MTH4100 Calculus I

Week 3 (Thomas' Calculus Sections 2.1 to 2.4)

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Revision of Lecture 6

- composition of functions
  note: $(f \circ g)(x)$ is *different* from $(f \cdot g)(x)$!
- shifting and scaling of functions: transform graph of

$$y = f(x)$$

to graph of

$$\boxed{y = cf(ax + b) + d}$$

- trigonometric functions

Reading Assignment

Reminder: read

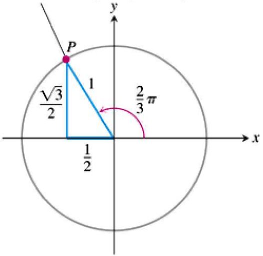
Thomas' Calculus:

- short **Paragraph** about ellipses, p.44/45
- **Section 1.6** about trigonometric functions, particularly about
  - symmetries
  - law of cosines
  - transformations of trigonometric graphs

Periodic functions

note: for angle of measure θ and angle of measure $\theta + 2\pi$ we have the *very same* trigonometric function values

example:



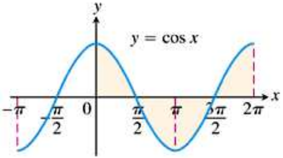
$$\sin(\theta + 2\pi) = \sin\theta$$
$$\cos(\theta + 2\pi) = \cos\theta$$
$$\tan(\theta + 2\pi) = \tan\theta$$

and so on

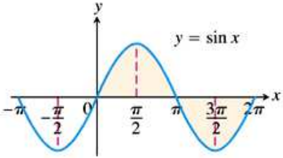
DEFINITION Periodic Function

A function $f(x)$ is **periodic** if there is a positive number p such that $f(x + p) = f(x)$ for every value of x. The smallest such value of p is the **period** of f.

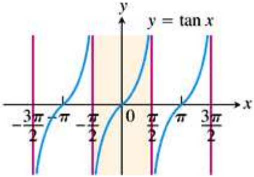
Graphs of trigonometric functions



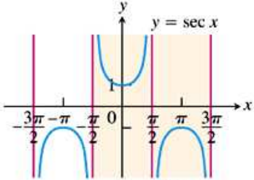
Domain: $-\infty < x < \infty$
Range: $-1 \le y \le 1$
Period: 2π
(a)



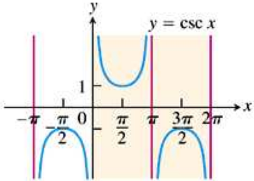
Domain: $-\infty < x < \infty$
Range: $-1 \le y \le 1$
Period: 2π
(b)



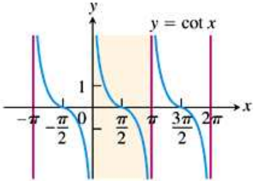
Domain: $x \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$
Range: $-\infty < y < \infty$
Period: π
(c)



Domain: $x \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$
Range: $y \le -1$ and $y \ge 1$
Period: 2π
(d)



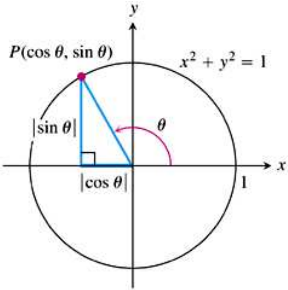
Domain: $x \neq 0, \pm\pi, \pm2\pi, \ldots$
Range: $y \le -1$ and $y \ge 1$
Period: 2π
(e)



Domain: $x \neq 0, \pm\pi, \pm2\pi, \ldots$
Range: $-\infty < y < \infty$
Period: π
(f)

An important trigonometric identity

Since $x = r\cos\theta$ and $y = r\sin\theta$ by definition, for a triangle with $r = 1$ we immediately have $\boxed{\cos^2\theta + \sin^2\theta = 1}$ (why?)



This is an example of an identity, i.e., an equation that remains true *regardless of the values of any variables that appear within it*.

counterexample:

$$\cos\theta = 1$$

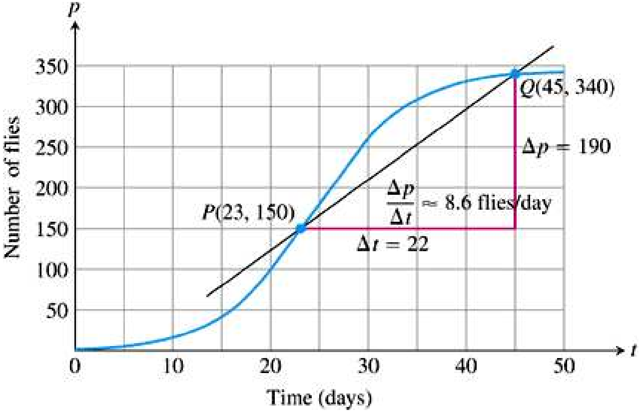
This is *not* an identity, because it is only true for *some* values of θ, not all.

First part of Chapter 2:

Limits

Average rate of change

example: growth of a fruit fly population measured experimentally

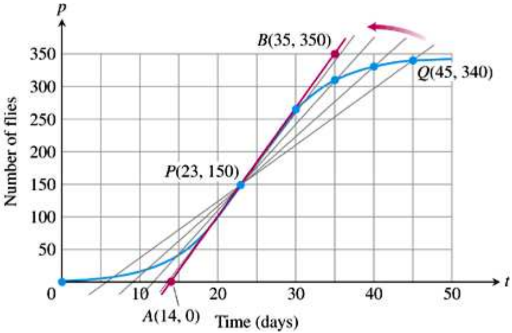


- average rate of change from day 23 to day 45?
- growth rate on day a specific day, e.g., day 23?

Growth rate on a specific day

study the average rates of change over increasingly short time intervals starting at day 23:

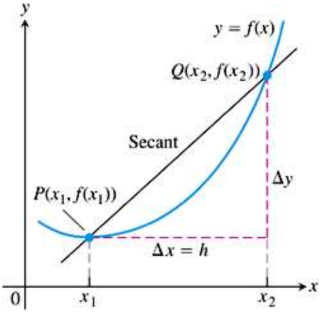
Q	Slope of $PQ = \Delta p / \Delta t$ (flies/day)
(45, 340)	$\frac{340 - 150}{45 - 23} \approx 8.6$
(40, 330)	$\frac{330 - 150}{40 - 23} \approx 10.6$
(35, 310)	$\frac{310 - 150}{35 - 23} \approx 13.3$
(30, 265)	$\frac{265 - 150}{30 - 23} \approx 16.4$



lines approach the red tangent at point P with slope

$$\frac{350 - 0}{35 - 14} \simeq 16.7 \text{ flies/day}$$

Summary: average rate of change and limit



$$x_2 = x_1 + h$$

DEFINITION **Average Rate of Change over an Interval**

The **average rate of change** of $y = f(x)$ with respect to x over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}, \qquad h \neq 0.$$

Animation!

Limits

To move from

average rates of change

to

instantaneous rates of change

we need to consider

limits

Informal definition of a limit

Definition (informal)

Let $f(x)$ be defined on an open interval about x_0 *except possibly at* x_0 *itself*. If $f(x)$ gets arbitrarily close to the number L (as close to L as we like) for all x sufficiently close to x_0, we say that f approaches the limit L as x approaches x_0, and we write

$$\lim_{x \longrightarrow x_0} f(x) = L \,,$$

which is read "the limit of $f(x)$ as x approaches x_0."

This is an *informal* definition, because:

What do "arbitrarily close" and "sufficiently close" mean?

This will be made mathematically precise later on ...

Behaviour of a function near a point

example: How does the function

$$f(x) = \frac{x^2 - 1}{x - 1}$$

behave near $x_0 = 1$?

- problem: $f(x)$ is not defined for $x_0 = 1$
- but: we can *simplify* for $x \neq 1$:

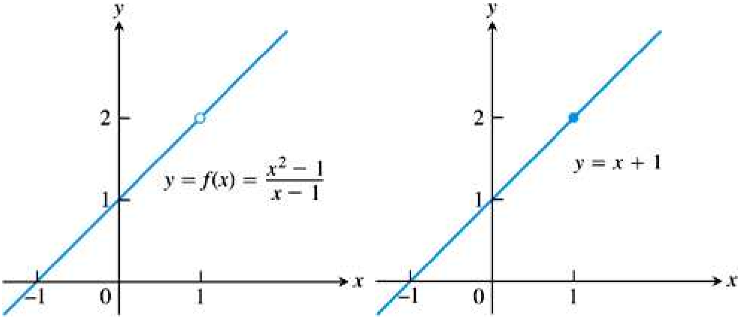
$$f(x) = \frac{(x-1)(x+1)}{x-1} = x + 1 \text{ for } x \neq 1$$

- this *suggests* that

$$\lim_{x \to 1} f(x) = 1 + 1 = 2$$

Limit: a geometric view

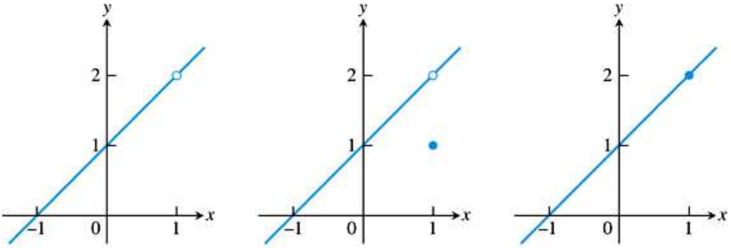
graphs of these two functions:



We say that $f(x)$ approaches the limit 2 as x approaches 1 and write

$$\lim_{x \to 1} f(x) = 2$$

The limit value does not depend on how the function is defined at x_0



(a) $f(x) = \frac{x^2 - 1}{x - 1}$ (b) $g(x) = \begin{cases} \frac{x^2-1}{x-1}, & x \neq 1 \\ 1, & x = 1 \end{cases}$ (c) $h(x) = x + 1$

All these functions have limit 2 as $x \to 1$!

However, only for h we have equality of limit and function value:

$$\lim_{x \to 1} h(x) = h(1)$$

Revision of Lecture 7

- periodicity of functions
- average rate of change
- intuitive approach to limits

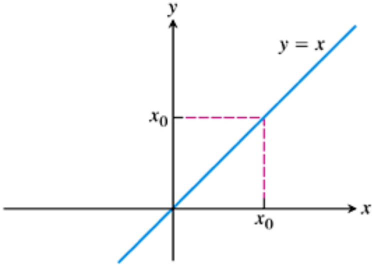
Recall our informal definition of limit

Definition (informal)

Let $f(x)$ be defined on an open interval about x_0 *except possibly at* x_0 *itself*. If $f(x)$ gets arbitrarily close to the number L (as close to L as we like) for all x sufficiently close to x_0, we say that f approaches the limit L as x approaches x_0, and we write

$$\lim_{x \longrightarrow x_0} f(x) = L \, .$$

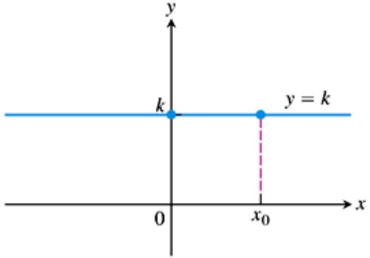
Limits at every point



for any value of x_0 we have

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} x = x_0$$

example: $\lim_{x \to 3} x = 3$



for any value of x_0 we have

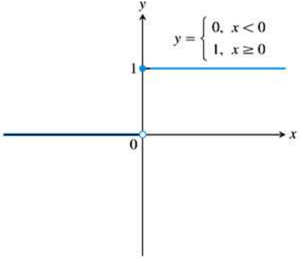
$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} k = k$$

example: for $k = 5$ we have

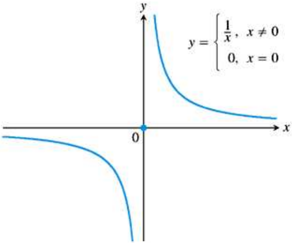
$$\lim_{x \to -12} 5 = \lim_{x \to 7} 5 = 5$$

Limits can fail to exist!

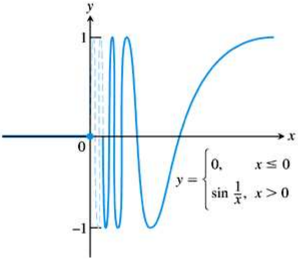
no limit — three different **examples:**



values that *jump*



values that *grow too large*



values that *oscillate too much*

Finding limits of simple functions

We have just "convinced ourselves" that for real constants k and c

$$\lim_{x \longrightarrow c} x = c$$

and

$$\lim_{x \longrightarrow c} k = k \quad .$$

The following important theorem provides the basis to calculate limits of functions that are arithmetic combinations of the above two functions (like polynomials, rational functions, powers):

Limit laws

Theorem

If L, M, c and k are real numbers and

$$\lim_{x\to c} f(x) = L \text{ and } \lim_{x\to c} g(x) = M \text{ , then}$$

1. *Sum Rule:* $\lim_{x\to c}(f(x)+g(x)) = L+M$
   The limit of the sum of two functions is the sum of their limits.
2. *Difference Rule:* $\lim_{x\to c}(f(x)-g(x)) = L-M$
3. *Product Rule:* $\lim_{x\to c}(f(x)\cdot g(x)) = L\cdot M$
4. *Constant Multiple Rule:* $\lim_{x\to c}(k\cdot f(x)) = k\cdot L$
5. *Quotient Rule:* $\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{L}{M}$, $M\neq 0$
6. *Power Rule: If s and r are integers with no common factor and $s\neq 0$, then*
   $$\lim_{x\to c}(f(x))^{r/s} = L^{r/s}$$
   provided that $L^{r/s}$ is a real number. (If s is even, we assume that $L>0$.)

Using limit laws

... concerning proofs of this theorem see later ...

examples:

- $\lim_{x\to c}(x^3 - 4x + 2) =$ (rules 1,2)
  $= \lim_{x\to c} x^3 - \lim_{x\to c} 4x + \lim_{x\to c} 2 =$ (rules 3 or 6,4)
  $= c^3 - 4c + 2$
- $\displaystyle\lim_{x\to c} \frac{x^4 + x^2 - 1}{x^2 + 5} = \frac{c^4 + c^2 - 1}{c^2 + 5}$ (rules 5,1,2,3 or 6)
- $\displaystyle\lim_{x\to -2} \sqrt{4x^2 - 3} = \sqrt{4(-2)^2 - 3} = \sqrt{13}$ (rules 6,2, 3 or 6,4)

So "sometimes" you can just *substitute the value of* x.

Some consequences of the limit laws theorem

THEOREM 2 Limits of Polynomials Can Be Found by Substitution

If $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, then

$$\lim_{x \to c} P(x) = P(c) = a_n c^n + a_{n-1}c^{n-1} + \cdots + a_0.$$

THEOREM 3 Limits of Rational Functions Can Be Found by Substitution If the Limit of the Denominator Is Not Zero

If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then

$$\lim_{x \to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

Eliminating zero denominators algebraically

example: Evaluate

$$\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - x}$$

- substitution of $x = 1$? *No!*
- *but* algebraic simplification is possible:

$$\frac{x^2 + x - 2}{x^2 - x} = \frac{(x+2)(x-1)}{x(x-1)} = \frac{x+2}{x}\,, \; x \neq 1$$

- therefore,

$$\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - x} = \lim_{x \to 1} \frac{x+2}{x} = 3$$

Creating and cancelling a common factor

$$\lim_{x\to 0} \frac{\sqrt{x^2+100}-10}{x^2}$$

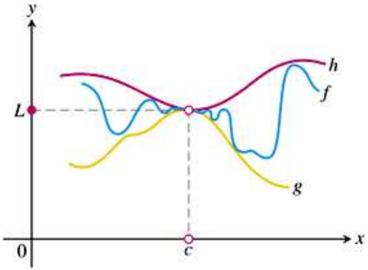
- substitution of $x = 0$?
- trick: algebraic simplification

$$\begin{aligned}
\frac{\sqrt{x^2+100}-10}{x^2} &= \frac{\sqrt{x^2+100}-10}{x^2}\,\frac{\sqrt{x^2+100}+10}{\sqrt{x^2+100}+10} \\
&= \frac{(x^2+100)-100}{x^2(\sqrt{x^2+100}+10)} \\
&= \frac{1}{\sqrt{x^2+100}+10}
\end{aligned}$$

- therefore

$$\lim_{x\to 0} \frac{\sqrt{x^2+100}-10}{x^2} = \lim_{x\to 0} \frac{1}{\sqrt{x^2+100}+10} = \frac{1}{20}$$

The Sandwich Theorem



function f sandwiched between g and h that have the same limit

THEOREM 4 The Sandwich Theorem

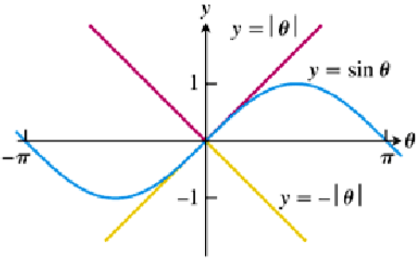
Suppose that $g(x) \le f(x) \le h(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself. Suppose also that

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L.$$

Then $\lim_{x \to c} f(x) = L$.

Application

example: Show that $\lim_{\theta \to 0} \sin\theta = 0$.

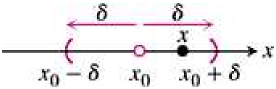


(a)

- From the definition of $\sin\theta$ it follows that
$$-|\theta| \le \sin\theta \le |\theta|$$
- We have
$$\lim_{\theta \to 0}(-|\theta|) = \lim_{\theta \to 0}|\theta| = 0$$
- Using the sandwich theorem, we therefore conclude that
$$\lim_{\theta \to 0} \sin\theta = 0$$
- Similarly, one can prove that $\lim_{\theta \to 0} \cos\theta = 1$

Limits: trying to be more precise

- We have used informal phrases such as "sufficiently close". But what do they mean?
- A picture might help:



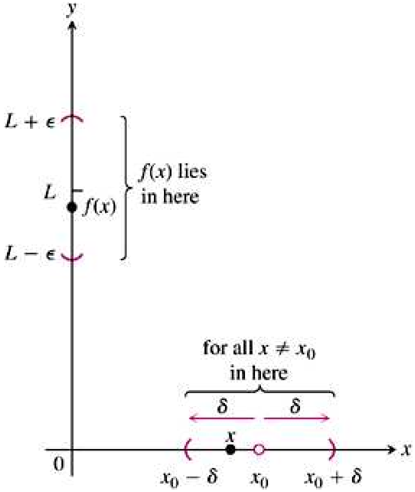
- Let's be precise: instead of

"for all x sufficiently close to x_0 ..."

write

"choose $\delta > 0$ such that for all x, $0 < |x - x_0| < \delta$ ..."

Revisiting the definition of limit



The **informal definition** was:

Let $f(x)$ be defined on an open interval about x_0 *except possibly at x_0 itself*. If $f(x)$ gets arbitrarily close to L for all x sufficiently close to x_0, we say that f approaches the limit L as x approaches x_0, and we write

$$\lim_{x \longrightarrow x_0} f(x) = L \,.$$

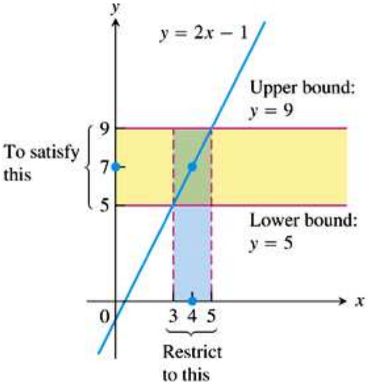
Think of a function as a machine and of ϵ as the desired *output tolerance* depending on the *input accuracy*.

Revision of Lecture 8

- limit laws
- Some useful "tricks"
- $\epsilon - \delta$ definition of limit

Output-input relation in limits

example: output-input tolerance for a given ϵ of a linear function



If we want to keep y within $\epsilon = 2$ units of $y_0 = 7$, we need to keep x within $\delta = 1$ unit of $x_0 = 4$.

The precise definition of a limit

Animation?! (or blackboard...)

DEFINITION Limit of a Function

Let $f(x)$ be defined on an open interval about x_0, except possibly at x_0 itself. We say that the **limit of $f(x)$ as x approaches x_0 is the number L**, and write

$$\lim_{x \to x_0} f(x) = L,$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x,

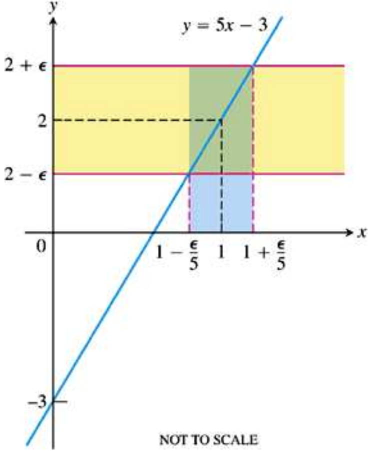
$$0 < |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

This is a crucial concept!!

If you have trouble to understand it: read p.91-93 for further details!

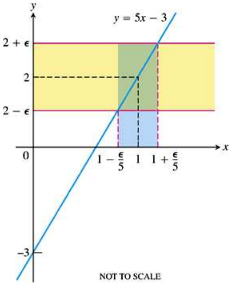
Testing the definition, part 1

example: show that $\lim_{x\to 1}(5x-3) = 2$; graphically:



Testing the definition, part 2

example: show that $\lim_{x\to 1}(5x-3)=2$; algebraically:



- $|f(x)-L|<\epsilon$: this is what we want to be fulfilled!
  substitute: $|(5x-3)-2|<\epsilon$
  $\Leftrightarrow \quad |5x-5|<\epsilon$
  $\Leftrightarrow \boxed{|x-1|<\frac{1}{5}\epsilon}$ (1)
- given this inequality, we now need to find a $\delta>0$ such that
  $0<|x-x_0|<\delta$ is fulfilled
  substitute: $\boxed{0<|x-1|<\delta}$ (2)
- *matching* (1) with (2) suggests to choose $\delta=\frac{1}{5}\epsilon$, because:
  if $0<|x-1|<\delta=\epsilon/5$, then $|f(x)-2|=5|x-1|<5\delta=\epsilon$
  for all ϵ.

General recipe of how to apply the definition

How to Find Algebraically a δ for a Given f, L, x_0, and $\epsilon > 0$

The process of finding a $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon$$

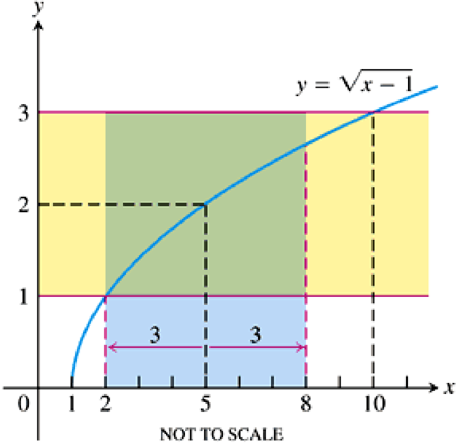
can be accomplished in two steps.

1. *Solve the inequality* $|f(x) - L| < \epsilon$ to find an open interval (a, b) containing x_0 on which the inequality holds for all $x \neq x_0$.
2. *Find a value of* $\delta > 0$ that places the open interval $(x_0 - \delta, x_0 + \delta)$ centered at x_0 inside the interval (a, b). The inequality $|f(x) - L| < \epsilon$ will hold for all $x \neq x_0$ in this δ-interval.

A slightly more complicated example, part 1

For the limit $\lim_{x\to 5}\sqrt{x-1} = 2$ and $\epsilon = 1$, find a $\delta > 0$ such that for all x

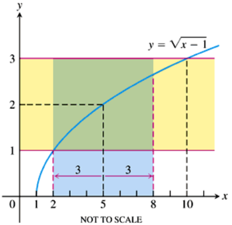
$$0 < |x-5| < \delta \Rightarrow |\sqrt{x-1} - 2| < 1$$



asymmetric preimage of the ϵ-interval!

A slightly more complicated example, part 2

Find a $\delta > 0$ such that $|\sqrt{x-1} - 2| < 1$ for all $0 < |x-5| < \delta$:



1. solve $|f(x) - L| < \epsilon$:
   substitute: $|\sqrt{x-1} - 2| < 1$
   $\Leftrightarrow \quad -1 < \sqrt{x-1} - 2 < 1$
   $\Leftrightarrow \quad 1 < \sqrt{x-1} < 3$
   $\Leftrightarrow \quad 2 < x < 10$
   therefore $(a, b) = (2, 10)$
2. find δ:
   find the distance from $x_0 = 5$ to the *nearest endpoint* of $(2, 10)$, which is $\delta = 3$. Then

   $$x \in (5 - \delta, 5 + \delta) = (2, 8) \subset (2, 10)$$

   means $0 < |x - 5| < 3$, which implies

   $$|\sqrt{x-1} - 2| < 1$$

Proof of the previous limit laws theorem

note:

the $\epsilon - \delta$ definition of limit can be used to *rigorously prove* our limit laws theorem

see p.97 for a proof of the Sum Rule,

$$\lim_{x \to c}(f(x) + g(x)) = L + M$$

and Appendix 2 for a proof of product and quotient rules

One-sided limits

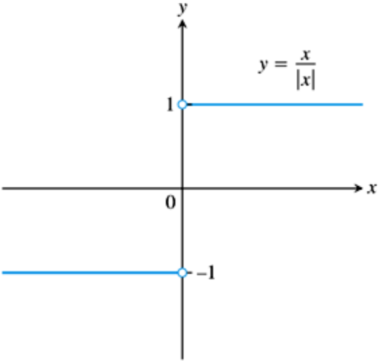
- To have a *limit* L as $x \to c$, a function f must be defined on both sides of c (two-sided limit)
- If f fails to have a limit as $x \to c$, it may still have a one-sided limit if the approach is only from the right (*right-hand limit*) or from the left (*left-hand limit*)
- We write

$$\boxed{\lim_{x\to c^+} f(x) = L} \text{ or } \boxed{\lim_{x\to c^-} f(x) = M}$$

- The symbol $x \to c^+$ means that we only consider values of x *greater than* c. The symbol $x \to c^-$ means that we only consider values of x *less than* c.

Jump function

example:



- $\lim_{x \to 0^+} f(x) = 1$
- $\lim_{x \to 0^-} f(x) = -1$
- $\lim_{x \to 0} f(x)$
  does not exist