MTH4100 Calculus I

Week 8 (Thomas' Calculus Sections 4.1 to 4.4)

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Revision of lecture 18

- linearisation
- extreme values
- midterm test

Calculating absolute extrema

How to Find the Absolute Extrema of a Continuous Function f on a Finite Closed Interval

1. Evaluate f at all critical points and endpoints.
2. Take the largest and smallest of these values.

why the above conditions? recall the *extreme value theorem*

example 1: Find the absolute extrema of $f(x) = x^2$ on $[-2, 1]$.

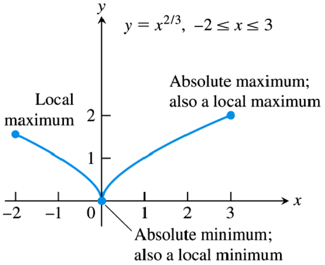
- f is differentiable on $[-2, 1]$ with $f'(x) = 2x$
- critical point: $f'(x) = 0 \quad \Rightarrow \quad x = 0$
- endpoints: $x = -2$ and $x = 1$
- $f(0) = 0$, $f(-2) = 4$, $f(1) = 1$

Therefore f has an absolute maximum value of 4 at $x = -2$ and an absolute minimum value of 0 at $x = 0$.

Absolute extrema with $f'(c)$ being undefined

example 2: Find the absolute extrema of $f(x) = x^{2/3}$ on $[-2, 3]$.

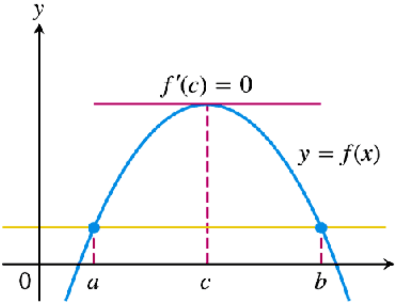
- f is differentiable with $f'(x) = \frac{2}{3}x^{-1/3}$ *except* at $x = 0$
- critical point: $f'(x) = 0$ or $f'(x)$ undefined $\Rightarrow \quad x = 0$
- endpoints: $x = -2$ and $x = 3$
- $f(-2) = \sqrt[3]{4}$, $f(0) = 0$, $f(3) = \sqrt[3]{9}$

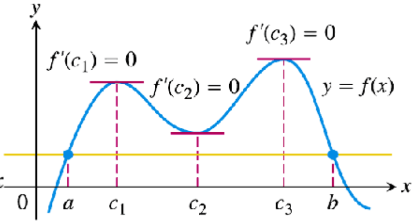


Therefore f has an absolute maximum value of $\sqrt[3]{9}$ at $x = 3$ and an absolute minimum value of 0 at $x = 0$.

Rolle's theorem

motivation:





Theorem

Let $f(x)$ be continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$ then there exists a $c \in (a, b)$ with

$$f'(c) = 0 .$$

Rolle's theorem: proof

Theorem

Let $f(x)$ be continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$ then there exists a $c \in (a, b)$ with

$$f'(c) = 0 \; .$$

Proof.

- extreme value theorem: f is continuous on $[a, b]$, so it has absolute maximum and minimum.
- these occur only at critical points or endpoints – here: at $f'(x) = 0$ on (a, b), or else at a or b.
- apply first derivative theorem for extrema: if one of them occurs at $c \in (a, b)$, then $f'(c) = 0$ (and we're done).
- If not, both must occur at the endpoints. But as $f(a) = f(b)$, $f(x)$ must then be constant and therefore $f'(x) = 0$ on $[a, b]$. □

Assumptions in Rolle's theorem

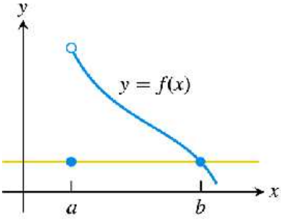
Theorem

Let $f(x)$ be continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$ then there exists a $c \in (a, b)$ with

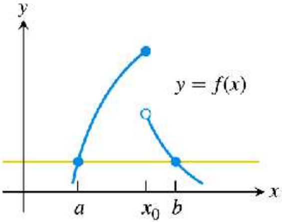
$$f'(c) = 0 \; .$$

It is essential that all of the hypotheses in the theorem are fulfilled!

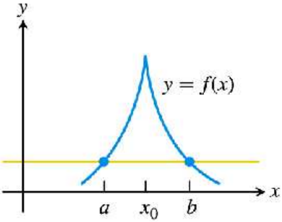
examples:



(a) Discontinuous at an endpoint of $[a, b]$



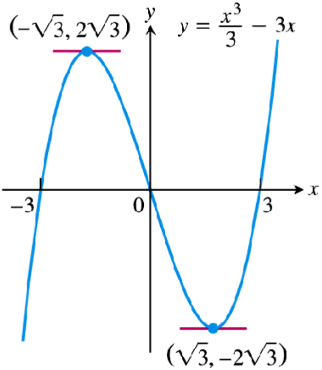
(b) Discontinuous at an interior point of $[a, b]$



(c) Continuous on $[a, b]$ but not differentiable at an interior point

Horizontal tangents of a cubic polynomial

example: Apply Rolle's theorem to $f(x) = \frac{x^3}{3} - 3x$ on $[-3, 3]$.

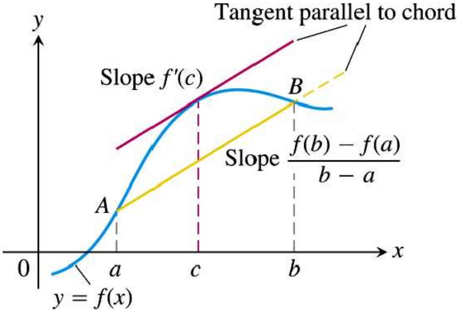


- polynomial f is continuous on $[-3, 3]$ and differentiable on $(-3, 3)$
- $f(-3) = f(3) = 0$
- by Rolle's theorem there exists (at least!) one $c \in [-3, 3]$ with $f'(c) = 0$

From $f'(x) = x^2 - 3 = 0$ we find that indeed $x = \pm\sqrt{3}$.

The mean value theorem

motivation: "slanted version of Rolle's theorem"



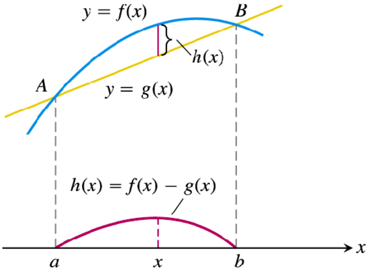
Theorem

Let $f(x)$ be continuous on $[a, b]$ and differentiable on (a, b). Then there exists a $c \in (a, b)$ with

$$f'(c) = \frac{f(b) - f(a)}{b - a} .$$

The mean value theorem: proof

basic idea:

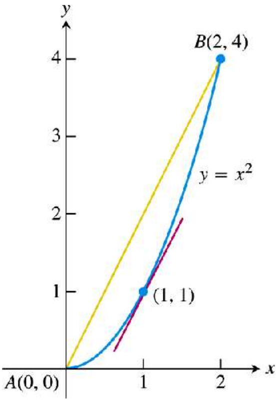


Proof.

- secant through $(a, f(a))$ and $(b, f(b))$ is given by
  $$g(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a) \text{ (point-slope form)}$$
- shift graph: for $h(x) = f(x) - g(x)$, both $h(a) = 0$ and $h(b) = 0$
- apply Rolle's theorem: there is a $c \in (a, b)$ with $h'(c) = 0$
- as $h'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$, this implies $f'(c) = \frac{f(b)-f(a)}{b-a}$. □

Applying the mean value theorem

example: Consider $f(x) = x^2$ on $[0, 2]$.



- $f(x)$ is continuous and differentiable on $[0, 2]$.
- Therefore there is a $c \in (0, 2)$ with

$$f'(c) = \frac{f(2) - f(0)}{2 - 0} = 2 .$$

- Since $f'(x) = 2x$ we find that $c = 1$.

Functions with zero derivatives are constant

know $f'(x) \Rightarrow$ know $f(x)$? **special case:**

Corollary

If $f'(x) = 0$ on (a, b) then $f(x) = C$ for all $x \in (a, b)$.

Proof.

For any $x_1, x_2 \in (a, b)$ with $x_1 < x_2$, f is differentiable and continuous on $[x_1, x_2]$. According to the mean value theorem there is thus a $c \in (x_1, x_2)$ with

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} .$$

But as $f'(c) = 0$ by assumption, it follows that

$$f(x_2) = f(x_1) .$$

As x_1 and x_2 are chosen arbitrarily in (a, b), $f(x)$ is constant for all $x \in (a, b)$. □

Functions with the same derivative differ by a constant

know $f'(x) = g'(x) \Rightarrow$ know relation between f and g?

Corollary

If $f'(x) = g'(x)$ *for all* $x \in (a, b)$, *then*

$$f(x) = g(x) + C \; .$$

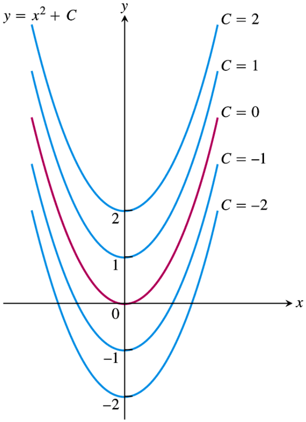
Proof.

Consider $h(x) = f(x) - g(x)$. As

$$h'(x) = f'(x) - g'(x) = 0$$

for all $x \in (a, b)$, $h(x) = C$ by the previous corollary and so $f(x) = g(x) + C$. □

example:



Application of the last corollary

example: Find the function $f(x)$ whose derivative is $\sin x$ and whose graph passes through the point $(0, 2)$.

- $g(x) = -\cos x$ satisfies

$$g'(x) = \sin x = f'(x)$$

- Therefore $f(x) = g(x) + C$, i.e.

$$f(x) = -\cos x + C$$

- $f(0) = 2$ gives

$$2 = -\cos 0 + C$$

so that $C = 3$.

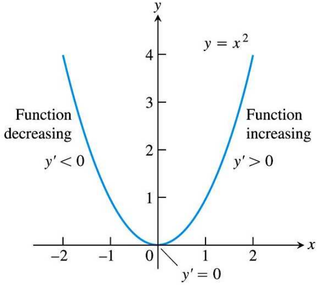
$$\Rightarrow \boxed{f(x) = 3 - \cos x}$$

Revision of lecture 19

- calculating absolute extrema
- Rolle's theorem
- mean value theorem

Increasing and decreasing functions

motivation:



- make increasing/decreasing mathematically precise
- clarify relation to positive/negative derivative

DEFINITIONS Increasing, Decreasing Function

Let f be a function defined on an interval I and let x_1 and x_2 be any two points in I.

1. If $f(x_1) < f(x_2)$ whenever $x_1 < x_2$, then f is said to be **increasing** on I.
2. If $f(x_2) < f(x_1)$ whenever $x_1 < x_2$, then f is said to be **decreasing** on I.

A function that is increasing or decreasing on I is called **monotonic** on I.

example: $f(x) = x^2$ decreases on $(-\infty, 0]$ and increases on $[0, \infty)$. It is monotonic on $(-\infty, 0]$ and $[0, \infty)$ but not monotonic on $(-\infty, 0\infty)$.

First derivative test for monotonic functions

Corollary (of the mean value theorem)

Suppose that f is continuous on $[a, b]$ and differentiable on (a, b).
If $f'(x) > 0$ at each point $x \in (a, b)$, then f is increasing on $[a, b]$.
If $f'(x) < 0$ at each point $x \in (a, b)$, then f is decreasing on $[a, b]$.

Proof.

Consider *any* $x_1, x_2 \in [a, b]$ with $x_1 < x_2$. According to the mean value theorem for f on $[x_1, x_2]$ we have

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

for some $c \in (x_1, x_2)$. Since $x_2 - x_1 > 0$ (why?), the sign of the right hand side is the same as the sign of $f'(c)$. Hence, $f(x_2) > f(x_1)$ if $f' > 0$ on (a, b) and $f(x_2) < f(x_1)$ if $f' < 0$ on (a, b). □

Using the first derivative test for monotonic functions

example: Find the critical points of $f(x) = x^3 - 12x - 5$ and identify the intervals on which f is increasing and decreasing.

$$f'(x) = 3x^2 - 12 = 3(x^2 - 4) = 3(x+2)(x-2)$$

$$\Rightarrow x_1 = -2, x_2 = 2$$

These critical points subdivide the natural domain into

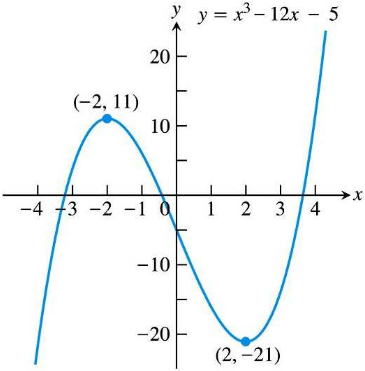
$$(-\infty, -2), (-2, 2), (2, \infty) \ .$$

rule: If $a < b$ are two nearby critical points for f, then f' must be positive on (a, b) *or* negative there. (proof relies on continuity of f').This implies that for finding the sign of f' it suffices to compute $f'(x)$ at *one* $x \in (a, b)$!

here: $f'(-3) = 15 \ , \ f'(0) = -12 \ , \ f'(3) = 15$

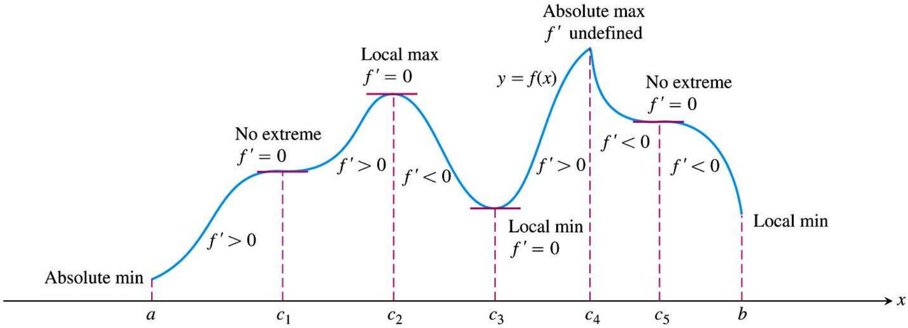
Summary: identifying monotonicity on subintervals

intervals	$-\infty < x < -2$	$-2 < x < 2$	$2 < x < \infty$
sign of f'	$+$	-	$+$
behaviour of f	increasing	decreasing	increasing



First derivatives and local extrema

example:



- whenever f has a minimum, $f' < 0$ to the left and $f' > 0$ to the right
- whenever f has a maximum, $f' > 0$ to the left and $f' < 0$ to the right

$\Rightarrow$ At local extrema, the sign of $f'(x)$ changes!

Checking for local extrema

First Derivative Test for Local Extrema

Suppose that c is a critical point of a continuous function f, and that f is differentiable at every point in some interval containing c except possibly at c itself. Moving across c from left to right,

1. if f' changes from negative to positive at c, then f has a local minimum at c;
2. if f' changes from positive to negative at c, then f has a local maximum at c;
3. if f' does not change sign at c (that is, f' is positive on both sides of c or negative on both sides), then f has no local extremum at c.

Using the first derivative test for local extrema

example: Find the critical points of $f(x) = x^{4/3} - 4x^{1/3}$. Identify the intervals on which f is increasing and decreasing. Find the function's extrema.

$$f'(x) = \frac{4}{3}x^{1/3} - \frac{4}{3}x^{-2/3} = \frac{4}{3}\frac{x-1}{x^{2/3}}$$

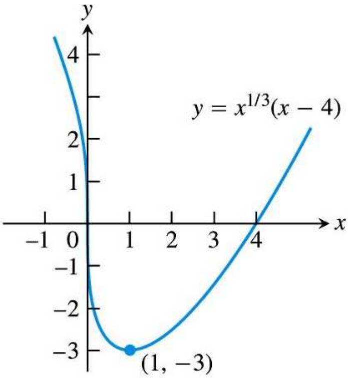
$$\Rightarrow x_1 = 1\ ,\ x_2 = 0$$

intervals	$x < 0$	$0 < x < 1$	$1 < x$
sign of f'	-	-	$+$
behaviour of f	decreasing	decreasing	increasing

Apply the first derivative test to identify local extrema:

- f' does not change sign at $x = 0 \Rightarrow$ no extremum
- f' changes from $-$ to $+$ $\Rightarrow$ local minimum

Summary: geometrical picture



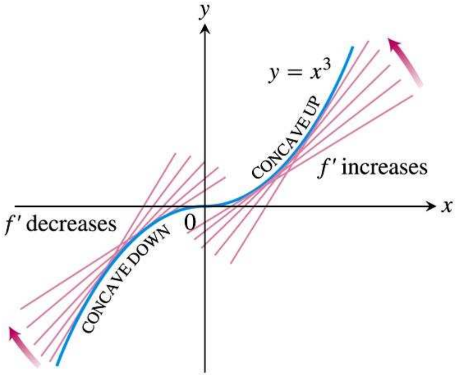
Since $\lim_{x \to \pm\infty} = \infty$, the minimum at $x = 1$ with $f(1) = -3$ is also an *absolute minimum*.
Note that $f'(0) = -\infty$!

Revision of lecture 20

- increasing, decreasing, monotonicity
- checking for local extrema

Concavity of a function

example:



intervals	$x < 0$	$0 < x$
turning of curve	turns to the *right*	turns to the *left*
tangent slopes	decreasing	increasing

The turning or bending behaviour defines the concavity of the curve.

Testing for concavity

DEFINITION Concave Up, Concave Down
The graph of a differentiable function $y = f(x)$ is

(a) **concave up** on an open interval I if f' is increasing on I

(b) **concave down** on an open interval I if f' is decreasing on I.

If f'' exists, the last corollary of the mean value theorem implies that f' *increases* if $f'' > 0$ on I and *decreases* if $f'' < 0$:

The Second Derivative Test for Concavity
Let $y = f(x)$ be twice-differentiable on an interval I.

1. If $f'' > 0$ on I, the graph of f over I is concave up.

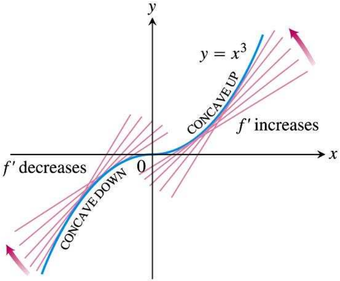
2. If $f'' < 0$ on I, the graph of f over I is concave down.

Applying the concavity test

example 1:

$$y = x^3 \Rightarrow y'' = 6x$$

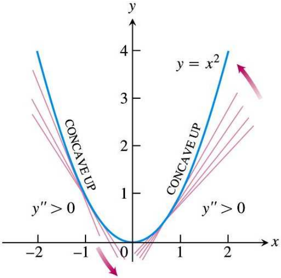
for $(-\infty, 0)$ it is $y'' < 0$:graph concave down;
for $(0, \infty)$ it is $y'' < 0$:
graph concave up



example 2:

$$y = x^2 \Rightarrow y'' = 2 > 0$$

graph is concave up everywhere



Point of inflection

motivation: $y = x^3$ changes concavity at the point $(0, 0)$; specify:

DEFINITION **Point of Inflection**
A point where the graph of a function has a tangent line and where the concavity changes is a **point of inflection**.

At a point of inflection it is $y'' > 0$ on one, $y'' < 0$ on the other side, and either $y'' = 0$ or undefined at such point.

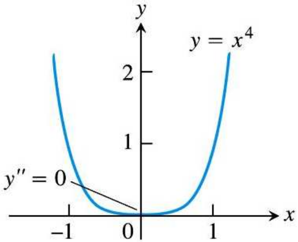
If y'' exists at an inflection point it is $y'' = 0$ and y' has a local maximum or minimum.

Types of inflection points

example 1:

$$y = x^4 \Rightarrow y'' = 12x^2$$

$y''(0) = 0$ but y'' does not change sign: no inflection point at $x = 0$!

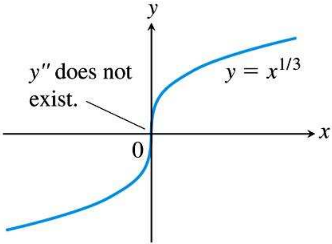


example 2:

$$y = x^{1/3}$$

$$y'' = \left(\frac{1}{3}x^{-\frac{2}{3}}\right)' = -\frac{2}{9}x^{-\frac{5}{3}}$$

$y''(0) = 0$ and y'' *does* change sign: inflection point at $x = 0$ *but* $y''(0)$ does not exist!



Second derivatives at extrema

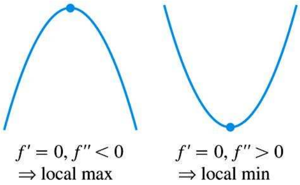
Look at second derivative instead of sign changes at critical points in order to test for local extrema:

THEOREM 5 Second Derivative Test for Local Extrema

Suppose f'' is continuous on an open interval that contains $x = c$.

1. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$.
2. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$.
3. If $f'(c) = 0$ and $f''(c) = 0$, then the test fails. The function f may have a local maximum, a local minimum, or neither.

proof of 1. and 2.:



proof of 3.:
consider $y = -x^4$, $y = x^4$ and $y = x^3$ as examples.
In this case use first derivative test to identify local extrema.

Summary: curve sketching

Strategy for Graphing $y = f(x)$

1. Identify the domain of f and any symmetries the curve may have.
2. Find y' and y''.
3. Find the critical points of f, and identify the function's behavior at each one.
4. Find where the curve is increasing and where it is decreasing.
5. Find the points of inflection, if any occur, and determine the concavity of the curve.
6. Identify any asymptotes.
7. Plot key points, such as the intercepts and the points found in Steps 3–5, and sketch the curve.

Application: curve sketching

example: Sketch the graph of $f(x) = \frac{(x+1)^2}{1+x^2}$.

1. The natural domain of f is$(-\infty, \infty)$; no symmetries about any axis.
2. calculate derivatives:

$$f'(x) = \text{[calculation on whiteboard]} = \frac{2(1-x^2)}{(1+x^2)^2}$$

$$f''(x) = \text{[calculation on whiteboard]} = \frac{4x(x^2-3)}{(1+x^2)^3}$$

3. critical points: f' exists on $(-\infty, \infty)$ with $f'(\pm 1) = 0$ and $f''(-1) = 1 > 0$, $f''(1) = -1 < 0$:
   $(-1, 0)$ is a local minimum and $(1, 2)$ a local maximum.

Example continued 1

4. On $(-\infty, -1)$ it is $f'(x) < 0$: curve decreasing; on $(-1, 1)$ it is $f'(x) > 0$: curve increasing; on $(1, \infty)$ it is $f'(x) < 0$: curve decreasing

5. $f''(x) = 0$ if $x = \pm\sqrt{3}$ or 0; $f'' < 0$ on $(-\infty, -\sqrt{3})$: concave down; $f'' > 0$ on $(-\sqrt{3}, 0)$: concave up; $f'' < 0$ on $(0, \sqrt{3})$: concave down; $f'' > 0$ on $(\sqrt{3}, \infty)$: concave up. Each point is a point of inflection.

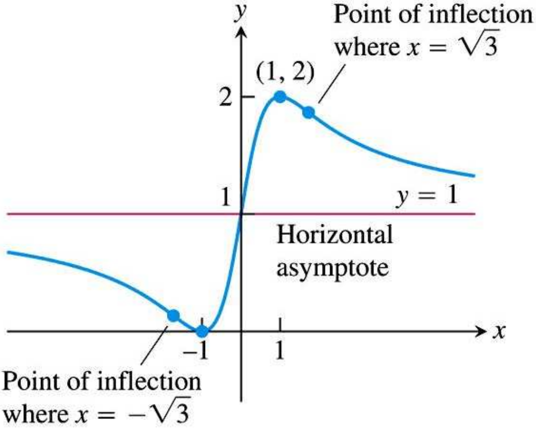
6. calculate asymptotes:

$$f(x) = \frac{(x+1)^2}{1+x^2} = \frac{x^2+2x+1}{1+x^2} = \frac{1+2/x+1/x^2}{1/x^2+1}$$

$f(x) \to 1^+$ as $x \to \infty$ and $f(x) \to 1^-$ as $x \to -\infty$: $y = 1$ is a horizontal asymptote. No vertical asymptotes.

Example continued 2

8. sketch the curve:



Learning about functions from derivatives