MTH4100 Calculus I

Week 2 (Thomas' Calculus Sections 1.3 to 1.6)

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Revision of Lecture 3

- Some absolute value properties and their proofs:

$$|-a| = |a|\ ,\ |ab| = |a|\,|b|\ ,\ \left|\frac{a}{b}\right| = \frac{|a|}{|b|} \text{ for } b \neq 0$$

- Three important inequalities and their proofs:
  - Triangle inequality

$$|a+b| \leq |a| + |b|$$

  - Arithmetic-geometric mean inequality

$$\sqrt{ab} \leq \frac{1}{2}(a+b) \qquad \text{for } a, b \geq 0$$

  - Cauchy-Schwarz inequality

$$(ac+bd)^2 \leq (a^2+b^2)(c^2+d^2)$$

Reading Assignment

Reminder: read

Thomas' Calculus, Section 1.2:

Lines, Circles, and Parabolas

What is a function?

examples:

height of the floor of the lecture hall depending on distance; stock market index depending on time; volume of a sphere depending on radius

What do we mean when we say

y is a function of x?

Symbolically, we write $y = f(x)$, where

- x is the independent variable (input value of f)
- y is the dependent variable (output value of f at x)
- f is a function ("rule that assigns x to y" – further specify!)

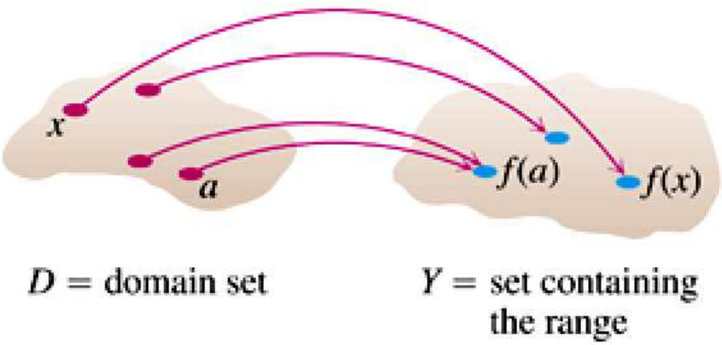
a function acts like a "little machine":

Important: uniqueness – only *one value* $f(x)$ for every x!

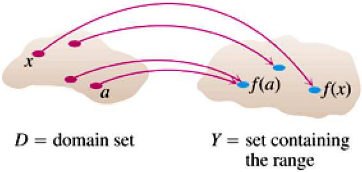
Definition of a function

Definition

A function from a set D to a set Y is a rule that assigns a *unique* (single) element $f(x) \in Y$ to each element $x \in D$.



Domain, range and some notation



- The set D of all possible *input values* is called the domain of f.
- The set R of all possible *output values* of $f(x)$ as x varies throughout D is called the range of f.

note: $R \subseteq Y$!

- We write f *maps* D *to* Y symbolically as

$$f : D \to Y$$

- We write f *maps* x *to* $y = f(x)$ symbolically as

$$f : x \mapsto y = f(x)$$

Note the different arrow symbols used!

Natural domain

The natural domain is the largest set of real x which the rule f can be applied to.

examples:

Function	**Domain** $x \in D$	**Range** $y \in R$
$y = x^2$	$(-\infty, \infty)$	$[0, \infty)$
$y = 1/x$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$
$y = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$y = \sqrt{1 - x^2}$	$[-1, 1]$	$[0, 1]$

note: A function is specified by the rule f *and* the domain D:

$$f : x \mapsto y = x^2 \ , \quad D(f) = [0, \infty)$$

and

$$f : x \mapsto y = x^2 \ , \quad D(f) = (-\infty, \infty)$$

are *different* functions!

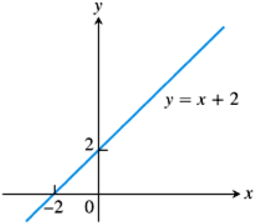
Graphs of functions

Definition

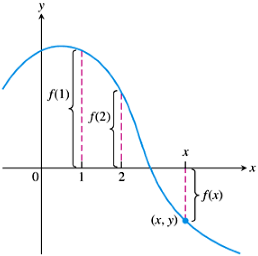
If f is a function with domain D, its graph consists of the points (x, y) whose coordinates are the input-output pairs for f:

$$\{(x, f(x)) | x \in D\}$$

examples:



given the function, one can *sketch* the graph

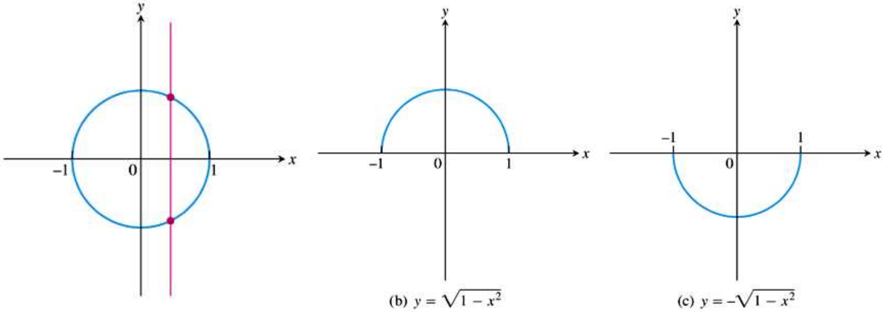


$y = f(x)$ is the *height* of the graph above/below x.

Arbitrary curves vs. graphs of functions

recall: A function f can have only one value $f(x)$ for each x in its domain! This leads to the vertical line test:

No vertical line can intersect the graph of a function *more than once*.



(a) $x^2 + y^2 = 1$

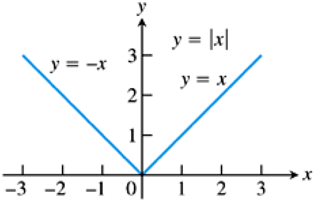
Piecewise defined functions

A piecewise defined function is a function that is is described by using different formulas on different parts of its domain.

examples:

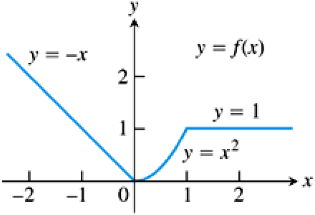
- the *absolute value function*

$$f(x) = |x| = \begin{cases} x & , x \geq 0 \\ -x & , x < 0 \end{cases}$$



- some other function

$$f(x) = \begin{cases} -x & , x < 0 \\ x^2 & , 0 \leq x \leq 1 \\ 1 & , x > 1 \end{cases}$$



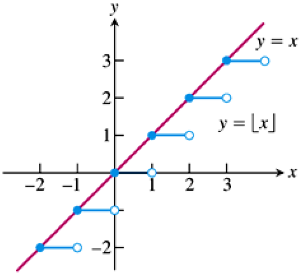
Floor and ceiling functions

- the floor function

$$f(x) = \lfloor x \rfloor$$

is given by the greatest integer less than or equal to x:

$$\lfloor 1.3 \rfloor = 1, \ \lfloor -2.7 \rfloor = -3$$

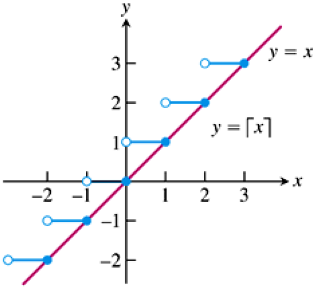


- the ceiling function

$$f(x) = \lceil x \rceil$$

is given by the smallest integer greater than or equal to x:

$$\lceil 3.5 \rceil = 4, \ \lceil -1.8 \rceil = -1$$

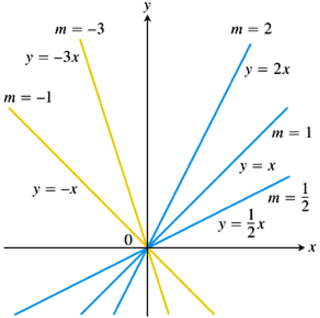


Revision of Lecture 4

- definition of a function
- domain and range of a function
- graph of a function
- piecewise defined functions

Some fundamental types of functions

- linear function $\boxed{f(x) = mx + b}$

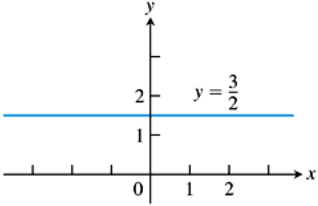


$b = 0$: all lines pass through the origin,

$$f(x) = mx$$

One also says "$y = f(x)$ is proportional to x" for some nonzero constant m.

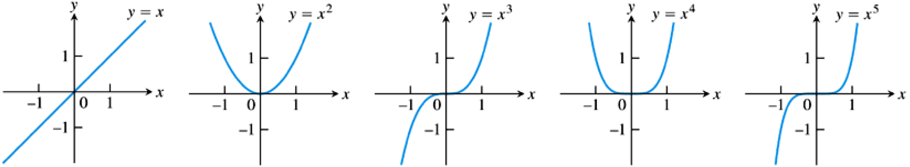
$m = 0$: constant function $\boxed{\text{f(x)=b}}$



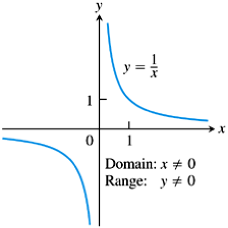
Power function I

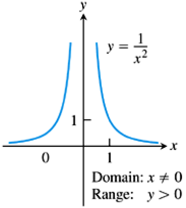
- power function $\boxed{f(x) = x^a}$

  $a = n \in \mathbb{N}$: graphs of $f(x)$ for $n = 1, 2, 3, 4, 5$



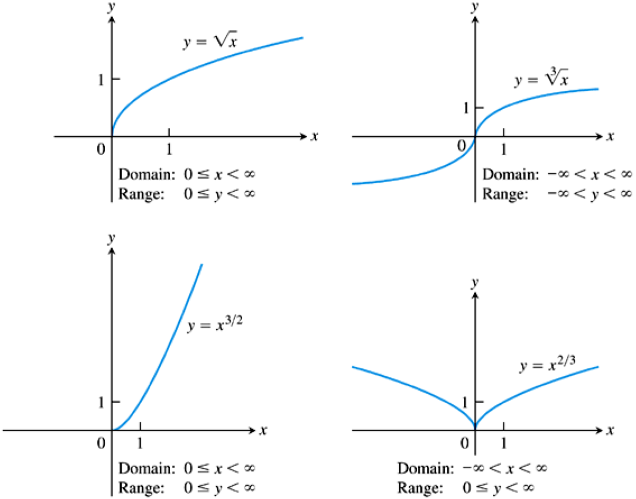
$a = -n$, $n \in \mathbb{N}$: graphs of $f(x)$
for $n = -1, -2$





Power function II

still power function $\boxed{f(x) = x^a}$, now for $a \in \mathbb{Q}$: graphs of $f(x)$ for $a = \frac{1}{2}, \frac{1}{3}, \frac{3}{2}, \frac{2}{3}$



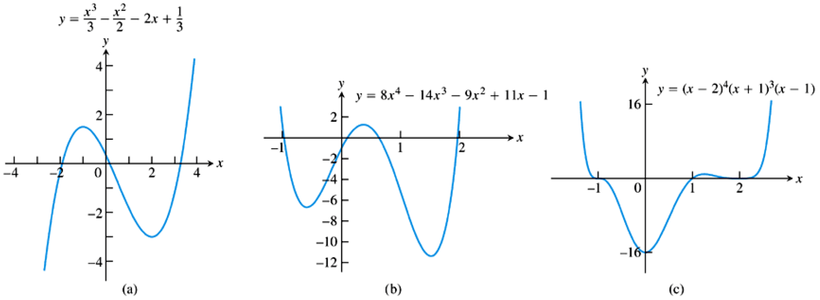
Polynomials

- polynomials

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 \ , \ n \in \mathbb{N}$$

with $a_n \neq 0$, *coefficients* $a_0, a_1, \ldots, a_{n-1}, a_n \in \mathbb{R}$ and domain $\mathbb{R}$
n is called the *degree* of the polynomial

examples: linear functions with $m \neq 0$ are polynomials of degree 1
three polynomial functions and their graphs



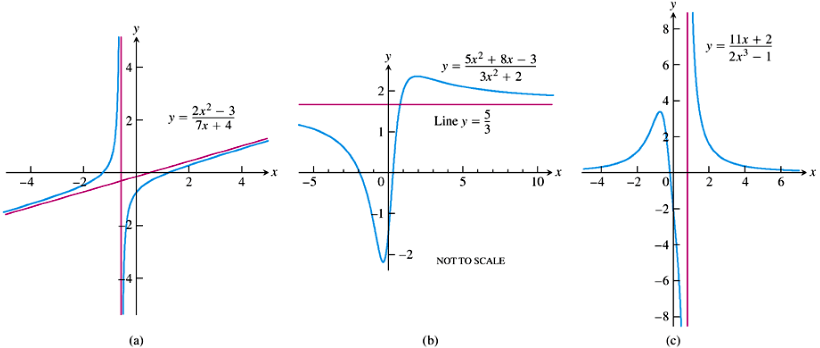
Rational functions

- rational functions

$$f(x) = \frac{p(x)}{q(x)}$$

with $p(x)$ and $q(x)$ polynomials and domain $\mathbb{R} \setminus \{x|q(x) = 0\}$ (never divide by zero!)

examples: three rational functions and their graphs

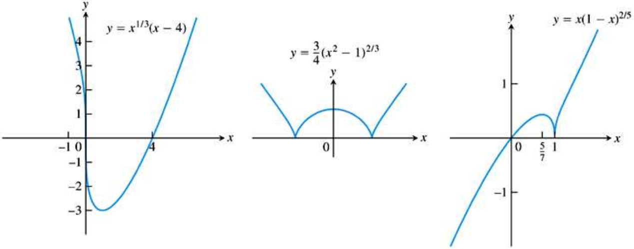


Even more types of functions

Other classes (to come later):

- algebraic functions: any function constructed from polynomials using algebraic operations (including taking roots)

  examples



(a) (b) (c)

- trigonometric functions
- exponential and logarithmic functions
- transcendental functions: any function that is not algebraic
  examples: trigonometric or exponential functions
- ...

Increasing/decreasing functions

Informally,

- a function is called increasing if the graph of the function "climbs" or "rises" as you move *from left to right*.
- a function is called decreasing if the graph of the function "descends" or "falls" as you move *from left to right*.

examples:

function	**where increasing**	**where decreasing**
$y = x^2$	$0 \le x < \infty$	$-\infty < x \le 0$
$y = 1/x$	nowhere	$-\infty < x < 0$ and $0 < x < \infty$
$y = 1/x^2$	$-\infty < x < 0$	$0 < x < \infty$
$y = x^{2/3}$	$0 \le x < \infty$	$-\infty < x \le 0$

Even/odd functions

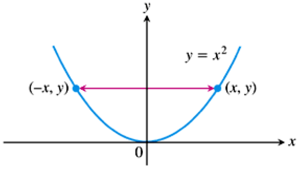
Definition

A function $y = f(x)$ is an

- even function of x if $f(-x) = f(x)$
- odd function of x if $f(-x) = -f(x)$

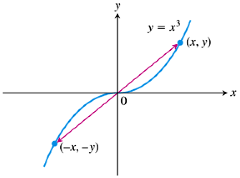
for every x in the function's domain.

examples:



(a)

$f(-x) = (-x)^2 = x^2 = f(x)$:
even function; graph is *symmetric about the y-axis*

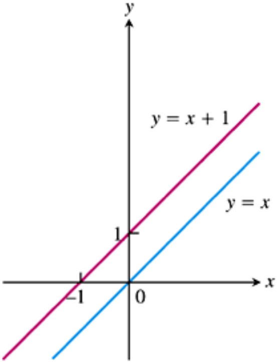


(b)

$f(-x) = (-x)^3 = -x^3 = -f(x)$:
odd function; graph is *symmetric about the origin*

Even/odd functions continued

further examples:



1. $f(-x) = -x = -f(x)$: odd function
2. $f(-x) = -x + 1 \neq f(x)$ and $-f(x) = -x - 1 \neq f(-x)$: neither even nor odd!

Sums, differences, products, quotients

If f and g are functions, then for every

$$x \in D(f) \cap D(g)$$

(that is, for every x that belongs to the domains of *both* f and g) we define

$$(f+g)(x) = f(x) + g(x)$$
$$(f-g)(x) = f(x) - g(x)$$
$$(fg)(x) = f(x)g(x)$$
$$(f/g)(x) = f(x)/g(x) \quad \text{if } g(x) \neq 0$$

algebraic operation on functions = algebraic operation on function values

Special case: multiplication by a constant $c \in \mathbb{R}$:

$$(cf)(x) = c\,f(x)$$

(take $g(x) = c$ constant function)

Combining functions algebraically

examples:

$$f(x) = \sqrt{x} \quad , \quad g(x) = \sqrt{1-x}$$

(natural) domains:

$$D(f) = [0, \infty) \qquad D(g) = (-\infty, 1]$$

intersection of both domains:

$$D(f) \cap D(g) = [0, \infty) \cap (-\infty, 1] = [0, 1]$$

function	**formula**	**domain**
$f+g$	$(f+g)(x) = \sqrt{x} + \sqrt{1-x}$	$[0,1] = D(f) \cap D(g)$
$f-g$	$(f-g)(x) = \sqrt{x} - \sqrt{1-x}$	$[0,1]$
$g-f$	$(g-f)(x) = \sqrt{1-x} - \sqrt{x}$	$[0,1]$
$f \cdot g$	$(f \cdot g)(x) = f(x)g(x) = \sqrt{x(1-x)}$	$[0,1]$
f/g	$\frac{f}{g}(x) = \frac{f(x)}{g(x)} = \sqrt{\frac{x}{1-x}}$	$[0,1)$ ($x=1$ excluded)
g/f	$\frac{g}{f}(x) = \frac{g(x)}{f(x)} = \sqrt{\frac{1-x}{x}}$	$(0,1]$ ($x=0$ excluded)

Revision of Lecture 5

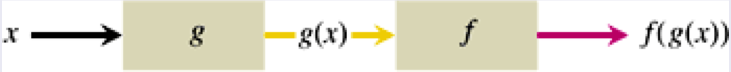
- classification of basic types of functions
- increasing/decreasing functions
- even/odd functions
- algebraic combinations of functions

Composition of functions

Definition

If f and g are functions, the composite function $f \circ g$ ("f composed with g") is defined by

$$(f \circ g)(x) = f(g(x))$$

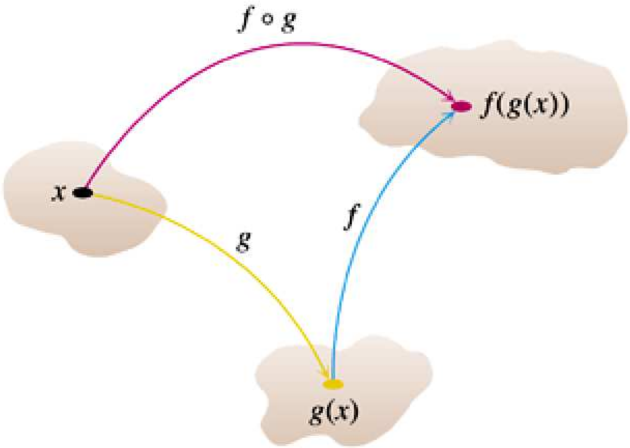


The *domain* of $f \circ g$ consists of the numbers x in the domain of g for which $g(x)$ lies in the domain of f, i.e.

$$D(f \circ g) = \{x | x \in D(g) \text{ and } g(x) \in D(f)\}$$

Arrow diagram for a composite function

$$D(f \circ g) = \{x | x \in D(g) \text{ and } g(x) \in D(f)\}$$



Finding formulas for composites

examples:

$$\begin{aligned} f(x) &= \sqrt{x} && \text{with} \quad D(f) = [0, \infty) \\ g(x) &= x + 1 && \text{with} \quad D(g) = (-\infty, \infty) \end{aligned}$$

composite	**domain**
$(f \circ g)(x) = f(g(x)) = \sqrt{g(x)} = \sqrt{x+1}$	$[-1, \infty)$
$(g \circ f)(x) = g(f(x)) = f(x) + 1 = \sqrt{x} + 1$	$[0, \infty)$
$(f \circ f)(x) = f(f(x)) = \sqrt{f(x)} = \sqrt{\sqrt{x}} = x^{1/4}$	$[0, \infty)$
$(g \circ g)(x) = g(g(x)) = g(x) + 1 = x + 2$	$(-\infty, \infty)$

The domain of composites

further examples:

$$\begin{array}{lll} f(x) & = \sqrt{x} \quad \text{with} & D(f) = [0, \infty) \\ g(x) & = x^2 \quad \text{with} & D(g) = (-\infty, \infty) \end{array}$$

composite	**domain**
$(f \circ g)(x) = \|x\|$	$(-\infty, \infty)$
$(g \circ f)(x) = x$	$[0, \infty)$

Shifting a graph of a function

Shift Formulas

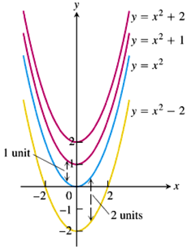
Vertical Shifts

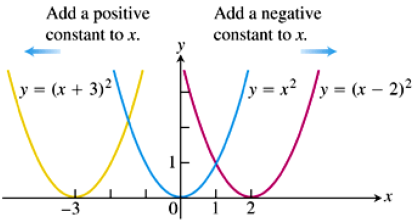
$y = f(x) + k$ — Shifts the graph of f *up* k units if $k > 0$
Shifts it *down* $|k|$ units if $k < 0$

Horizontal Shifts

$y = f(x + h)$ — Shifts the graph of f *left* h units if $h > 0$
Shifts it *right* $|h|$ units if $h < 0$

examples:



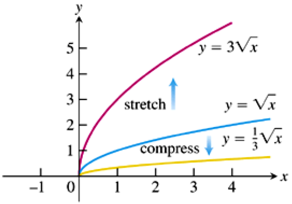


Scaling a graph of a function

For $c > 1$,

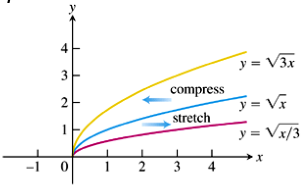
$y = cf(x)$ stretches the graph of f along the y-axis by a factor of c

$y = \frac{1}{c}f(x)$ compresses the graph of f along the y-axis by a factor of c



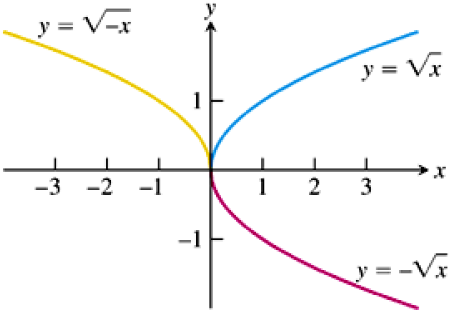
$y = f(cx)$ compresses the graph of f along the x-axis by a factor of c

$y = f(x/c)$ stretches the graph of f along the x-axis by a factor of c



Reflecting a graph of a function

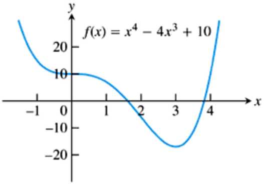
For $c = -1$,
$y = -f(x)$ reflects the graph of f across the x-axis



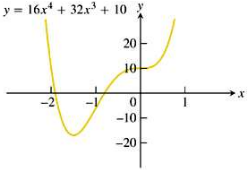
$y = f(-x)$ reflects the graph of f across the y-axis

Combining scalings and reflections

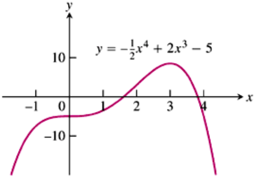
the **original graph** of
$y = f(x)$:



horizontal compression by a factor of 2: $y = f(2x)$
followed by a reflection across the y-axis: $y = f(-2x)$



vertical compression by a factor of 2: $y = \frac{1}{2} f(x)$
followed by a reflection across the x-axis: $y = -\frac{1}{2} f(x)$



Reading Assignment

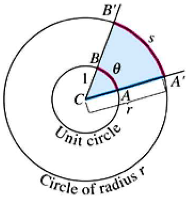
Read

Thomas' Calculus:

- short **Paragraph** about ellipses, p.44/45
- **Section 1.6** about trigonometric functions, *especially* trigonometric identities

You will need this for Coursework 2!

Radian measure



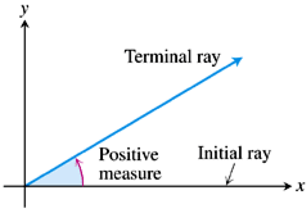
The radian measure of the angle ACB is the length θ of arc AB on the unit circle.

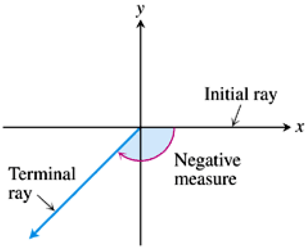
$s = r\theta$ is the length of arc on a circle of radius r when θ is measured in radians.

conversion formula degrees $\leftrightarrow$ radians:

$$360^\circ \text{ corresponds to } 2\pi \Rightarrow \boxed{\frac{\text{angle in radians}}{\text{angle in degrees}} = \frac{\pi}{180}}$$

Signed angles



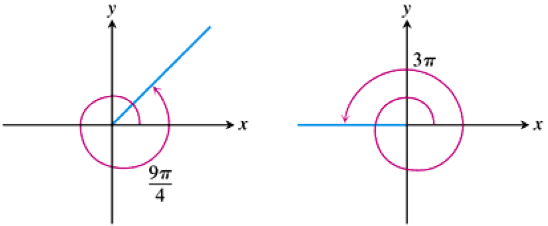


- angles are oriented
- positive angle: counter-clockwise
- negative angle: clockwise

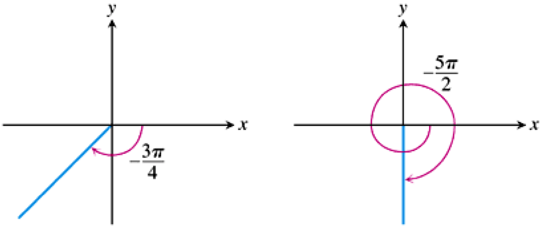
Large angles

note: angles can be larger than 2π:

counter-clockwise:

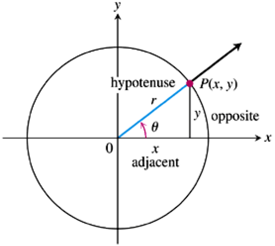


clockwise:



Trigonometric functions

reminder: the six basic trigonometric functions

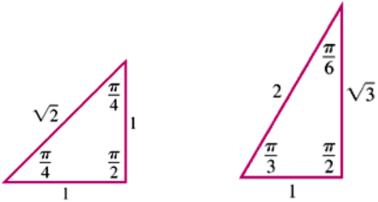


sine:	$\sin\theta = \frac{y}{r}$	**cosecant:**	$\csc\theta = \frac{r}{y}$
cosine:	$\cos\theta = \frac{x}{r}$	**secant:**	$\sec\theta = \frac{r}{x}$
tangent:	$\tan\theta = \frac{y}{x}$	**cotangent:**	$\cot\theta = \frac{x}{y}$

note: These definitions hold not only for $0 \le \theta \le \pi$ but also for $\theta < 0$ and $\theta > \pi/2$.

Finding trigonometric function values

recommended to memorize the following two triangles:



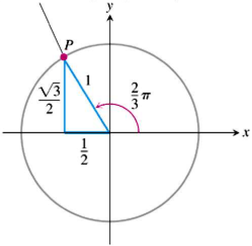
because exact values of trigonometric ratios can be read from them

example:

$$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} \quad ; \quad \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

Finding extended trigonometric function values

a more non-trivial **example:**



$\sin\frac{2}{3}\pi = \frac{y}{r} = \sin\left(\pi - \frac{2}{3}\pi\right) = \sin\frac{\pi}{3}$

see previous triangle: $\sin\frac{\pi}{3} = \sqrt{3}/2$

here $r = 1 \Rightarrow x = -1/2\,,\ y = \sqrt{3}/2$ (why?)

from the above triangle we can now read off the values of all trigonometric functions:

$$\sin\left(\frac{2}{3}\pi\right) = \frac{y}{r} = \frac{\sqrt{3}}{2} \qquad \csc\left(\frac{2}{3}\pi\right) = \frac{r}{y} = \frac{2}{\sqrt{3}}$$

$$\cos\left(\frac{2}{3}\pi\right) = \frac{x}{r} = -\frac{1}{2} \qquad \sec\left(\tfrac{2}{3}\pi\right) = \tfrac{r}{x} = -2$$

$$\tan\left(\frac{2}{3}\pi\right) = \frac{y}{x} = -\sqrt{3} \qquad \cot\left(\frac{2}{3}\pi\right) = \frac{x}{y} = -\frac{1}{\sqrt{3}}$$