Indeterminate forms and L'Hôpital's Rule

If $f(a) = g(a) = 0$, $f(a)/g(a)$ is a meaningless, indeterminate form.

$$\Rightarrow \quad \lim_{x \to a} \frac{f(x)}{g(x)} \text{ cannot be found by substitution } x = a.$$

Under certain conditions, we can nevertheless calculate it:

> **Theorem:** L'Hôpital's Rule (first form)
>
> Suppose that $f(a) = g(a) = 0$, that $f'(a)$ and $g'(a)$ exist, and that $g'(a) \neq 0$. Then $\displaystyle\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$

<u>Proof:</u> proceed rhs $\to$ lhs

$$\underline{\frac{f'(a)}{g'(a)}} \underset{\text{def } f', g'}{=} \frac{\lim_{x \to a} \frac{f(x) - f(a)}{x - a}}{\lim_{x \to a} \frac{g(x) - g(a)}{x - a}} \underset{\text{limit laws}}{=} \lim_{x \to a} \frac{\frac{f(x) - f(a)}{\cancel{x - a}}}{\frac{g(x) - g(a)}{\cancel{x - a}}} =$$

$$= \lim_{x \to a} \frac{f(x) - \cancel{f(a)}^{=0}}{g(x) - \cancel{g(a)}^{=0}} \underset{\text{hyp. thm}}{=} \underline{\lim_{x \to a} \frac{f(x)}{g(x)}} \quad \blacksquare$$

<u>WARNING:</u>

- Always check for "0/0", i.e., $f(a) = g(a) = 0$, before using L'Hôpital!
- Do <u>not</u> compute $\left(\frac{f}{g}\right)'(x)$ but $\frac{f'(x)}{g'(x)}$!

examples:

① $\lim_{x\to 0} \frac{3x - \sin x}{x} = \frac{3 - \cos x}{1}\Big|_{x=0} = \underline{\underline{2}}$

② $\lim_{x\to 0} \frac{1 + \sin x}{1 - x} = \ldots$ (this does not fulfill the assumptions of L'Hôpital!)

$= \frac{1}{1} = \underline{\underline{1}}$ (by substitution)

③ $\lim_{x\to 0} \frac{x - \sin x}{x^3} = \frac{1 - \cos x}{3x^2}\Big|_{x=0} = \frac{0}{0}$ ↯ doesn't work!

can be handled with

Theorem: L'Hôpital's Rule (stronger form)

Suppose that $f(a) = g(a) = 0$, that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I <u>if</u> $x \neq a$. Then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$, assuming that the limit on the rhs exists.

(<u>Proof</u>: see p. 293 ff book, via Cauchy's mean value theorem)

example: finish ③ above,

$\lim_{x\to 0} \frac{x - \sin x}{x^3} = \lim_{x\to 0} \frac{1 - \cos x}{3x^2} = \lim_{x\to 0} \frac{\sin x}{6x} = \lim_{x\to 0} \frac{\cos x}{6} = \underline{\frac{1}{6}}$

Summary: Using L'Hôpital's Rule

To find $\lim_{x\to a} f(x)/g(x)$ by using L'Hôpital, first check "0/0". Then continue to differentiate f and g, so long as we still get 0/0 at $x = a$. As soon as this is not the case anymore, stop differentiating.

remark: L'Hôpital also applies to one-sided limits (see proof of previous thm.).

example: $$\lim_{x\to 0^\pm} \frac{\sin x}{x^2} = \lim_{x\to 0^\pm} \frac{\cos x}{2x} = \pm\infty$$

What's about limits involving other indeterminate forms like ∞/∞, $\infty\cdot 0$ or $\infty - \infty$?

(1) ∞/∞: can be proven that if $f(x), g(x) \to \pm\infty$ as $x\to a$, then
$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

So use L'Hôpital same way as before for "∞/∞"!

example: $$\lim_{x\to\infty} \frac{x-2x^2}{3x^2+7x} = \lim_{x\to\infty} \frac{1-2x}{3x+7} = \lim_{x\to\infty} \frac{-2}{3} = -\frac{2}{3}$$

(2) $\infty\cdot 0$: use $$\lim_{x\to a} (f(x)\cdot g(x)) = \lim_{x\to a} \frac{g(x)}{1/f(x)}$$

example: $$\lim_{x\to\infty}\left(x\cdot \sin\frac{1}{x}\right) = \lim_{x\to\infty} \frac{\sin\frac{1}{x}}{\frac{1}{x}} \underset{h=1/x}{=} \lim_{h\to 0^+} \frac{\sin h}{h} = \lim_{h\to 0^+} \frac{\cos h}{1} = 1$$

(3) $\infty - \infty$: best demonstrated by an

example: $\lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right) = \lim_{x\to 0} \frac{x - \sin x}{x \sin x} =$

$$= \lim_{x\to 0} \frac{1-\cos x}{\sin x + x\cos x} = \lim_{x\to 0} \frac{\sin x}{\cos x + \cos x + x(-\sin x)} = \underline{\underline{0}}$$

Antiderivatives

aim: given $f(x)$ and $f(x) = F'(x)$, find $F(x)$.

Definition: Antiderivative

A function F is an antiderivative of f on an interval I if $F'(x) = f(x)$ for all $x \in I$.

examples:
- $f(x) = 2x \Rightarrow F(x) = x^2$
- $g(x) = \sin x \Rightarrow F(x) = -\cos x$

But these are not the only solutions:

Corollary: (of the mean value thm.)

If $G'(x) = F'(x)$ on (a,b), then $G(x) = F(x) + C$ for all $x \in (a,b)$.

which implies:

If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is $F(x) + C$, $C = \text{const}$.

Some antiderivative formulas are shown in Table 4.2.

examples: ① $f(x) = x^4 \Rightarrow F(x) = \frac{x^5}{5} + C$

② $h(x) = \cos 5x \Rightarrow H(x) = \frac{\sin 5x}{5} + C$

The rules shown in Table 4.3 are easily proven by differentiation.

(more advanced techniques see later)

example: Find the general antiderivative of

$$f(x) = \frac{3}{\sqrt{x}} + \sin 2x$$

fct. is of the form $f(x) = 3\,g(x) + h(x)$ with $g(x) = x^{-\frac{1}{2}}$ and $h(x) = \sin 2x$.

$\Rightarrow G(x) = 2\sqrt{x} + C_1$ which satisfies $G'(x) = g(x)$

$H(x) = -\frac{1}{2}\cos 2x + C_2$, " " $H'(x) = h(x)$

Hence, $F(x) = 6\sqrt{x} - \frac{1}{2}\cos 2x + C$, $C = C_1 + C_2$

A special symbol is used to denote the collection of all antiderivatives of f:

Definition: Indefinite Integral, Integrand

The set of all antiderivatives of f is the indefinite integral of f with respect to x, denoted by

$$\int f(x)\,dx.$$

$$(= F(x) + C)$$

The symbol $\int$ is an integral sign. The function f is the integrand of the integral, and x is the variable of integration.

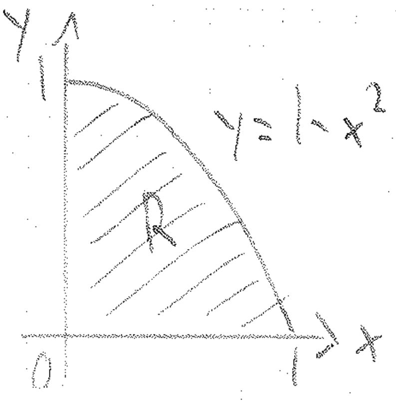
examples: ① $\int 2x\,dx = x^2 + C$

② $\int \cos x\,dx = \sin x + C$

INTEGRATION

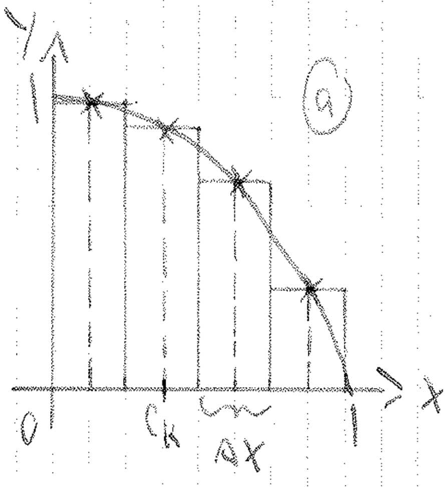
Estimating with finite sums

example:

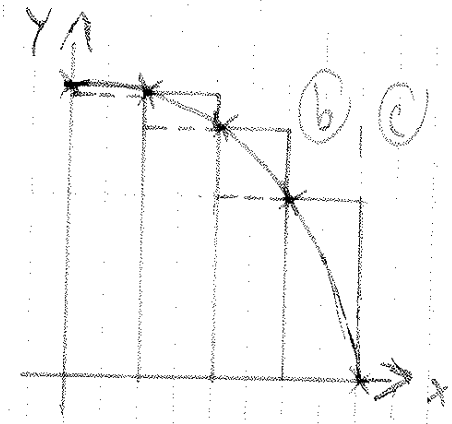


How can we compute the shaded area R?

=> animation!!



- subdivide the interval $[a,b]$ into n subintervals of equal width $\Delta x = (a-b)/n$
- choose point c_k in the k-th subinterval



- construct rectangles:

  ⓐ midpoint rule: choose c_k in the middle of the interval

  ⓑ upper sum: choose c_k s.t. $f(c_k)$ is maximal

  ⓒ lower sum: " " " " " minimal

- form the sum $f(c_1)\Delta x + f(c_2)\Delta x + \dots + f(c_n)\Delta x$
- refine your approximation by choosing more rectangles: Table 6.1

To handle sums with many terms, we need a better notation:

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + \dots + a_n$$

with summation symbol (Greek letter Sigma)

$$\sum_{k=1}^{n} a_k$$

n ← index k ends at $k=n$

a_k ← formula for k-th term

$k=1$ ← index k starts at $k=1$

examples: ① $f(c_1)\Delta x + f(c_2)\Delta x + \dots + f(c_n)\Delta x = \sum_{k=1}^{n} f(c_k)\Delta x$

⑤ $1+3+5+7+9 = \sum_{k=1}^{5} (2k-1)$

$(k = u+1) = \sum_{u=0}^{4} (2u+1)$

$(u = t+3) = \sum_{t=-3}^{1} (2t+7) = 25$

② $\sum_{k=1}^{5} k = 1+2+3+4+5 = 15$

③ $\sum_{k=1}^{3} (-1)^k k = (-1)^1\cdot 1 + (-1)^2 \cdot 2 + (-1)^3 \cdot 3$
$= -1 + 2 - 3 = -2$

④ $\sum_{k=1}^{2} \frac{k}{k+1} = \frac{1}{1+1} + \frac{2}{2+1} = \frac{1}{2} + \frac{2}{3} = \frac{7}{6}$

Algebra rules for finite sums: (proof via induction)

① sum/difference rule: $\sum_{k=1}^{n} (a_k \pm b_k) = \sum_{k=1}^{n} a_k \pm \sum_{k=1}^{n} b_k$

② constant multiple rule: $\sum_{k=1}^{n} c \cdot a_k = c \cdot \sum_{k=1}^{n} a_k \,, \; c \in \mathbb{R}$

③ constant value rule: $\sum_{k=1}^{n} c = n \cdot c \,, \; c \in \mathbb{R}$

<u>example:</u> $\sum_{k=1}^{n} (3k - k^2) = 3 \underbrace{\sum_{k=1}^{n} k}_{} - \underbrace{\sum_{k=1}^{n} k^2}_{}$ (with ①, ②)

can we calculate these sums?

<u>example:</u> $\sum_{k=1}^{n} k = 1 + 2 + 3 + \dots + (n-1) + n$
$= n + (n-1) + (n-2) + \dots + 2 + 1$

$\Rightarrow 2 \sum_{k=1}^{n} k = (n+1) \cdot n$, or

$$\underline{\sum_{k=1}^{n} k = \frac{n(n+1)}{2}}$$

It also holds: $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

(can be proven by mathematical induction, see App. 1 book)

Limits of finite sums

<u>example</u>: Compute the area R below the graph of $y = 1 - x^2$ and above the interval $[0,1]$.

- subdivide the interval into n subintervals of width $\Delta x = \frac{1}{n}$:
  $[0, \frac{1}{n}], [\frac{1}{n}, \frac{2}{n}], \dots, [\frac{n-1}{n}, \frac{n}{n}]$
- choose lower sum: $c_k = \frac{k}{n}$, $k \in \mathbb{N}$, is rightmost point
- do the summation:

$$f\left(\frac{1}{n}\right)\cdot\frac{1}{n} + f\left(\frac{2}{n}\right)\cdot\frac{1}{n} + \dots + f\left(\frac{n}{n}\right)\cdot\frac{1}{n} =$$

$$= \sum_{k=1}^{n} f\left(\frac{k}{n}\right)\cdot\frac{1}{n} = \sum_{k=1}^{n}\left(1 - \left(\frac{k}{n}\right)^2\right)\cdot\frac{1}{n} =$$

$$= \sum_{k=1}^{n}\left(\frac{1}{n} - \frac{k^2}{n^3}\right) = \frac{1}{n}\sum_{k=1}^{n} 1 - \frac{1}{n^3}\sum_{k=1}^{n} k^2 =$$

$$= \frac{1}{n}\cdot n - \frac{1}{n^3}\,\frac{n(n+1)(2n+1)}{6} =$$

$$= 1 - \frac{2n^3 + 3n^2 + n}{6n^3} = \underline{\underline{\frac{2}{3} - \frac{1}{2n} - \frac{1}{6n^2}}}$$

S/2

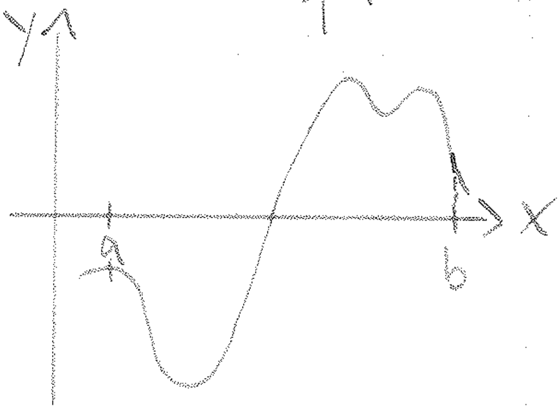
- lower sum: $R \geq \frac{2}{3} - \frac{1}{2n} - \frac{1}{6n^2}$
- upper sum: $R \leq \frac{2}{3} + \frac{1}{2n} - \frac{1}{6n^2}$ (not done here)
- as $n \to \infty$, both sums converge to $\frac{2}{3}$. Therefore,

$$\underline{R = \frac{2}{3}}$$

<u>note</u>: Any other choice of c_k would give the same result! (why?)

Riemann sums

Consider a typical continuous fct. over $[a,b]$:



Partition the interval $[a,b]$ by choosing $n-1$ points between $[a,b]$: $a = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = b$

i.e., $\Delta x_k = x_k - x_{k-1}$, the width of the subinterval $[x_{k-1}, x_k]$, may vary.