MTH4100 Calculus I

Week 6 (Thomas' Calculus Sections 3.5 to 4.2)

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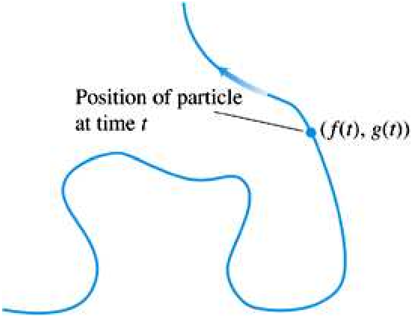
Revision of lecture 15

- differentiation rules
- higher-order derivatives
- derivatives of trigonometric functions
- the chain rule

Parametric equations

example:



Describe a point moving in the *xy*-plane as a function of a parameter t ("time") by two functions

$$x = x(t)\ , \quad y = y(t)\ .$$

This *may* be the graph of a function, but it need not be.

Parametric curve

DEFINITION Parametric Curve
If x and y are given as functions

$$x = f(t), \qquad y = g(t)$$

over an interval of t-values, then the set of points $(x, y) = (f(t), g(t))$ defined by these equations is a **parametric curve**. The equations are **parametric equations** for the curve.

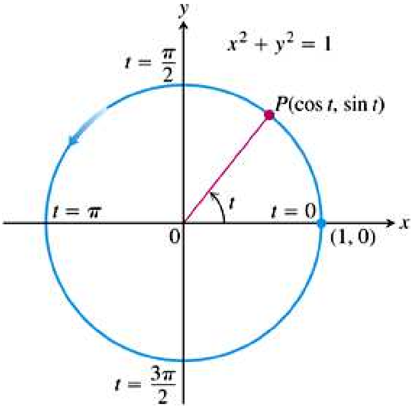
The variable t is a parameter for the curve.
If $t \in [a, b]$, which is called a parameter interval, then

$$(f(a), g(a)) \text{ is the initial point, and}$$
$$(f(b), g(b)) \text{ is the terminal point.}$$

Equations and interval constitute a parametrisation of the curve.

Motion on a circle

example: parametrisation $\boxed{x = \cos t\ , \quad y = \sin t\ , \quad 0 \le t \le 2\pi}$



The above parametric equations describe motion on the unit circle:

The motion starts at initial point $(1, 0)$ at $t = 0$ and traverses the circle $x^2 + y^2 = 1$ counterclockwise once, ending at the terminal point $(1, 0)$ at $t = 2\pi$.

Moving along a parabola

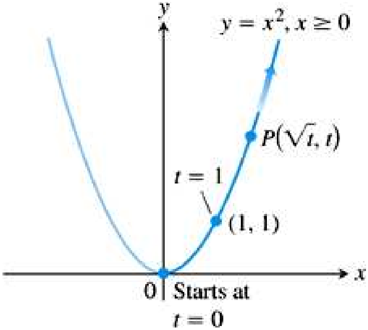
example: parametrisation $\boxed{x = \sqrt{t}\ , \quad y = t\ , \quad t \geq 0}$

What is the path defined by these equations?

Solve for $y = f(x)$:

$$y = t\ ,\ x^2 = t\ \Rightarrow y = x^2$$

Note that the domain of f is *only* $[0, \infty)$!



Parametrising a line segment

example:

Find a parametrisation for the line segment from $(-2,1)$ to $(3,5)$.

- Start at $(-2,1)$ for $t=0$ by making the ansatz ("educated guess")

$$x = -2 + at \ , \quad y = 1 + bt \ .$$

- Implement the terminal point at $(3,5)$ for $t=1$:

$$3 = -2 + a \ , \quad 5 = 1 + b \ .$$

- We conclude that $a = 5$, $b = 4$.
- Therefore, the solution *based on our ansatz* is:

$$\boxed{x = -2 + 5t \ , \ y = 1 + 4t \ , \ 0 \leq t \leq 1} \ ,$$

which indeed defines a straight line.

Slopes of parametrised curves

A parametrised curve $x = f(t)$, $y = g(t)$ is differentiable at t if f and g are differentiable at t.
If y is a differentiable function of x, say $y = h(x)$, then $y = h(x(t))$ and by the chain rule

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} .$$

Solving for dy/dx yields the

Parametric Formula for dy/dx
If all three derivatives exist and $dx/dt \neq 0$,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} .$$

Moving along an ellipse

example: Describe the motion of a particle whose position $P(x, y)$ at time t is given by

$$\boxed{x = a\cos t\ , \quad y = b\sin t\ , \quad 0 \le t \le 2\pi}$$

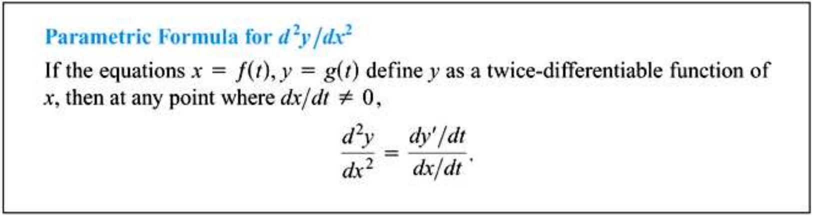
and compute the slope at P.

- Find the equation in (x, y) by eliminating t: Using $\cos t = x/a$, $\sin t = y/b$ and $\cos^2 t + \sin^2 t = 1$ we obtain
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\ ,$$
which is the equation of an ellipse.
- With $\frac{dx}{dt} = -a\sin t$ and $\frac{dy}{dt} = b\cos t$ the parametric formula yields
$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{b\cos t}{-a\sin t}\ .$$
Eliminating t again we obtain $\quad \frac{dy}{dx} = -\frac{b^2}{a^2}\frac{x}{y}.$

Higher-order derivatives

$$\textbf{motivation:}\ y' = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \ \Rightarrow\ y'' = ?$$

Remember $y'' = (y')'$: *put y' in place of y*


Parametric Formula for d^2y/dx^2

If the equations $x = f(t), y = g(t)$ define y as a twice-differentiable function of x, then at any point where $dx/dt \neq 0$,

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}.$$

example about ellipse continued: $y' = -\frac{b}{a}\frac{\cos t}{\sin t}$ gives

$$y'' = \frac{\frac{d}{dt}\left[-\frac{b}{a}\frac{\cos t}{\sin t}\right]}{-a\sin t} = -\frac{b}{a^2}\frac{1}{\sin^3 t} = -\frac{b^4}{a^2}\frac{1}{y^3}$$

Revision of lecture 16

- parametric equations
- parametric differentiation

Implicit differentiation

problem: We want to compute y' but do not have an explicit relation $y = f(x)$ available. Rather, we have an implicit relation

$$F(x, y) = 0$$

between x and y.

example:

$$F(x, y) = x^2 + y^2 - 1 = 0 \, .$$

solutions:

1. Use *parametrisation*, for example, $x = \cos t$, $y = \sin t$ for the unit circle: see previous lecture.
2. If no obvious parametrisation of $F(x, y) = 0$ is possible:

use $\boxed{\text{implicit differentiation}}$

Differentiating implicitly

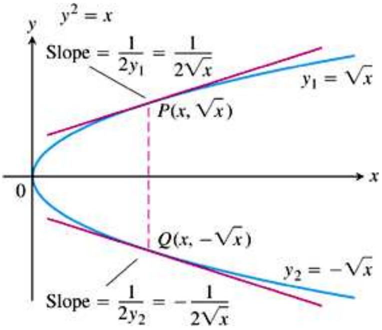
example: Given $y^2 = x$, compute y'.

new method by differentiating *implicitly*:

- Differentiating *both sides* of the equation gives $2yy' = 1$.
- Solving for y' we get $\boxed{y' = \frac{1}{2y}}$.

Compare with differentiating *explicitly*:

- For $y^2 = x$ we have the two *explicit solutions* $|y| = \sqrt{x} \Rightarrow y_{1,2} = \pm\sqrt{x}$ with derivatives $\boxed{y'_{1,2} = \pm\frac{1}{2\sqrt{x}}}$.
- Compare with solution above: substituting $y = y_{1,2} = \pm\sqrt{x}$ therein reproduces the explicit result.



General recipe

Implicit Differentiation

1. Differentiate both sides of the equation with respect to x, treating y as a differentiable function of x.
2. Collect the terms with dy/dx on one side of the equation.
3. Solve for dy/dx.

example: the ellipse again, $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

1. $\dfrac{2x}{a^2} + \dfrac{2yy'}{b^2} = 0$
2. $\dfrac{2yy'}{b^2} = -\dfrac{2x}{a^2}$
3. $y' = -\dfrac{b^2}{a^2}\dfrac{x}{y}$, as obtained via parametrisation in the previous lecture.

Higher-order derivatives

Implicit differentiation also works for higher-order derivatives.
example:

- For the ellipse we had after differentiation:

$$\frac{2x}{a^2} + \frac{2yy'}{b^2} = 0$$

- Differentiate again:

$$\frac{2}{a^2} + \frac{2(y'^2 + yy'')}{b^2} = 0$$

- Now substitute our previous result $y' = -\frac{b^2}{a^2}\frac{x}{y}$ and simplify (this takes a few steps):

$$y'' = -\frac{b^4}{a^2}\frac{1}{y^3} \ ,$$

as also obtained via parametrisation in the previous lecture.

Power rule for rational powers

Another application: Differentiate $y = x^{\frac{p}{q}}$ using implicit differentiation.

- write $$y^q = x^p$$
- differentiate: $$qy^{q-1}y' = px^{p-1}$$
- solve for y' as a function of x:

$$y' = \frac{p}{q}\frac{x^{p-1}}{y^{q-1}} = \frac{p}{q}\frac{x^p}{y^q}\frac{y}{x} = \frac{p}{q}\frac{y}{x} = \frac{p}{q}\frac{x^{\frac{p}{q}}}{x} = \frac{p}{q}x^{\frac{p}{q}-1}$$

THEOREM 4 Power Rule for Rational Powers

If p/q is a rational number, then $x^{p/q}$ is differentiable at every interior point of the domain of $x^{(p/q)-1}$, and

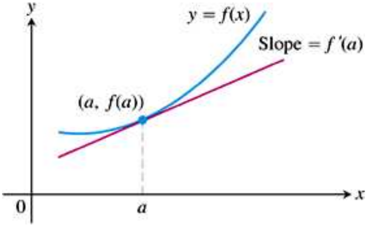
$$\frac{d}{dx}x^{p/q} = \frac{p}{q}x^{(p/q)-1}.$$

note: Above we have silently assumed that y' exists! Therefore we have 'motivated' but not (yet) proven the theorem.

Revision of lecture 17

- implicit differentiation
- application to higher-order derivatives
- power rule for rational powers

Linearisation



"Close to" the point $(a, f(a))$, the tangent

$$y = f(a) + f'(a)(x - a)$$

(point-slope form)

is a "good" approximation for $y = f(x)$.

DEFINITIONS Linearization, Standard Linear Approximation

If f is differentiable at $x = a$, then the approximating function

$$L(x) = f(a) + f'(a)(x - a)$$

is the **linearization** of f at a. The approximation

$$f(x) \approx L(x)$$

of f by L is the **standard linear approximation** of f at a. The point $x = a$ is the **center** of the approximation.

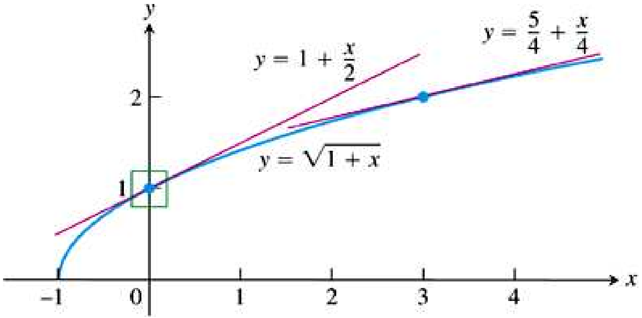
Finding a linearisation

example: Compute the linearisation for $f(x) = \sqrt{1+x}$ at $a = 0$.
Use

$$\boxed{L(x) = f(a) + f'(a)(x-a)}:$$

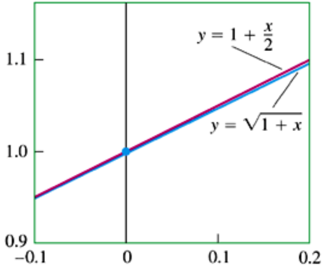
We have $f(0) = 1$ and with $f'(x) = \frac{1}{2}(1+x)^{-1/2}$ we get $f'(0) = \frac{1}{2}$, so

$$L(x) = 1 + \frac{1}{2}x\ .$$



How accurate is this approximation?

Magnify region around $x = 0$:



Approximation	True value	|True value − approximation|
$\sqrt{1.2} \approx 1 + \frac{0.2}{2} = 1.10$	1.095445	$<10^{-2}$
$\sqrt{1.05} \approx 1 + \frac{0.05}{2} = 1.025$	1.024695	$<10^{-3}$
$\sqrt{1.005} \approx 1 + \frac{0.005}{2} = 1.00250$	1.002497	$<10^{-5}$

Applications of linearisations and further theory

- **why useful?** simplify problems, solve equations analytically, ...
- Make phrases like *"close to a point* $(a, f(a))$ *the linearisation is a good approximation"* mathematically precise in terms of **differentials**.

Extreme values of functions

DEFINITIONS **Absolute Maximum, Absolute Minimum**

Let f be a function with domain D. Then f has an **absolute maximum** value on D at a point c if

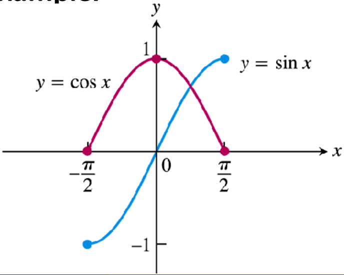
$$f(x) \leq f(c) \qquad \text{for all } x \text{ in } D$$

and an **absolute minimum** value on D at c if

$$f(x) \geq f(c) \qquad \text{for all } x \text{ in } D.$$

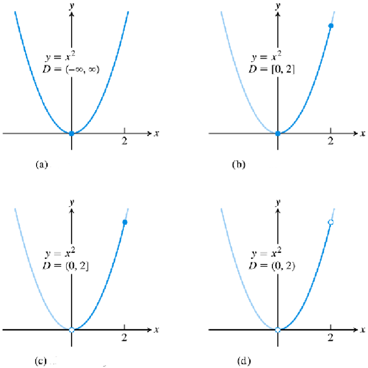
These values are also called absolute **extrema**, or **global** extrema.

example:



Same rule for different domains yields different extrema

example:



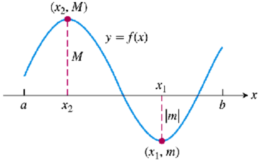
	Domain	abs. max.	abs. min.
(a)	$(-\infty, \infty)$	none	0, at 0
(b)	$[0, 2]$	4, at 2	0, at 0
(c)	$(0, 2]$	4, at 2	none
(d)	$(0, 2)$	none	none

Existence of a global maximum and minimum

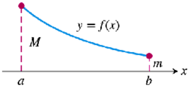
THEOREM 1 The Extreme Value Theorem

If f is continuous on a closed interval $[a, b]$, then f attains both an absolute maximum value M and an absolute minimum value m in $[a, b]$. That is, there are numbers x_1 and x_2 in $[a, b]$ with $f(x_1) = m$, $f(x_2) = M$, and $m \leq f(x) \leq M$ for every other x in $[a, b]$ (Figure 4.3).

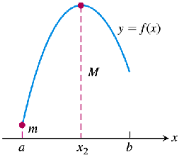
examples:



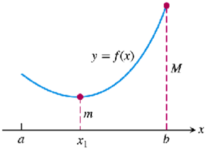
Maximum and minimum at interior points



Maximum and minimum at endpoints



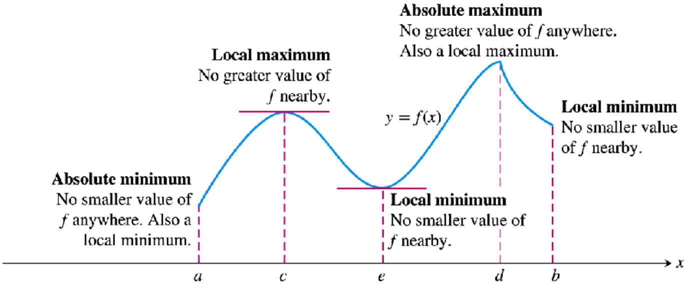
Maximum at interior point, minimum at endpoint



Minimum at interior point, maximum at endpoint

counterexamples?

Local (relative) extreme values



DEFINITIONS **Local Maximum, Local Minimum**

A function f has a **local maximum** value at an interior point c of its domain if

$$f(x) \leq f(c) \qquad \text{for all } x \text{ in some open interval containing } c.$$

A function f has a **local minimum** value at an interior point c of its domain if

$$f(x) \geq f(c) \qquad \text{for all } x \text{ in some open interval containing } c.$$

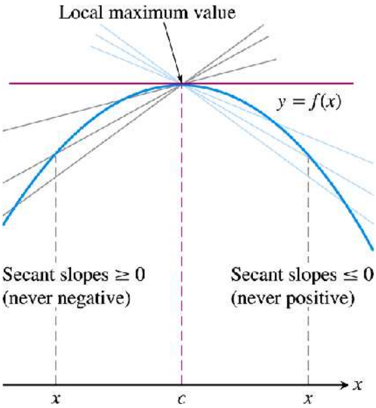
and extension of def. to endpoints via half-open intervals at endpoints

Finding extreme values

Theorem

If f has a local maximum or minimum value at an interior point c of its domain, and if f' is defined at c, then $f'(c) = 0$.

basic idea of the proof:



First derivative theorem for local extrema: proof

Theorem

If f has a local maximum or minimum value at an interior point c of its domain, and if f' is defined at c, then $f'(c) = 0$.

Proof.

If at a local *maximum* c the derivative

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

exists, then

$$f'(c) = \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h} \leq 0$$

and

$$f'(c) = \lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h} \geq 0$$

so that $f'(c) = 0$. (Similarly for *minimum*.) □

note: the converse is false! *(counterexample)*

Conditions for extreme values

Where can a function f possibly have an extreme value? Recall the

Theorem

If f has a local maximum or minimum value at an interior point c of its domain, and if f' is defined at c, then $f'(c) = 0$.

answer:

1. at interior points where $f' = 0$
2. at interior points where f' is not defined
3. at endpoints of the domain of f.

combine 1 and 2:

DEFINITION Critical Point

An interior point of the domain of a function f where f' is zero or undefined is a **critical point** of f.