\theta) = r^2\cos 2\theta = 1$

In some cases polar coordinates are a lot simpler, in others they are not.

examples:

1. Cartesian $\longrightarrow$ polar for circle



$$\begin{aligned} x^2 + (y-3)^2 =&\; 9 \\ (x^2 + y^2) - 6y + 9 =&\; 9 \\ r^2 - 6r\sin\theta =&\; 0 \\ r = 0 \text{ or }&\; r = 6\sin\theta \\ &\;\text{(which includes } r = 0\text{)} \end{aligned}$$

2. polar $\longrightarrow$ Cartesian:

$$r = \frac{4}{2\cos\theta - \sin\theta}$$

is equivalent to

$$2r\cos\theta - r\sin\theta = 4$$

or $2x - y = 4$, which is the equation of a line,

$$y = 2x - 4 \; .$$

The End