MTH4100 Calculus I

Lecture notes for Week 1

Thomas' Calculus, Section 1.1

Rainer Klages

School of Mathematical Sciences
Queen Mary University of London

Autumn 2009

What is Calculus?

Calculus may be considered as "advanced algebra and geometry", with the goal to set up mathematics as a *formal language*. Fundamental for Calculus are the *real numbers*. They enable the study of *functions* of real variables:

- for one real variable see Calculus I
- for many variables see Calculus II

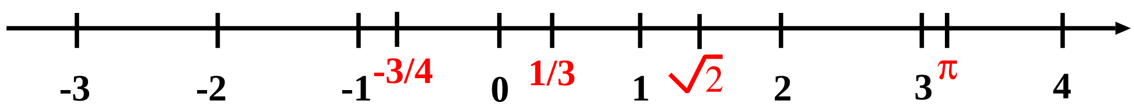
The geometric view of Calculus concerns the *graph* of a function covering aspects like

- continuity properties
- slope $\leftrightarrow$ derivative
- area $\leftrightarrow$ integral

You will learn many techniques, based on *algebraic manipulations* with many applications in *all branches of modern society*. The next level of mathematical abstraction is called *analysis*.

Real numbers and the real line

Think of the real numbers, e.g., as all decimals.
examples: $-\frac{3}{4} = -0.7500\ldots \quad ; \quad \frac{1}{3} = 0.333\ldots \quad ; \quad \sqrt{2} = 1.4142\ldots$
The real numbers $\mathbb{R}$ can be represented as points on the *real line*:



Real numbers are characterized by three fundamental *properties*:

- **algebraic** means formalisations of the rules of calculation (addition, subtraction, multiplication, division).
  example: $2(3+5) = 2\cdot 3 + 2\cdot 5 = 6 + 10 = 16$
- **order** denotes inequalities (for a geometric picture see the real line).
  example: $-\frac{3}{4} < \frac{1}{3} \quad \Rightarrow -\frac{1}{3} < \frac{3}{4}$
- **completeness** implies that there are "no gaps" on the real line

1. Algebraic properties of the reals for *addition* ($a, b, c \in \mathbb{R}$) are:

(A1) $a + (b + c) = (a + b) + c$ *associativity*
(A2) $a + b = b + a$ *commutativity*
(A3) There is a 0 such that $a + 0 = a$. *identity*
(A4) There is an x such that $a + x = 0$. *inverse*

Why these rules? They define an *algebraic structure* (commutative group).

Now define analogous algebraic properties for *multiplication*:

(M1) $a(bc) = (ab)c$
(M2) $ab = ba$
(M3) There is a 1 such that $a\,1 = a$.
(M4) There is an x such that $a\,x = 1$ (for $a \neq 0$).

Finally, connect *multiplication with addition*:

(D) $a(b + c) = ab + ac$ *distributivity*

These 9 rules define an algebraic structure called a *field*.

2. Order properties of the reals are:

(O1) for any $a, b \in \mathbb{R}$, $a \leq b$ or $b \leq a$ *totality of ordering I*
(O2) if $a \leq b$ and $b \leq a$ then $a = b$ *totality of ordering II*
(O3) if $a \leq b$ and $b \leq c$ then $a \leq c$ *transitivity*
(O4) if $a \leq b$ then $a + c \leq b + c$ *order under addition*
(O5) if $a \leq b$ and $0 \leq c$ then $a\,c \leq b\,c$ *order under multiplication*

Some useful rules for calculations with inequalities (practise in exercises) are:

Rules for Inequalities
If a, b, and c are real numbers, then:

1. $a < b \Rightarrow a + c < b + c$
2. $a < b \Rightarrow a - c < b - c$
3. $a < b$ and $c > 0 \Rightarrow ac < bc$
4. $a < b$ and $c < 0 \Rightarrow bc < ac$
   Special case: $a < b \Rightarrow -b < -a$
5. $a > 0 \Rightarrow \dfrac{1}{a} > 0$
6. If a and b are both positive or both negative, then $a < b \Rightarrow \dfrac{1}{b} < \dfrac{1}{a}$

These rules can all be *proved* by using (O1) to (O5): 1. to 3. follow straightforwardly, 4. to 6. are more tricky.

3. The **completeness property** can be understood by the following *construction* of the real numbers: (using set notation!)
Start with "counting numbers" $1, 2, 3, \ldots$

- $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$ *natural numbers* $\rightarrow$ Can we solve $a + x = b$ for x?

- $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ *integers* $\rightarrow$ Can we solve $ax = b$ for x?
- $\mathbb{Q} = \{\frac{p}{q} | p, q \in \mathbb{Z},\ q \neq 0\}$ *rational numbers* $\rightarrow$ Can we solve $x^2 = 2$ for x?
- $\mathbb{R}$ *real numbers*
  example: The positive solution to the equation $x^2 = 2$ is $\sqrt{2}$. This is an *irrational number* whose decimal representation is not eventually repeating: $\sqrt{2} = 1.414\ldots$ Another example is $\pi = 3.141\ldots$

$$\Rightarrow \boxed{\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}}$$

In fact, one has to "prove" this:

Theorem 1 *$x^2 = 2$ has no solution $x \in \mathbb{Q}$.*

Proof: *Assume there is* an $x \in \mathbb{Q}$ with $x^2 = 2$.
This must be of the form $x = \frac{p}{q}$, $p, q \in \mathbb{Z},\ q \neq 0$, and we can assume that **p and q have no common factors** (otherwise cancel them).
$x^2 = 2$ then implies that $(\frac{p}{q})^2 = 2$, or $\boxed{p^2 = 2q^2}$, so p^2 is even.
However, p^2 even implies that **p is even** (...requires proof...).
Write $p = 2p_1$, so that $p^2 = (2p_1)^2$, or $4p_1^2 = 2q^2$, or $\boxed{2p_1^2 = q^2}$.
This implies that q^2 is even, so **q is even** as well.
We have now shown that both p and q must be even, so they **share a common factor 2**.
This is a contradiction! Therefore the assumption must be wrong. q.e.d.

In summary, the real numbers $\mathbb{R}$ are *complete* in the sense that they correspond to all points on the real line, i.e., there are no "holes" or "gaps", whereas the rationals have "holes" (namely the irrationals).
See your textbook Appendix 4 for details. The "proof"of completeness of $\mathbb{R}$ is covered in MTH5104 Convergence and Continuity, a 2nd year "analysis" module.

University mathematics is built upon

- basic properties (Definitions, Axioms)
- statements deduced from these (Lemma, Proposition, Theorem, Corollary, ...) in form of *proofs*!

example: The technique in the previous proof is called *Proof by Contradiction.*

Many different ones to come! For details about the logic behind proofs see MTH5117, Mathematical Writing.
This formal framework is illustrated in Calculus 1 by many examples, exercises, applications.

Intervals

Definition 1 *A subset of the real line is called an* interval *if it contains at least two numbers and all the real numbers between any two of its elements.*

examples:

- $x > -2$ defines an *infinite interval*. Geometrically, it corresponds to a *ray* on the real line
- $3 \leq x \leq 6$ defines a *finite interval*. Geometrically, it corresponds to a *line segment* on the real line

So we can distinguish between two basic types of intervals – let's further classify:

TABLE 1.1 Types of intervals

	Notation	Set description	Type	Picture
Finite:	(a, b)	$\{x \mid a < x < b\}$	Open	
	$[a, b]$	$\{x \mid a \leq x \leq b\}$	Closed	
	$[a, b)$	$\{x \mid a \leq x < b\}$	Half-open	
	$(a, b]$	$\{x \mid a < x \leq b\}$	Half-open	
Infinite:	(a, ∞)	$\{x \mid x > a\}$	Open	
	$[a, \infty)$	$\{x \mid x \geq a\}$	Closed	
	$(-\infty, b)$	$\{x \mid x < b\}$	Open	
	$(-\infty, b]$	$\{x \mid x \leq b\}$	Closed	
	$(-\infty, \infty)$	$\mathbb{R}$ (set of all real numbers)	Both open and closed	

Solve inequalities to find intervals of $x \in \mathbb{R}$.

examples:

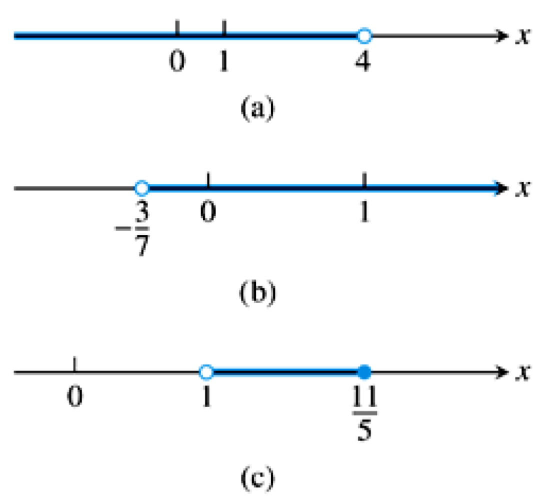
$$\begin{aligned} (a) \quad 2x - 1 &< x + 3 \\ 2x &< x + 4 \\ x &< 4 \end{aligned}$$

$$\begin{aligned} (b) \quad -\frac{x}{3} &< 2x + 1 \\ -x &< 6x + 3 \\ -\frac{3}{7} &< x \end{aligned}$$

(c) $\quad \frac{6}{x-1} \geq 5$: must hold $x > 1$!

$$\begin{aligned} 6 &\geq 5x - 5 \\ \frac{11}{5} &\geq x \end{aligned}$$

solution sets on the real line:



Absolute value

Definition 2 *The* absolute value *(or* modulus*) of a real number x is*

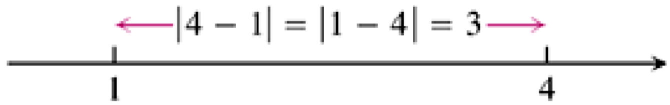
$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases} .$$

Geometrically, $|x|$ is the *distance* between x and 0.

example:

$|x - y|$ is the distance between x and y.

example:



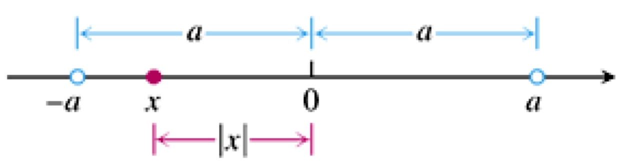
An alternative definition of $|x|$ is

$$|x| = \sqrt{x^2} \quad ,$$

since taking the square root always gives a *non-negative* result!
$|x|$ in an inequality:

$$|x| < a \qquad \Leftrightarrow \qquad -a < x < a \quad \text{(why?)}$$

The distance from x to 0 is less than $a > 0 \Leftrightarrow x$ must lie between a and $-a$.



Absolute value properties are:

1. $|-a| = |a|$
2. $|ab| = |a|\,|b|$
3. $|\frac{a}{b}| = \frac{|a|}{|b|}$ for $b \neq 0$
4. $|a + b| \leq |a| + |b|$, the *triangle inequality*

Prove these statements! *Key idea*: use $|x| = \sqrt{x^2}$.

1. Proof of $|-a| = |a|$: $\quad |-a| = \sqrt{(-a)^2} = \sqrt{a^2} = |a|$

   We have used a *direct proof*: We started on the left hand side of the equation and transformed it step by step until we have arrived at the right hand side.

2. Proof of $|ab| = |a|\,|b|$: $\quad |ab| = \sqrt{(ab)^2} = \sqrt{a^2b^2} = \sqrt{a^2}\sqrt{b^2} = |a|\,|b|$

3. Proof of $|\frac{a}{b}| = \frac{|a|}{|b|}$ for $b \neq 0$: exercise!

4. Proof of the triangle inequality $|a+b| \leq |a| + |b|$:

   Use a little trick and prove instead: $\quad |a+b|^2 \leq (|a|+|b|)^2$

$$
\begin{aligned}
|a+b|^2 &= \left(\sqrt{(a+b)^2}\right)^2 && \text{(with } |x| = \sqrt{x^2}\text{)} \\
&= (a+b)^2 \\
&= a^2 + 2ab + b^2 \\
&\leq a^2 + 2|a|\,|b| + b^2 && \text{(because } ab \leq |ab| = |a||b|\text{)} \\
&= |a|^2 + 2|a|\,|b| + |b|^2 && \text{(see above)} \\
&= (|a|+|b|)^2
\end{aligned}
$$

Now take the square root and observe that the arguments of both roots are positive – we are done.

Absolute Values and Intervals

If a is any positive number, then

5. $|x| = a$ if and only if $x = \pm a$

6. $|x| < a$ if and only if $-a < x < a$

7. $|x| > a$ if and only if $x > a$ or $x < -a$

8. $|x| \leq a$ if and only if $-a \leq x \leq a$

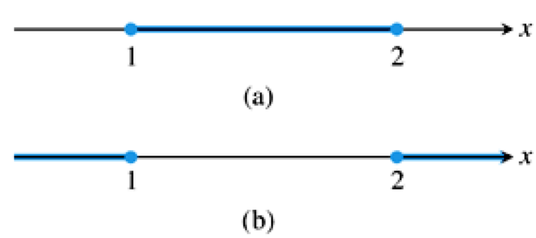
9. $|x| \geq a$ if and only if $x \geq a$ or $x \leq -a$

note: "if and only if" is often abbreviated by the sign "$\Leftrightarrow$"

examples

(a) $|2x - 3| \leq 1$

(b) $|2x - 3| \geq 1$



Reading Assignment: read

Thomas' Calculus, Chapter 1.2: Lines, Circles, and Parabolas