



# Lecture Notes for MAS424/MTHM021: Introduction to Dynamical Systems

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## Contents

|                                                                            |           |
|----------------------------------------------------------------------------|-----------|
| <b>1 Preliminaries</b>                                                     | <b>3</b>  |
| <b>2 What is a ‘dynamical system’?</b>                                     | <b>3</b>  |
| 2.1 Realistic examples of dynamical systems . . . . .                      | 3         |
| 2.1.1 Driven nonlinear pendulum . . . . .                                  | 3         |
| 2.1.2 Bouncing ball . . . . .                                              | 5         |
| 2.1.3 Billiards . . . . .                                                  | 5         |
| 2.1.4 Simulation of a very similar system . . . . .                        | 7         |
| 2.2 Definition of dynamical systems . . . . .                              | 7         |
| 2.2.1 Time-continuous dynamical systems . . . . .                          | 7         |
| 2.2.2 Time-discrete dynamical systems . . . . .                            | 11        |
| 2.2.3 Relation between time-discrete and time-continuous dynamical systems | 14        |
| <b>3 Some elementary properties of one-dimensional dynamics</b>            | <b>18</b> |
| 3.1 Homeomorphism, diffeomorphisms . . . . .                               | 20        |
| 3.2 Periodic points . . . . .                                              | 20        |
| 3.3 Dense sets, Bernoulli shift . . . . .                                  | 23        |

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|          |                                           |           |
|----------|-------------------------------------------|-----------|
| <b>4</b> | <b>Stability</b>                          | <b>27</b> |
| <b>5</b> | <b>Definitions of deterministic chaos</b> | <b>30</b> |
| 5.1      | Devaney chaos . . . . .                   | 30        |
| 5.2      | Sensitivity . . . . .                     | 30        |
| 5.3      | Ljapunov exponent . . . . .               | 32        |
| <b>6</b> | <b>The Frobenius-Perron equation</b>      | <b>34</b> |
| <b>7</b> | <b>Partitions and transition matrices</b> | <b>39</b> |
| <b>8</b> | <b>Probability Measures</b>               | <b>45</b> |
| <b>9</b> | <b>Basics of ergodic theory</b>           | <b>51</b> |

# 1 Preliminaries

There exist two essentially different approaches to the study of dynamical systems.

- Time-continuous nonlinear differential equations.
- Time-discrete maps.

It is in principle possible to develop the full theory of both from either perspective, but for the bulk of this course, we shall follow the latter route. This allows a generally more simple way of introducing the important concepts, which can usually be carried over to a more complex and physically realistic context.

It is important to say that the spirit of this course and thus these notes are in that of an applied mathematician. Though we shall not dwell overly long on how interesting equations of motion can be derived from physical principles, some physical motivation for studying particular systems will usually be given. Similarly, for the early part of this course especially, we shall not go into great mathematical detail but point to the literature for further reading.

That said, it is also not intended to present an introduction to the context and history of the subject. However, this is well worth studying, the field now encompassing over a hundred years of activity. The book by Gleick [6] is an excellent starting point for such purposes.

The format of these notes is currently somewhat sparse, and it is expected that they will require substantial annotation to clarify points presented in more detail during the actual lectures. Please treat them merely as a study aid rather than a comprehensive syllabus.

## 2 What is a ‘dynamical system’?

### 2.1 Realistic examples of dynamical systems

#### 2.1.1 Driven nonlinear pendulum

Figure 1 above shows a pendulum system subject to a torque (the rotational equivalent of a force). The angle with the vertical in a positive sense is denoted by  $\theta = \theta(t)$ , where  $t \in \mathbb{R}$  is the ‘time’ for the system. It is clear that

$$-\pi \leq \theta < \pi$$

and we write

$$\dot{\theta} = \frac{d\theta}{dt}$$

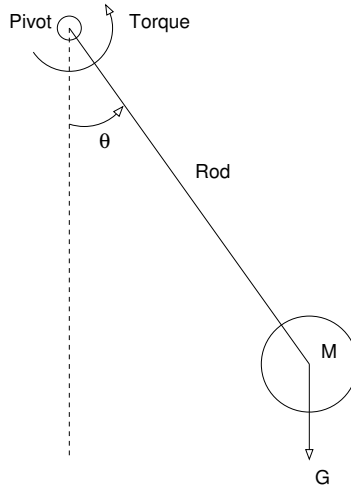


Figure 1: Driven pendulum of mass  $M$  with a torque applied at the pivot and subject to gravity.

to denote the derivative of  $\theta$  with respect to time, which can be called an ‘angular velocity’.

As explained above, without worrying too much about how one can use physics to obtain an equation of motion for this system, we move straight to the equation itself and merely indicate whence each term arises.

$$\begin{array}{ccccccc}
 \text{Equation of motion} & \nu \ddot{\theta} & + & \dot{\theta} & + & \sin \theta & = & A \sin(2\pi f t) . \\
 & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 \text{Balance of forces} & \text{inertia} & + & \text{friction} & + & \text{gravity} & = & \text{torque. (periodic)} \\
 & & & & & & & (1)
 \end{array}$$

- Control parameters  $\nu, A, f$ .
- Online simulation at <http://monet.unibas.ch/~elmer/pendulum/index.html>

It is unfortunately in general impossible to solve such complicated nonlinear equations of motion such as (1), but numerical methods such as Runge-Kutta allow production of such simulations as that linked above. Playing around with varying the parameters there, one finds several types of behaviour.

1. No torque, no damping, small angles  $\theta \ll 1$  - called ‘libration’ or the ‘harmonic oscillator’.
2.  $-\pi \leq \theta < \pi$ . Fixed points exist at  $\theta = 0$  (stable) and  $\theta = \pi$  (unstable). If the initial condition  $\theta(0) = \pi$  is taken with non-zero initial velocity, continual ‘rotation’ is seen.

3. Damping - pendulum comes to rest for  $t \rightarrow \infty$ .
4. Torque
  - sensitive to initial conditions ‘chaotic’ type of motion.
  - complex, apparently unpredictable.

The driven nonlinear pendulum is a paradigmatic example of a non-trivial dynamical system.

### 2.1.2 Bouncing ball

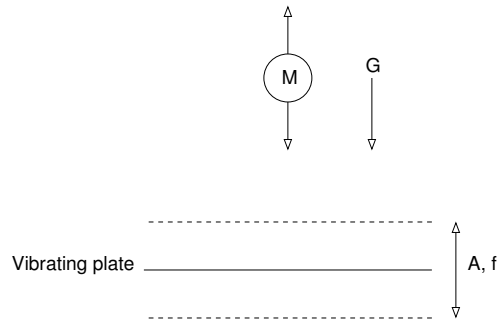


Figure 2: A ball subject to gravity, that elastically bounces off a vibrating plate.

Figure 2 shows an elastic ball falling under gravity, that bounces off a plate that vibrates, thus allowing transfer of energy to and from the ball system.

- $\exists$  ‘frequency locking’ (simple periodic behaviour).
- $\exists$  transitions to complicated ‘chaotic’ behaviour.

### 2.1.3 Billiards

- 2 dimensions.
- ‘Sinai billiard’ - see figure 3.

Hard disk, radius  $r$  fixed at centre of square box.

Point particle collides elastically with disk and walls; constant speed.

For elastic collisions  $\theta = \theta'$ .

Sensitive dependence to initial conditions.

See <http://www.dynamical-systems.org/sinai/index.html> for a simulation (with optional gravity).

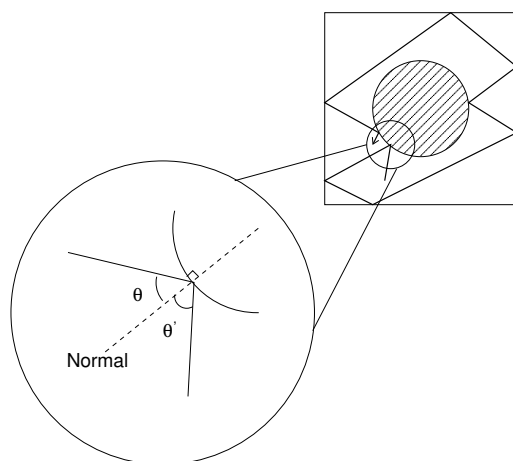


Figure 3: Sinai billiard system.

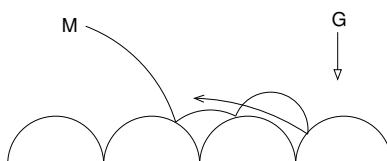


Figure 4: Bouncing off semicircles on a plate.

#### 2.1.4 Simulation of a very similar system

- See figure 4.
- Diffusion - view spreading in time of an ensemble of particles starting from the same point, but with varied velocity angles.
- 2 important mechanisms in the dynamics of many particles are illustrated - see figure 5.

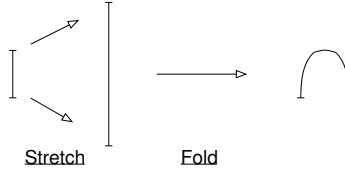


Figure 5: Stretch and fold mechanisms.

This is like kneading dough (see Baker's map) - 'mixing' behaviour.

## 2.2 Definition of dynamical systems

- Definition A dynamical system may be understood as a mathematical prescription for evolving the state of a system in time (do a web search for various similar definitions).
- It is defined by a phase (or state) space  $P$  (in this course  $P \subset \mathbb{R}^N$ ) and a one-parameter family of mappings,  $\phi_t : P \rightarrow P$ , where  $t$  (time)  $\in \mathbb{R}$  or  $t \in \mathbb{Z}$  (time-continuous/discrete).

For arbitrary  $\underline{x} \in P$  the following must hold

1.  $\phi_0(\underline{x}) = \underline{x}$  - identity and
2.  $\phi_t(\phi_s(\underline{x})) = \phi_{t+s}(\underline{x})$  - additivity  $\forall t, s \in \mathbb{R}$ .

### 2.2.1 Time-continuous dynamical systems

Let  $P \subset \mathbb{R}^N$ ,  $N \in \mathbb{N}$ ,  $\underline{x} = (x^1, x^2, \dots, x^N) \in P$ ,  $t \in \mathbb{R}$ . Then

$$\underline{F} : P \rightarrow P, \dot{\underline{x}} = \underline{F}(\underline{x}(t)) = \underline{F}(\underline{x}) \quad (2)$$

is called a vector field, which can be written as a system of  $N$  first order, autonomous, ordinary differential equations,

$$\begin{aligned}
\frac{dx^1}{dt} &= F^1(x^1, x^2, \dots, x^N); \\
\frac{dx^2}{dt} &= F^2(x^1, x^2, \dots, x^N); \\
&\dots \\
\frac{dx^N}{dt} &= F^N(x^1, x^2, \dots, x^N).
\end{aligned}
\tag{3}$$

$$\tag{4}$$

The formal solution of Eq. (2) (if  $\exists$ ),

$$\underline{x}(t) = \underline{\phi}_t(\underline{x}(0)),$$

is called the flow of the vector field.

A single path in phase space followed by  $\underline{x}$  in time is called the trajectory or orbit of the dynamical system; see figure 6 for example.

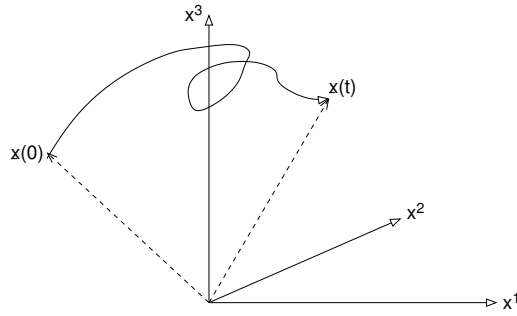


Figure 6: Example of a trajectory in 3-dimensional phase space.

In the following we will assume the existence and uniqueness of solutions of the vector field  $\underline{F}$ . A proof of this exists if  $\underline{F}$  is smooth (continuously differentiable,  $C^1$ ) - see e.g. the book by Strogatz [13].

Corollary: *2 distinct trajectories cannot intersect, nor can a single trajectory cross itself, for  $t < \infty$ .*

Proof: Assume they intersect: Starting with initial condition at the point of intersection there would be two choices of direction to proceed in - contradicts uniqueness.  $\square$

If  $\exists$  uniqueness,  $\underline{x}(0)$  determines the outcome of a flow after time  $t$  : we have determinism!

Question: What is the general ‘Character’ of a flow  $\phi_t(\underline{x})$ ? Is it ‘simple’ or ‘complicated’?

- Simple:  $\exists$  fixed points  $\underline{x}$  constant, respectively  $\dot{\underline{x}} = \underline{0}$  or  $\exists$  limit cycles/periodic orbits  $\underline{x}(t + \tau) = \underline{x}(t), \tau \in \mathbb{R}$ , for  $t \rightarrow \infty$ .  
2-d e.g. figure 7.

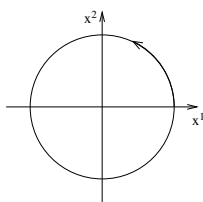


Figure 7: Circular periodic orbit.

Note: Linear stability analysis should be done in order to determine/classify periodic orbits, e.g. see figure 8.

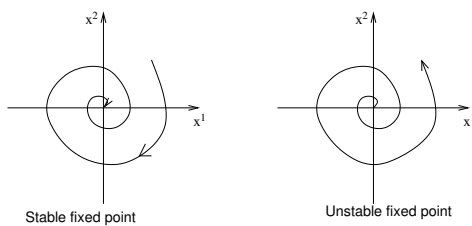


Figure 8: Stable (left) and unstable (right) fixed points.

- Complicated:  $\phi_t(\underline{x})$  is not simple (!) - see figure 9.

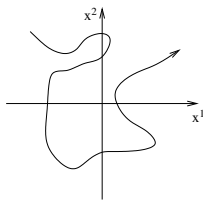


Figure 9: An orbit that isn't simple.

Question: How large does  $N$  have to be for complicated behaviour to be possible?

Theorem (Poincaré-Bendixson): Let  $\dot{\underline{x}} = \underline{F}(\underline{x}) \in \mathbb{R}^2$  ( $N = 2$ ,  $\underline{x} = \underline{x}(t)$ , but  $\underline{F}$  does not explicitly depend on  $t$  - autonomous) be a continuously differentiable vector field such that the flow  $\phi_t(\underline{x})$  is bounded in the plane. Let  $R$  be a region such that all trajectories starting in  $R$  remain in  $R$ . Then any trajectory starting in  $R$  either approaches a fixed point or a limit cycle for  $t \rightarrow \infty$ .

These two possibilities are illustrated in figure 10.

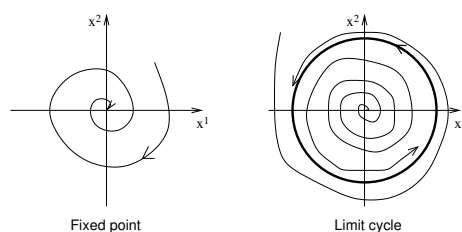


Figure 10: Approach to a stable fixed point (left) and to a limit cycle (right).

Proof: Alligood/Sauer/Yorke [1] (p. 336 ff. )

‘Idea’: uniqueness of solutions, i.e. , trajectories do not cross in 2 dimensions (see figure 11).

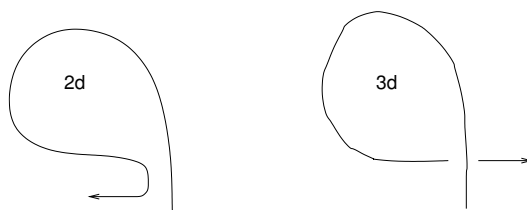


Figure 11: Trajectories in 2-d (left) cannot cross whereas in 3-d (right), they can.

Corollary: Let  $\underline{F} \in \mathbb{R}^N$  and the conditions of the Poincaré-Bendixson theorem be fulfilled.

$N \leq 2 \Rightarrow$  solutions simple.

Solutions complicated  $\Rightarrow N > 2$ .

i.e. ,  $N \geq 3$  implies ‘anything can happen’.

How to check this for a given dynamical system?

Example: Driven pendulum equation,

$$\ddot{\theta} + \nu\dot{\theta} + \sin \theta = A \sin(2\pi ft)$$

Rewrite this dynamical system in the form of a vector field.

$$x^1 := \dot{\theta}, \quad x^2 := \theta, \quad x^3 := 2\pi ft.$$

The ‘trick’ of incorporating the time dependence of the differential equation as a third coordinate allows us to make the vector field autonomous.

$$\begin{aligned} \dot{x}^1 &= \ddot{\theta} = -\nu x^1 - \sin x^2 + A \sin x^3, \\ \dot{x}^2 &= x^1, \\ \dot{x}^3 &= 2\pi f. \end{aligned}$$

So  $N = 3$ : complicated dynamics are possible.

### 2.2.2 Time-discrete dynamical systems

Let  $P \subset \mathbb{R}^N$ ,  $N \in \mathbb{N}$ ,  $\underline{x}_n \in P$ ,  $n \in \mathbb{Z}$ .

Then  $\underline{M} : P \rightarrow P$ ,  $\underline{x}_{n+1} = \underline{M}(\underline{x}_n)$  is called a time-discrete map.

$\underline{x}_{n+1} = \underline{M}(\underline{x}_n)$  are sometimes called the equations of motion of the dynamical system.

Choosing the initial condition  $\underline{x}_0$  determines the outcome after discrete time  $n$  (determinism!) in the following way:

$$\underline{x}_1 = \underline{M}(\underline{x}_0) = \underline{M}^1(\underline{x}_0),$$

$$\underline{x}_2 = \underline{M}(\underline{x}_1) = \underline{M}(\underline{M}(\underline{x}_0)) = \underline{M}^2(\underline{x}_0) \neq \underline{M}(\underline{x}_0)\underline{M}(\underline{x}_0).$$

$$\boxed{\underline{M}^m(\underline{x}_0) \quad := \quad \underbrace{\underline{M} \circ \underline{M} \circ \cdots \circ \underline{M}}_{\text{m-fold composed map}}(\underline{x}_0).}$$

$\exists$  a formal solution to the equations of motion:  $\underline{x}_n = \underline{M}(\underline{x}_{n-1}) = \dots = \underline{M}^n(\underline{x}_0)$ .  
cp. to flow for  $t \in \mathbb{R}$ .

Example: (following Ott [11])  $N = 1$ ; let  $z_n \in \mathbb{N}_0$  be the number of insects hatching out of eggs in year  $n \in \mathbb{N}_0$  and  $r > 0$  be the number of eggs laid per insect (on average).

- Ideal case:  $z_{n+1} = rz_n = r^2 z_{n-1} = \dots = r^{n+1} z_0 = \exp((n+1) \ln r) z_0$  (insects only live one year).

Exponentially increasing ( $r > 1$ )/decreasing ( $r < 1$ ) population.

- Malthus (1798): incorporate the effect of enemies/overcrowding by replacing  $r \rightarrow r(1 - \frac{z_n}{\tilde{z}})$ , where  $\tilde{z}$  models food supply,  $0 \leq z_n \leq \tilde{z}$ ;

If  $z_n = \tilde{z}$ , food is exhausted and all die!

$$z_{n+1} = r \rightarrow r(1 - \frac{z_n}{\tilde{z}}) z_n.$$

Divide both sides by  $\tilde{z}$  and put  $x_n := \frac{z_n}{\tilde{z}}$ .

$$\boxed{x_{n+1} = rx_n(1 - x_n), 0 \leq x_n \leq 1, r > 0.}$$

‘Logistic map’ (logistic growth).

Dynamics of logistic map? Look at the set of all iterates of a given initial condition  $x_0$ , ie.  $\{x_0, x_1, x_2, \dots, x_n\}$ , which is the trajectory/orbit of  $M(x_0)$ .

Graphical representation: cobweb plot.

Example: logistic map,  $1 < r < 2, 0 \leq x \leq 1$ .

$$[\frac{d}{dx}(rx(1-x)) = r - 2rx; = 0 @ x = 1/2 - M(1-x) = M(x).]$$

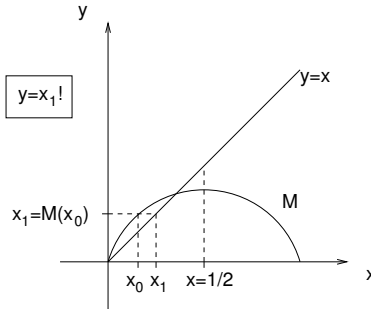


Figure 12: Logistic map with  $1 < r < 2$ .

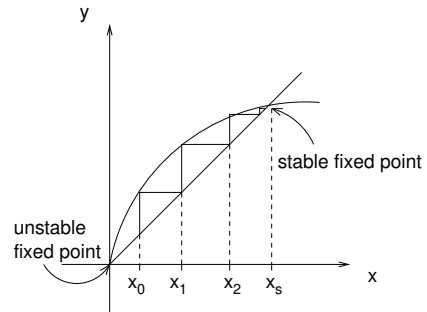


Figure 13: Cobweb plot for logistic map.

Magnify:

Definition:  $x$  is a fixed point of  $M$  if  $x = M(x)$ .

Here: 'Points move away from  $x = 0$  and converge to  $x_s$ .'

$\Rightarrow x_u = 0$  is an unstable/repelling fixed point whereas  $x_s$  is stable/attracting.

Cobweb plot yields information about the existence of fixed points  $x = M(x)$  and their stability.

Definition: A map  $M$  is called invertible if  $\exists$  an inverse  $M^{-1}$  with  $\underline{x}_n = M^{-1}(\underline{x}_{n+1})$ .

Examples:

1. Logistic map

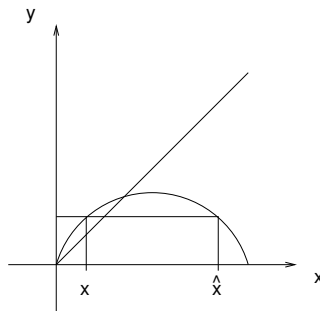


Figure 14: The logistic map is not invertible.

Not invertible since one image has two preimages.

2.  $x_{n+1} = f(x_n) = ky_n, k \neq 0$   
 $y_{n+1} = x_n.$

$$f(x) = A - x^2, A \in \mathbb{R}.$$

This is the Hénon map (1976), derived from the motion of a star in a galaxy.

$$\begin{aligned} \text{Invertible? } x_n &= y_{n+1} \\ y_n &= \frac{1}{k}(f(x_n) - x_{n+1}). \\ &= \frac{1}{k}(f(y_{n+1}) - x_{n+1}) \Rightarrow \text{invertible!} \end{aligned}$$

Condition on  $N$  for existence of ‘complicated solutions’?

Corollary: (‘Poincaré-Bendixson discretized’)

- Map  $\underline{M} \in \mathbb{R}^N$  smooth, defined on a bounded set.
- $\underline{M}$  invertible: 

|                                                                                          |
|------------------------------------------------------------------------------------------|
| $N = 1 \Rightarrow$ solutions simple<br>$\text{solutions complicated} \Rightarrow N > 1$ |
|------------------------------------------------------------------------------------------|

  
*i.e. for  $N \geq 2$  ‘anything can happen’.*
- $\underline{M}$  not invertible: 

|                                                                                             |
|---------------------------------------------------------------------------------------------|
| $N < 1 \Rightarrow$ solutions simple<br>$\text{solutions complicated} \Rightarrow N \geq 1$ |
|---------------------------------------------------------------------------------------------|

  
*i.e. for  $N \geq 1$  ‘anything can happen’.*
- For explanations of these results, see following discussions

Examples:

1. Logistic map: not invertible,  $N = 1 \Rightarrow$  complicated solutions possible.
2. Hénon map: invertible,  $N = 2 \Rightarrow$  complicated solutions possible.

### 2.2.3 Relation between time-discrete and time-continuous dynamical systems

Reduction of a  $N$ -dimensional flow to an  $(N - 1)$  dimensional map:

Poincaré surface of section; 2 types

1. In space: example for  $N = 3$ .

$\dot{\underline{x}} = \underline{F}(\underline{x})$ ; choose, e.g.  $x^3 = K = \text{const.}$  (‘appropriately’)

Poincaré map defined uniquely by iteration  $\begin{pmatrix} x_{n+1}^1 \\ x_{n+1}^2 \end{pmatrix} = \underline{M} \begin{pmatrix} x_n^1 \\ x_n^2 \end{pmatrix}$

from  $n^{\text{th}} \rightarrow (n + 1)^{\text{st}}$  piercing under condition  $x^3 = \text{const.}$

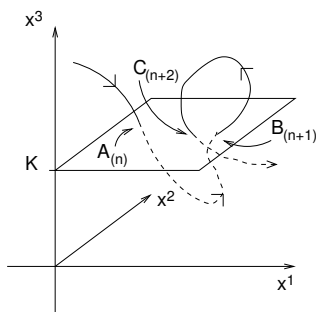


Figure 15: A continuous trajectory in space pierces the plane  $x^3 = K$  at several discrete points.

Problem 1: nice numerical tool, but typically this map is very hard to construct analytically.

Problem 2: If the times  $\Delta t$  between two piercings are constant, then ‘trivial information’ is separated out by using the Poincaré surface of section. But if not: full dynamics defined by suspended flow;

$$\boxed{\begin{array}{lcl} \underline{x}_{n+1} & = & \underline{M}(\underline{x}_n), \quad \underline{x}_n, \underline{M} \in \mathbb{R}^{N-1} \\ t_{n+1} & = & t_n + T(\underline{x}_n), \quad T: \mathbb{R}^{N-1} \rightarrow \mathbb{R} \end{array}}$$

Allows some idea of how Poincaré-Bendixson applies to reduced-dimension discrete systems.

## 2. Poincaré surface of section in time:

Sampling not in space but at discrete times

$$t_n = t_0 + n\tau, n \in \mathbb{N}.$$

i.e.  $T(\underline{x}_n) = n\tau$  : stroboscopic sampling.

example: driven pendulum; choose  $\tau = \frac{1}{f} = P$  as the period of the driving force  $\underline{F} = A \sin(2\pi ft)$ ,  $f = \frac{1}{P}$ .

## Interlude: ‘ $\delta$ -function’

$$\delta(x) := \begin{cases} 0, x \neq 0 \\ \infty, x = 0 \end{cases} ;$$

$$\text{e.g. } \delta_\gamma(x) := \frac{1}{\gamma\sqrt{2\pi}} \exp\left(-\frac{x^2}{2\gamma^2}\right) \rightarrow \delta(x) \quad [\gamma \rightarrow 0].$$

Other series representations exist (theory of distributions) e.g. Lorentzians, Fourier transforms.

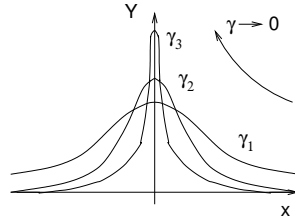


Figure 16: Sequence of functions approaching the  $\delta$ -function.

Normalised:  $\forall \epsilon > 0 \int_{-\epsilon}^{\epsilon} dx \delta(x) = 1$

and  $\int_{-\infty}^{\infty} dx f(x) \delta(x) = f(0)$ .

example: kicked rotor.

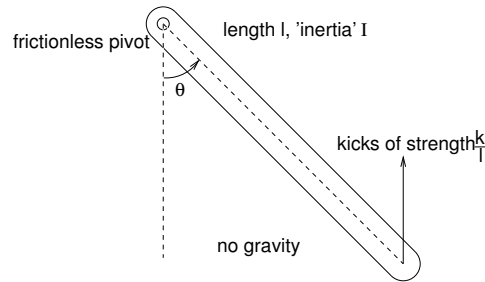


Figure 17: Kicked rotor.

Equation of motion:  $\ddot{\theta} = k \sin \theta \sum_m \delta(t - m\tau)$  (from simple physics)

$$\dot{\theta} = L$$

$$\dot{L} = k \sin \theta \sum_m \delta(t - m\tau)$$

Note:  $l = \text{constant}$  between kicks, changes discontinuously at kicks.

$\theta \propto t$  between kicks, changes continuously at kicks.

Construct Poincaré-s.o.s. in time.

Let  $L_n = L(t)$ ,  $\theta_n = \theta(t)$  at  $t = n\tau + 0^+$  ( $0^+$  is a positive infinitesimal) i.e. right after each kick.

$$\int_{n\tau+0^+}^{(n+1)\tau+0^+} dt \dot{\theta} = \theta_{n+1} - \theta_n = L_n \tau = L_n \text{ (set } \tau = 1).$$

$$\text{Similarly } L_{n+1} - L_n = k \int_{n\tau+0^+}^{(n+1)\tau+0^+} dt \sin \theta \sum_m \delta(t - m\tau) = k \sin \theta_{n+1}.$$

$$\boxed{\begin{array}{lcl} \Rightarrow \theta_{n+1} & = & \theta_n + L_n \quad [\text{mod } 2\pi] \\ L_{n+1} & = & L_n + k \sin \theta_{n+1} \end{array}}$$

2-dimensional (invertible [ $L_n = L_{n+1} - k \sin \theta_{n+1}$ ;  $\theta_n = \theta_{n+1} + k \sin \theta_{n+1} - L_{n+1}$ ]) standard (Chirikov-Taylor) map. (Standard Hamiltonian mapping)

Even further reduction

$$L_{n+1} = L_n + k \sin(\theta_n + L_n).$$

If  $\theta_n \ll L_n$

$$\boxed{L_{n+1} \approx L_n + k \sin L_n}$$

1-dimensional, non-invertible climbing sine map.

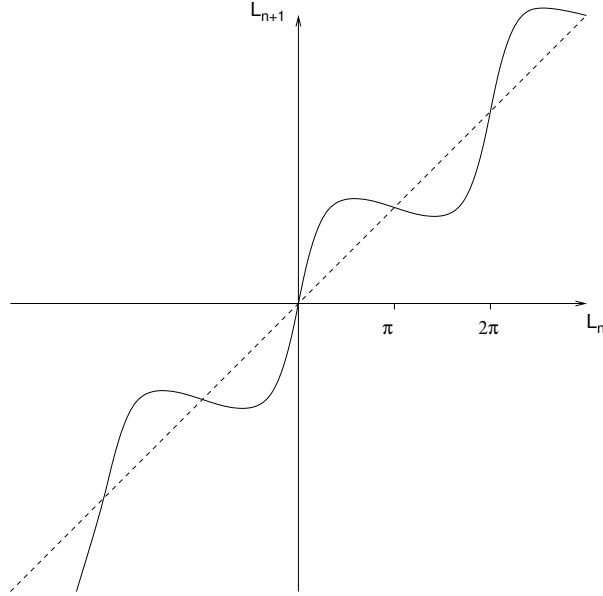


Figure 18: Climbing sine map.

### 3 Some elementary properties of one-dimensional dynamics

Motivation: One-dimensional non-invertible maps are the simplest systems capable of ‘chaotic’ motion (nevertheless, there is a lot we can learn from them!).

Example: Tent map.

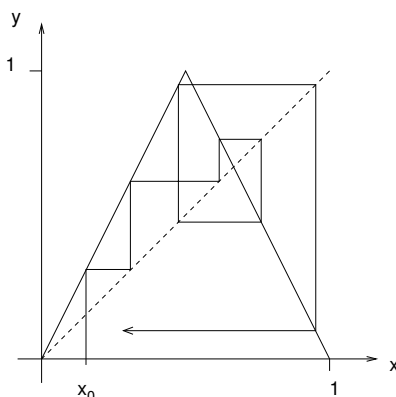


Figure 19: Cobweb plot for the tent map.

$$M : [0, 1] \rightarrow [0, 1]$$

$$M(x) = 1 - 2|x - 1/2| = \begin{cases} 2x, & 0 \leq x \leq 1/2 \\ 2 - 2x, & 1/2 < x \leq 1 \end{cases}$$

Equations of motion  $x_{n+1} = M(x_n)$  bounded for  $x_0 \in [0, 1]$ .

(Why can a piece-wise linear map exhibit ‘chaos’?)

(Piecewise)linear↔nonlinear:  $\exists$  point of non-differentiability.

Such mappings exhibit the main ‘physical’ mechanism of ‘chaos’ - compare with bouncing ball simulations (cf. figure 5).

Consider uniform ensemble of points:

1. Stretching: divergence of nearby trajectories.
2. Fold: bounded motion.

⇒simple model of essentially an nonlinear mechanism generating ‘chaos’.

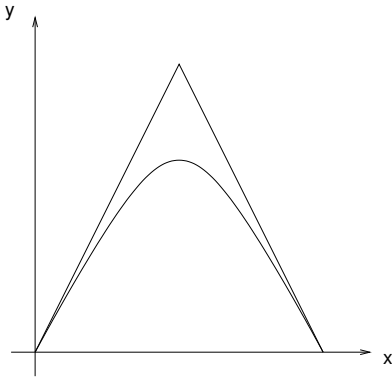


Figure 20: Similarity of the tent map to the logistic map.

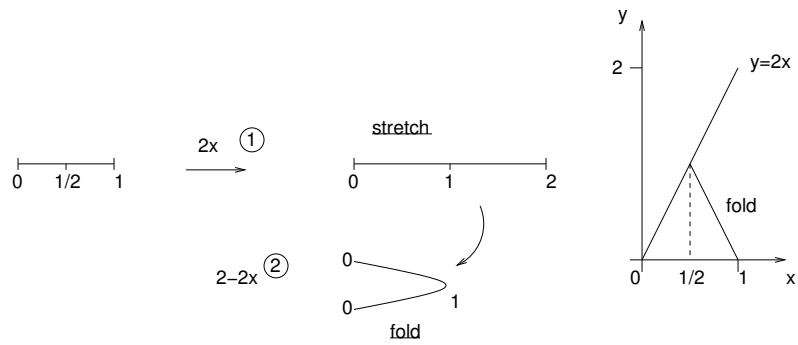


Figure 21: Stretch and fold mechanism in the tent map.

(Think of kneading dough!) - can have stretch and  $\begin{cases} \text{fold} \\ \text{cut} \\ \text{twist and fold} \end{cases}$ .

### 3.1 Homeomorphism, diffeomorphisms

Let  $F : I \rightarrow J$ ,  $I, J \subseteq \mathbb{R}$ ,  $x \mapsto y = f(x)$  be a function.

Definition:  $F(x)$  is a homeomorphism if  $F$  is bijective, continuous and  $\exists$  continuous inverse  $F^{-1}(x)$ .

Example:  $F : (-\pi/2, \pi/2) \rightarrow \mathbb{R}$ ,  $x \mapsto F(x) = \tan(x)$ .

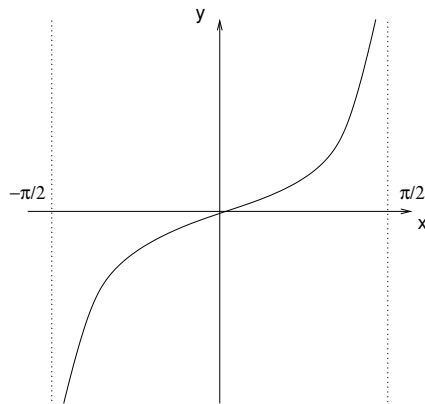


Figure 22:  $\tan(x)$  as a homeomorphism.

$F^{-1} : \mathbb{R} \rightarrow (-\pi/2, \pi/2)$ ,  $x \mapsto \arctan(x)$ .

Definition:  $F(x)$  is called a  $C^r$  diffeomorphism ( $r$ -times continuously differentiable) if  $F$  is a  $C^r$ -homeomorphism such that  $F^{-1}(x)$  is also  $C^r$ .

Examples:

- $F(x) = \tan(x)$  is a  $C^\infty$ -diffeomorphism (check forms of  $\arctan'$ ,  $\tan'$ ).
- $F(x) = x^3$  is a homeomorphism but not a diffeomorphism, because  $F^{-1}(x) = x^{1/3}$ ,  $(F^{-1})'(x) = \frac{1}{3}x^{-2/3}$ ;  $(F^{-1})'(0) = \infty$  : non-existent.

### 3.2 Periodic points

Definition: The point  $x$  is a periodic point of period  $n$  of  $F$  if  $F^n(x) = x$ . The least positive  $n$  for which  $F^n(x) = x$  is called the prime period of  $x$ .

$\mathrm{Per}_n(F)$  denotes the set of periodic points of (not necessarily prime!) period  $n$ .

The set of fixed points  $F(x) = x$  is  $\mathrm{Fix}(F) = \mathrm{Per}_1(F)$ .

The set of all iterates of a periodic point forms a periodic orbit.

Note: If  $x$  is a fixed point of  $F$ , i.e.  $F(x) = x$ , then

- $F^2(x) = F(F(x)) = F(x) = x$ .
- $\Rightarrow x \in \mathrm{Per}_1(F) \Rightarrow x \in \mathrm{Per}_2(F)$ .
- $\Rightarrow x \in \mathrm{Per}_n(F)! \quad \forall n \in \mathbb{N}$ .

Hence the definition of prime period.

Fixed points are the points where the graph  $\{x, F(x)\}$  of  $F$  intersects the diagonal  $\{(x, x)\}$ , see cobweb plot!

Examples:  $F(x) = x = \mathrm{Id}(x) : \quad \mathrm{Fix}(F) = \mathbb{R}$ .

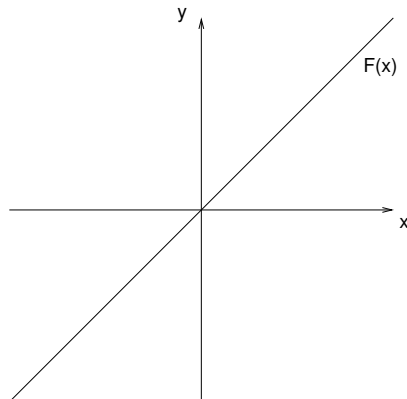


Figure 23:  $F(x) = x$ .

$F(x) = -x : \mathrm{Fix}(F) = 0 \quad (F(x) = x = -x \Rightarrow 2x = 0, x = 0)$ .

- $F^2(x) = F(F(x)) = x$  (fixed point condition).
- $= F(-x) = x$ .
- $\Rightarrow \mathrm{Per}_2(F) = \mathbb{R}$ .

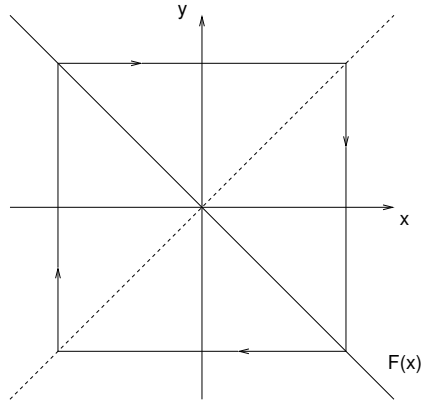


Figure 24:  $F(x) = -x$ .

- Prime  $\text{Per}_2(F) = \mathbb{R} \setminus \{0\}$ !

Note: Typically,  $\exists$  isolated fixed points/periodic orbits and ‘more complicated’ orbits in between, see ff.

$\exists$  ‘fixed point theorem’: a continuous function  $F$  mapping an interval onto itself has at least one fixed point (see Devaney [4]).

Example:  $F(x) = \frac{3x-x^3}{2}$  ( $x_{n+1} = F(x_n)$ )

- Fixed points:  $F(x) = x = \frac{3x-x^3}{2}$ .  
 $2x = 3x - x^3$ .  
 $x - x^3 = 0 = x(1 - x^2) \Rightarrow \underline{\text{Fix}(F) = \{0, \pm 1\}}$ .
- Cobweb: note that  $F'(x) = \frac{1}{2}(3 - 3x^2) = 0 \rightarrow x = \pm 1$  (stationary condition).  
and  $F(1) = 1, F(-1) = -1$ .  
 $F(x) = 0 = \frac{3x-x^3}{2} \rightarrow x(3 - x^2) = 0 \rightarrow x \in \{0, \pm\sqrt{3}\}$ .
- Period 2:  
 $F^2(x) = x$ : solve polynomial equations for roots.  
Note that  $\text{Fix}(F) \subset \text{Per}_2(F)$ !  
 $\rightarrow$  separate these solutions from polynomial equation, defining  $\text{Per}_2(F)$  by division.

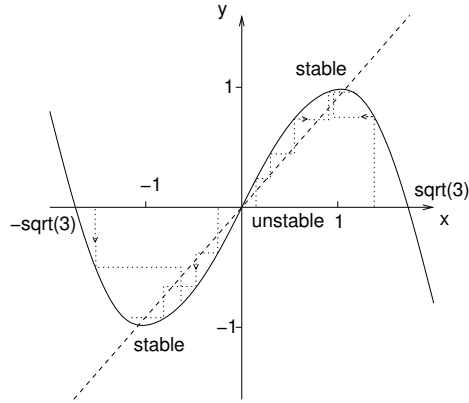


Figure 25: Cobweb plots for  $F(x) = \frac{3x-x^3}{2}$ .

Definition: A point  $x$  is called eventually periodic of period  $n$  if  $x$  is not periodic but  $\exists m > 0$  such that

$$\forall i \geq m \quad F^{n+i}(x) = F^i(x), \text{ that is } F^i(x) = p \text{ is periodic for } i \geq m, F^n(p) = p.$$

Examples:  $F(x) = x^2, F(1) = 1$  is a fixed point, whereas  $F(-1) = 1$  is eventually periodic with respect to the fixed point!

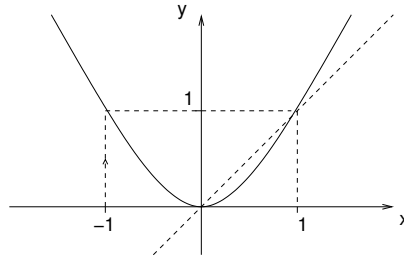


Figure 26: Eventually periodic orbit for  $F(x) = x^2$ .

Construct eventually periodic orbits via backward iteration.

### 3.3 Dense sets, Bernoulli shift

Note: In the following, we will work on metric spaces  $(I, d)$  where  $I$  is a set and  $d$  a metric or distance function.

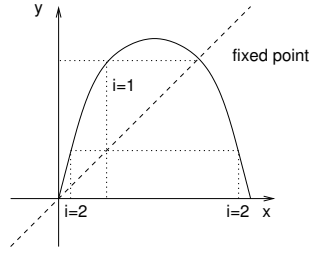
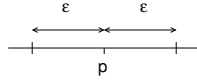


Figure 27: Construction of eventually periodic orbits via backward iteration.

Usually, here  $I \subseteq \mathbb{R}$  and  $d(x, y) = |x - y|$ , the Euclidean metric.

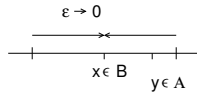
Definition: The epsilon neighbourhood of a point  $p \in \mathbb{R}$  is the interval of numbers  $N_\epsilon(p) := \{x \in \mathbb{R} / |x - p| < \epsilon\}$  for given  $\epsilon > 0$ ,

i.e.  $\forall x \in \mathbb{R}$  within a given distance  $\epsilon$  of  $p$ :



Definition: Let  $A, B \subset \mathbb{R}$  and  $A$  be a subset of  $B$ ,  $A \subset B$ .

$A$  is called dense in  $B$  if arbitrarily close to each point in  $B$  there is a point in  $A$ , i.e.  $\forall x \in B \quad \forall \epsilon > 0 \quad N_\epsilon(x) \cap A \neq \emptyset$ .

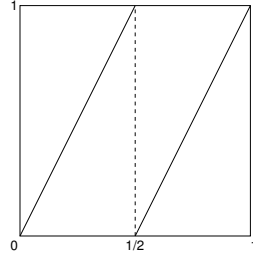


Proposition: *The rationals are dense on the unit interval.*

Proof: Let  $x \in [0, 1] \subset \mathbb{R}$ . For given  $\epsilon > 0$  choose  $n \in \mathbb{N}$  sufficiently large such that  $10^{-n} < \epsilon$ . Let  $\{a_1, a_2, a_3, \dots, a_n\}$  be the first  $n$  digits of  $x$  in decimal representation. Then  $|x - 0.a_1a_2a_3 \dots a_n| < \epsilon \Rightarrow y := 0.a_1a_2a_3 \dots a_n \in \mathbb{Q}$  is in  $N_\epsilon(x)$   $\square$ .

Example: Bernoulli shift

$$B : [0, 1] \rightarrow [0, 1]$$

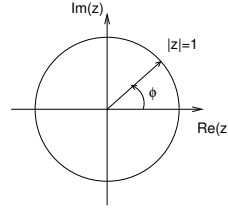


$$B(x) := 2x \mod 1 = \begin{cases} 2x, & 0 \leq x < 1/2 \\ 2 - 2x, & 1/2 \leq x < 1 \end{cases}$$

Proposition: The cardinality  $|\text{Per}_n(B)|$  is equal to  $2^n - 1$  and the periodic points are dense on  $[0, 1]$ .

Proof: Use ‘alternative representation’ of  $B(x)$  on the circle.

Let  $S^1 := \{z \in \mathbb{C} / |z| = 1\} = \{\exp(2\pi i \phi) / \phi \in \mathbb{R}\}$  denote the unit circle in the complex plane.



Then define  $B : S^1 \rightarrow S^1$ ,  $B(z) := z^2 = (\exp(2\pi i \phi))^2 = \exp(2\pi i (2\phi))$ .

Note:  $B$  on the circle is  $C^0$  while  $2x \mod 1$  is discontinuous at  $x = 1/2$ .

Calculate periodic orbits for  $B(z)$ . Let  $n \in \mathbb{N}$  and  $B^n(z) = \underbrace{((z^2)^2 \dots)^2}_{n \text{ times}}$

$= z^{2^n} = z$ . Let  $z = \exp(2\pi i \phi) \rightarrow \exp(2^n 2\pi i \phi) = \exp(2\pi i \phi) = \exp(2\pi i (\phi + k))$ ,  $k \in \mathbb{Z}$  called a ‘phase’.

$$\Rightarrow 2^n \phi = \phi + k \Rightarrow \boxed{\phi = \frac{k}{2^n - 1}}.$$

Now let’s restrict onto  $0 \leq \phi < 1$ , compare with  $2x \mod 1 : 0 \leq k < 2^n - 1$ ,  $k \in \mathbb{N}_0$ .

Example:  $n = 1 \rightarrow 0 \leq k < 1 \rightarrow k = 0, \phi = 0, \text{Fix}(B) = \{0\}$ .

$n = 2 \rightarrow 0 \leq k < 3 \rightarrow \phi \in \{0, \frac{1}{3}, \frac{2}{3}\} = \text{Per}_2(B)$ .

The roots of  $z^{2^n} - 1 = 1$  are periodic points of  $B$  of period  $n$ .

$\exists$  exactly  $2^n - 1$  of them, i.e.  $|\text{Per}_n(B)| = 2^n - 1$ , and they are uniformly spread over the circle with equal intervals of size  $\delta \rightarrow 0$  ( $n \rightarrow \infty$ )  $\Rightarrow$  they are dense.  
□

Definition: A map  $F : J \rightarrow J, J \subseteq \mathbb{R}$ , is topologically transitive on  $J$  if there  $\exists x \in J$  such that its orbit is dense in  $J$ .

Interpretation:  $F$  has points that eventually move under iteration from one arbitrarily small neighbourhood to any other.

$\Rightarrow J$  under the action of  $F$  cannot be decomposed into two disjoint open sets.

Theorem (Birkhoff transitivity theorem): Let  $J$  be a compact subset of  $\mathbb{R}$  and  $F$  be continuous. Then  $F$  is topologically transitive iff ( $\iff$ ) for any two open sets  $U, V \subset J$ ,  $\exists N \in \mathbb{N}$  such that  $F^N(U) \cap V \neq \emptyset$ .

Proof: See Hasselblatt/Katok (HK), Robinson [8; 12]

Example: The Bernoulli shift  $B(x)$  is topologically transitive.

Idea of proof: Bernoulli shift on circle such that  $B \in C^0$ .

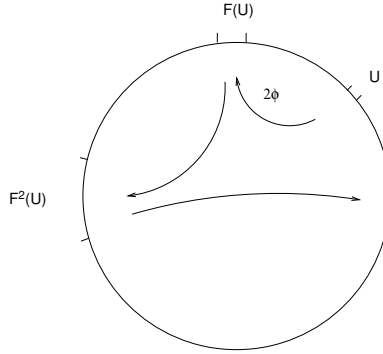


Figure 28: Bernoulli shift operating on a subset of the unit circle.

$\exists B^N(U) \supseteq S^1, N \in \mathbb{N}$ . cf. ‘mixing’.

Other proofs exist, e.g. via symbolic dynamics.

Definition:  $F$  is called minimal if  $\forall x \in J$  the orbit is dense in  $J$ .

Examples:

- Bernoulli shift not minimal.
- Rotation on circle  $R_\lambda(\theta)$  minimal if  $\lambda$  is irrational.

Proof: HK, Devaney [8; 4].

Notes:

- Minimality is rare in dynamical systems, because it allows no existence of periodic orbits.
- Minimality  $\Rightarrow$  topological transitivity.

## 4 Stability

Make qualitative assessment of stability with respect to cobweb plots quantitative:

Definition: Let  $p$  be a fixed point of the map  $F$  acting on  $\mathbb{R}$  (and  $F \in C^1$ ).

If  $|F'(p)| > 1$  then  $p$  is called a sink or attracting fixed point.

If  $|F'(p)| < 1$  then  $p$  is called a source or repelling fixed point.

If  $|F'(p)| = 1$  then  $p$  is called a marginally stable or indifferent, neutral fixed point.

Note:

- This means that for a sink,  $\forall$  points sufficiently close to  $p$  are attracted to  $p$ ; accordingly, for a source all such points are repelled.
- Here we follow Devaney [4] with definition of stability for  $\exists, F'(p)$ ; dig deeper via  $\epsilon$ -environment if  $\nexists F'(p)$ , and then prove theorem.

Definition: The set of initial conditions whose orbits converge to the sink,  $\lim_{n \rightarrow \infty} F^n(x) = p$ , is called the stable set of  $p$ , or the basin of  $p$ .

It is denoted as  $W^s(p)$ .  $W^u(p)$  denotes the ‘opposite set’ of initial conditions, whose orbits are repelled from a source [can be made more precise for invertible maps (via reverse iteration)].

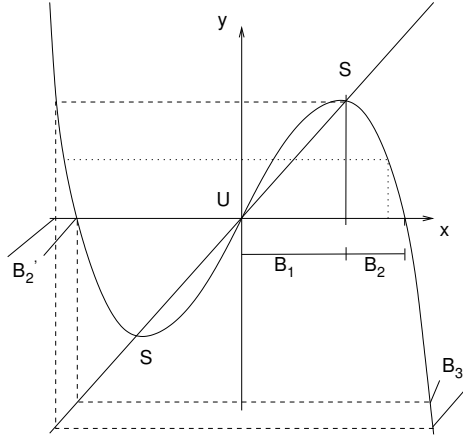


Figure 29: Preimages generating basin structure.

Example:  $F(x) = \frac{3x-x^3}{2}$ , see earlier (figure 25).

$$\text{Fix}(F) = \{0, -1, 1\}, \quad F'(x) = \frac{1}{2}(3 - 3x^2)$$

$$F'(0) = \frac{3}{2} > 1 \rightarrow \text{source}; \quad F'(\pm 1) = 0 < 1 \rightarrow \text{sinks}.$$

$\exists$  complicated basin structure for sinks!

Corollary: Let  $p$  be a periodic point of  $F$  of period  $k$ .

If  $|(F^k)'(p)| < 1$  then  $p$  is a periodic sink.

If  $|(F^k)'(p)| > 1$  then  $p$  is a periodic source.

If  $|(F^k)'(p)| = 1$  then  $p$  is a marginally stable periodic point.

Proof:  $G(x) = FK(x)$ ;  $G(p) = p \iff p$  is an attracting/repelling/marginal fixed point of  $G \iff p$  is an attracting/repelling/marginal periodic orbit of  $F$ .

Definition: Let  $p$  be a periodic point of prime period  $k$ . The point  $p$  is hyperbolic if  $|(F^k)'(p)| \neq 1$ .

Definition: A map  $F$  on  $J$ ,  $F \in C^1$  is called hyperbolic if  $\exists N \in \mathbb{N}$ ,  $N > 0$ , such that  $\forall x \in J \quad \forall n \geq N, \quad |(F^n)'(x)| \neq 1$  (that is, we wish to rule out marginal stability).

Definition:  $F \in C^1$  is called expanding if  $\forall x \in J, \quad |F'(x)| > 1$ .

$F \in C^1$  is called contracting if  $\forall x \in J, \quad |F'(x)| < 1$ .

Note:

- There is the related notion of expansiveness for  $F \notin C^1$ .
- $F$  expanding or contracting  $\Rightarrow$  hyperbolicity.
- If  $F \notin C^1$  but piecewise  $C^1$ , we speak of piecewise expanding/contracting/hyperbolic etc.

Examples:

- Tent map piecewise  $C^1$ .

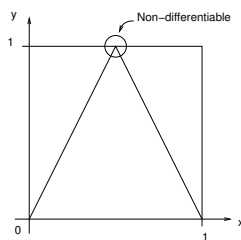


Figure 30: Tent map.

piecewise expanding  $\Rightarrow$  piecewise hyperbolic.

- Logistic map:  $L(x) = rx(1 - x), \quad x \in [0, 1], \quad r > 0$ .

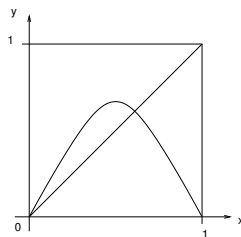


Figure 31: Logistic map.

$$L'(x) = r(1 - 2x), \quad L'(1/2) = 0 \forall r \Rightarrow \text{not expanding.}$$

Contracting?  $\rightarrow$  see problem set!

## 5 Definitions of deterministic chaos

### 5.1 Devaney chaos

Now: First definition of ‘complicated dynamics’.

Definition: Chaos in the sense of Devaney.

A map  $F : J \rightarrow J$ ,  $J \subseteq \mathbb{R}$ , is said to be D-chaotic on  $J$  if

1.  $F$  is topologically transitive.
2. The periodic points are dense in  $J$ .

That is, there are two ingredients:

1. Dense periodic orbits  $\rightarrow$  element of regularity.
2. Topological transitivity  $\rightarrow$  irregularity and indecomposability (sets intertwined).

Example: Bernoulli shift is topologically transitive and  $\exists$  dense set of periodic orbits  $\Rightarrow$  D-chaotic.

### 5.2 Sensitivity

(Guekenheimer [7])

Definition: A map  $F : J \rightarrow J$ ,  $J \subseteq \mathbb{R}$ , has sensitive dependence on initial conditions if  $\exists \delta > 0$  (independent of  $x$ !) such that  $\forall x \in J \quad \forall N_\epsilon(x) \quad \exists y \in N_\epsilon(x)$  and  $\exists n \in \mathbb{N}$  such that  $|F^n(x) - F^n(y)| > \delta$ .

$\forall$  neighbours of  $x$  (as close as desired) eventually move away a distance  $\delta$  from  $F^n(x)$  for  $n$  sufficiently large (see Banks [2]):

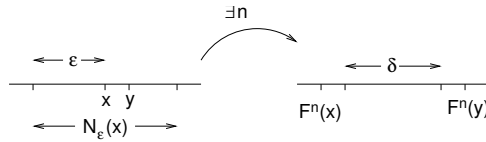
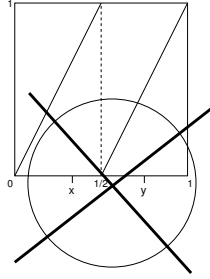


Figure 32: Sensitive dependence on initial conditions.

Example: Bernoulli shift on  $x \in [0, 1)$ ,  $B(x) = 2x \bmod 1$  is sensitive.

Proof: (simplified by excluding point of discontinuity)



Choose  $y$  close enough to  $x$  such that  $x, y$  do not hit different branches of  $B(x)$ ,

[include point using circle metric.]

New ‘reverse’ approach.

$$\delta = |B^m(x) - B^m(y)| = (\text{no discontinuity})|2^m x - 2^m y| = 2^m |x - y|.$$

Given  $\delta$ , choose  $y$  and  $m$ .

$$2^m = \frac{\delta}{|x-y|}, \quad x \neq y \Leftrightarrow m \ln 2 = \ln \frac{\delta}{|x-y|} \Leftrightarrow m = \frac{\ln \frac{\delta}{|x-y|}}{\ln 2}.$$

Choose  $\delta > 0$ ;  $\forall x$  and  $\forall N_\epsilon(x) \quad \exists y \in N_\epsilon(x)$  such that after  $n > m$  iterations  $|B^n(x) - B^n(y)| = 2^{n-m+m}|x - y| = 2^{n-m}2^m|x - y| = 2^{n-m}\delta > \delta, n > m \quad \square$ .

Definition: Chaos in the sense of Wiggins

A map  $F : J \rightarrow J, \quad J \subseteq \mathbb{R}$  is said to be W-chaotic on  $J$  if

1.  $F$  is topologically transitive.
2. It has sensitive dependence on initial conditions.

Again 2 ingredients (cf. simulations in section 2.1):

1. Sensitivity: stretching mechanism.
2. Topologically transitive:  $\exists$  a dense orbit, i.e. return to  $\epsilon$ -neighbourhoods  $\Rightarrow$  folding.

Theorem: Banks et al. [2], Glasner, Weiss [5]

If  $F$  is topologically transitive and  $\exists$  a dense set of periodic orbits  $\Rightarrow F$  sensitive  
( $D$ -chaos  $\Rightarrow W$ -chaos).

Example: Bernoulli shift is topologically transitive and has a dense set of periodic orbits  $\Rightarrow W$ -chaotic.

### 5.3 Ljapunov exponent

(Lyapunov=Ljapunov=Ljapunov)

Example: Bernoulli shift on  $[0, 1)$ ;

Consider 2 points that are originally displaced by  $\delta x_0 := |x'_0 - x_0|$  with  $\delta x_0$  ‘very small’ such that  $x_0, x'_0$  do not hit different branches around  $x = 1/2$ .

$$\delta x_n = |x'_n - x_n| = 2\delta x_{n-1} = 2^2\delta x_{n-2} = \dots = 2^n\delta x_0 = \underline{\exp(n \ln 2)\delta x_0};$$

Ljapunov exponent of  $B(x_0)$ ,  $\lambda(x_0) := \ln 2$ .

Exponential rate of separation of two nearby points as they follow their trajectories.

Generalisation  $\delta x_n = |x'_n - x_n| = |M^n(x'_0) - M^n(x_0)| =:$  (assuming  $\exists$  exponential separation!)  $\delta x_0 \exp(n\lambda(x_0))$ ,  $\delta x_0 \rightarrow 0$

$$\Rightarrow \lambda(x_0) = \lim_{n \rightarrow \infty} \lim_{\delta x_0 \rightarrow 0} \frac{1}{n} \ln \frac{\delta x_n}{\delta x_0} = \lim_{n \rightarrow \infty} \lim_{\delta x_0 \rightarrow 0} \frac{1}{n} \ln \frac{|M^n(x_0 + \delta x_0) - M^n(x_0)|}{\delta x_0} =$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \ln \left| \frac{dM^n(x)}{dx} \right|_{x=x_0} \quad \text{if } \exists \frac{dM(x)}{dx}.$$

$$\text{Use } \frac{dM^n(x)}{dx} \Big|_{x=x_0} = M'(x_{n-1}) \cdot M'(x_{n-2}) \cdot \dots \cdot M'(x_0).$$

$$\lambda(x_0) = \lim_{n \rightarrow \infty} \frac{1}{n} \ln \left| \prod_{i=0}^{n-1} M'(x_i) \right| = \underline{\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln |M'(x_i)|}.$$

Defines a time average: sum up  $n$  terms along the trajectory with initial condition  $x_0$  and average over  $n$  (Birkhoff average).

Definition: Let  $M \in C^1$  be a map of the real line.

The local Ljapunov number  $L(x_0)$  of the orbit  $\{x_0, x_1, \dots, x_{n-1}\}$  is defined as

$$\boxed{L(x_0) := \lim_{n \rightarrow \infty} \left| \prod_{i=0}^{n-1} M'(x_i) \right|} \quad \text{if } \exists \text{ limit (Ljapunov, PhD thesis 1892 [10]).}$$

Note:

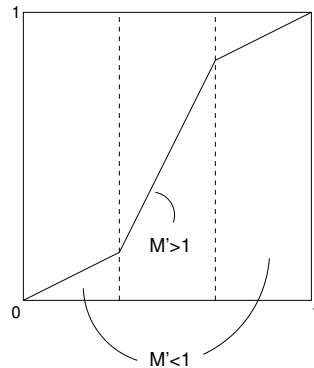
- If  $M$  is not  $\in C^1$  it may be piecewise  $C^1$  and the definition can still be applied by excluding single points.
- $\exists \lambda \Leftrightarrow \exists L \neq 0$  or  $\infty$ ; then  $\ln L = \lambda$ .
- If  $M'(x_i) = 0 \Rightarrow$  no  $\lambda(x)$  (typically, this concerns only a finite set of points).
- If  $M$  expanding,  $\forall x \quad |M'(x)| > 1 \Rightarrow \lambda(x) > 0$ .
- If  $M$  contracting,  $\forall x \quad |M'(x)| < 1 \Rightarrow \lambda(x) < 0$ .

Definition: Chaos in the sense of Ljapunov.

A map  $F : J \rightarrow J, \quad J \subseteq \mathbb{R}, \quad F \in C^1$  is said to be L-chaotic on  $J$  if

1.  $F$  is topologically transitive
2. It has a positive Ljapunov exponent for a typical initial condition  $x_0$ .

Why ‘typical’?



Fixed point  $x = 1/2$  +ve  $\lambda(x)$  i.e.  $\lambda(1/2) > 0$ . All other points attracted to  $\{0, 1\}$ !

Note:

- ‘Typical’ means ‘almost everywhere’ (i.e. except at points of Lebesgue measure zero).
- So far only local Ljapunov exponent defined,  $\lambda(x)$ ; later global  $\lambda$  will be defined such that topological transitivity and typicality are incorporated.

- ‘ $\lambda$  is a number for assessing whether a map is chaotic in the sense of exhibiting an exponential dynamical instability.’
- It is the case that  $\lambda > 0 \Rightarrow$  map sensitive, but ‘ $\Leftarrow$ ’ does not hold! That is, sensitivity is more general than L-chaos; not always exponential separation.

Ljapunov exponent for periodic points:

- Let  $p \in \mathbb{R}$  be a periodic point of period  $n$ . Then  $\lambda(p) = \frac{1}{n} \sum_{i=0}^{n-1} \ln |F'(p_i)|$ ,  $p = p_0; p_i = F^i(p)$ .
- Special case:  $p$  as a fixed point,  $\lambda(p) = \ln |F'(p)|$ .

Example 1: Bernoulli shift,  $B(x) = 2x \bmod 1$ .

$$B'(x) = 2 \quad \forall x \in [0, 1), x \neq \frac{1}{2}$$

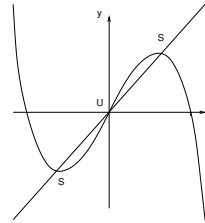
$$\Rightarrow \lambda(x) = \ln 2 \quad \forall x \neq \frac{1}{2}.$$

Example 2:  $F(x) = \frac{3x-x^3}{2}$  (see previously (figure 25).)

$$\text{Fix}(F) = \{0, \pm 1\}$$

$$\lambda\text{-exponents: } F'(x) = \frac{3}{2} - \frac{3}{2}x^2; \quad F'(0) = \frac{3}{2} \Rightarrow \lambda(0) = \ln \frac{3}{2} > 0.$$

$$F'(\pm 1) = 0 \Rightarrow \lambda(\pm 1) = \text{not defined!}$$



Summary (‘tentative’): Assume that map  $F \in C^1$  and topologically transitive. See figure 33.

Definition:  $F$  is topologically mixing if for any two open sets  $U, V \subset J \quad \exists N \in \mathbb{N}$  such that  $\forall n > N, n \in \mathbb{N}, \quad F^n(U) \cap V \neq \emptyset$ .

## 6 The Frobenius-Perron equation

Example: Do statistics on logistic map,  $L(x) = 4x(1-x)$ ,  $x \in [0, 1]$ , (see figure 34).

- Take an ensemble of  $N$  points with initial conditions  $x_0^i$ ,  $i = 1 \dots N$ , e.g. uniformly distributed on  $[0, 1]$ .

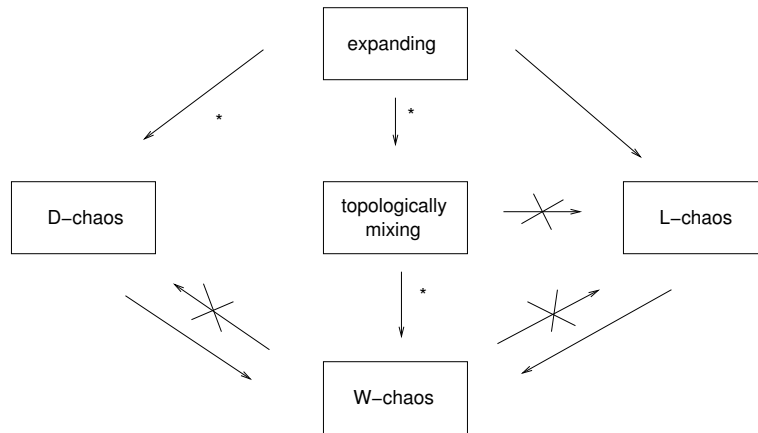


Figure 33:  $\star$ : See Hasselblatt/Katok [8] - proofs may depend on different technical requirements.

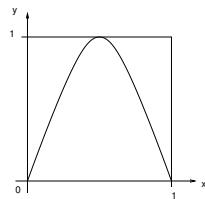


Figure 34: Logistic map.

- Iterate  $\forall x_0^i$  according to  $x_{n+1}^i = L(x_n^i)$ .
- Split  $[0, 1]$  into bins  $[\frac{j-1}{m}, \frac{j}{m})$ ,  $j = 1 \dots M$ .
- Construct histogram: how many of the  $N$  points are in the  $j^{\text{th}}$  bin at time step  $n$ ?

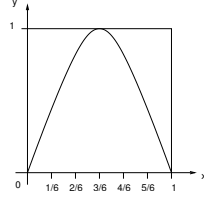


Figure 35: Binning of logistic map for  $m = 6$ .

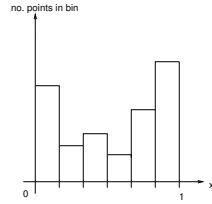


Figure 36: Histogram for logistic map at, e.g.  $n = 5$ .

Definition: Probability density  $\rho_n(x_j) := [\text{number of points in } j^{\text{th}} \text{ bin at position } x_j := \frac{2j-1}{2m}] / [(\text{total number of points } N) \cdot (\text{bin width } \delta x = \frac{1}{m})]$ .

Note:  $\rho$  is normalised;  $\sum_{j=1}^m \rho_n(x_j) \delta x = \frac{\sum_j N_j}{N \cdot \delta x} = \frac{N}{N} = 1$ . ( $N_j$  = number of points in bin  $j$ ) and  $\rho_n(x_j) \geq 0$ .

For  $m \rightarrow \infty$ ,  $(\delta x \rightarrow 0)$ ,  $N \rightarrow \infty$ :  $\sum_{j=1}^m \rho_n(x_j) \delta x \rightarrow \int_0^1 dx \rho_n(x) = 1$ .

Question:  $\exists$  equation to describe the dynamics of  $\rho_n(x)$ , i.e. variation under time  $n$  by iterating the map?

Answer: Frobenius-Perron equation.

Derivation: Basic idea is the conservation of the number of points of the ensemble.

Conservation of probability:  $\rho(y) \delta y = \rho(x) \delta x$ ,  $y = M(x)$  if map one-to-one!

Example: Bernoulli shift

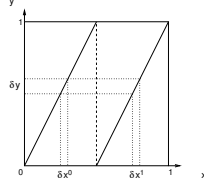


Figure 37: Bernoulli shift

If not one-to-one:  $\rho(y)\delta y = \sum_{y=M(x^i)} \rho(x^i)\delta x^i$

$$\Rightarrow \rho(y) = \sum_i \rho(x^i) \frac{\delta x^i}{\delta y} = \sum_i \rho(x^i) \frac{1}{\delta y / \delta x^i} = (\delta x^i \rightarrow 0) \sum_i \rho(x^i) \frac{1}{|M'(x^i)|}, \quad M'(x^i) \neq 0.$$

Identify  $\rho_{n+1}(x) = \rho(y), \quad \rho_n(x) = \rho(x)$

$$\Rightarrow \boxed{\rho_{n+1}(x) = \sum_{x=M(x^i)} \rho_n(x^i) |M'(x^i)|^{-1}}$$

is called the Frobenius-Perron equation (FPE). (Cp. with  $x_{n+1} = M(x_n)$ .)

The FPE can be represented by an operator applied onto the probability density function (PDF)  $\rho_n(x)$

$$\boxed{\rho_{n+1}(x) = P\rho_n(x)}, \text{ where } P \text{ is the Frobenius-Perron operator acting on } \rho_n : \\ P\rho_n(x) := \sum_{x=M(x^i)} |M'(x_n)|^{-1} \rho_n(x^i).$$

2 important properties of  $P$  :

1. Linearity:  $P(\lambda_1 \rho^1 + \lambda_2 \rho^2) = \lambda_1 P\rho^1 + \lambda_2 P\rho^2, \quad \lambda_1, \lambda_2 \in \mathbb{R}.$
2. Positivity:  $P\rho \geq 0$  if  $\rho \geq 0$ .

Note: Mathematically rigorously, define space of functions  $\rho$ , further (group) properties of  $P$  : see A. Lasota, M. C. Mackay, Chaos, fractals and noise [9].

Important question according to the following

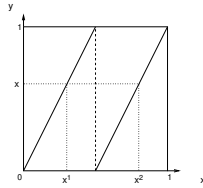
definition: Any function  $\rho^*(x)$  that is invariant under  $P$ ,

$\rho^*(x) = P\rho^*(x)$ , is called a fixed point of  $P$ .

$\rho^*(x)$  is also called an invariant density (does not change with time  $n$ ).

Crucial problem: to calculate  $\rho^*(x)$  for a given map, because  $\rho^*(x)$  provides statistical characteristics of such a map - what eventually happens to an ensemble of points.

Example:  $B(x) = 2x \bmod 1$ ;  $P\rho(x) = \sum_{x=M(x^i)} |M'(x^i)|^{-1} \rho(x^i)$ .



$$P\rho(x) = \frac{1}{|B'(x^1)|} \rho(x^1) + \frac{1}{|B'(x^2)|} \rho(x^2).$$

$$x = B(x^i), \quad i = 1, 2 = \begin{cases} 2x^1, & 0 \leq x^1 < \frac{1}{2} \\ 2x^2 - 1, & \frac{1}{2} \leq x^2 < 1 \end{cases}$$

Simplify further:

- $B'(x) = 2, \quad \forall x (x \neq \frac{1}{2})$ .

- Replace  $x^i = B^{-1}(x)$  piecewise:

$$x^1 = B^{-1}(x) = \frac{x}{2}, \quad 0 \leq x^1 < \frac{1}{2}.$$

$$x^2 = B^{-1}(x) = \frac{x+1}{2}, \quad \frac{1}{2} \leq x^2 < 1.$$

$$\Rightarrow \boxed{P\rho(x) = \frac{1}{2}\rho\left(\frac{x}{2}\right) + \frac{1}{2}\rho\left(\frac{x+1}{2}\right)}.$$

Explicit form of the Frobenius-Perron operator for  $B(x)$ .

Fixed point condition:  $\underline{\rho^*(x)} = P\rho^*(x) = \frac{1}{2}\underline{\rho\left(\frac{x}{2}\right)} + \frac{1}{2}\underline{\rho\left(\frac{x+1}{2}\right)}$ .

‘Hypothesis’:  $\rho^*(x) = 1$  (uniform distribution) check the above equation:

$$\rho^*(x) = 1 = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 1 = \underline{1} \text{ fulfilled.}$$

$\Rightarrow \rho^*(1)$  is a fixed point solution of this equation! (i.e., invariant density of  $B(x)$  is  $\rho^*(x) = 1$ .)

Question 1: Uniqueness of this solution? Or dependence on initial density  $\rho_0(x)$ ?

Example: (figure 38.)

or ... (figure 39.)

$\Rightarrow$  2 examples of maps for which there is no unique solution  $\rho^*(x)$ .

Question 2:  $\exists$  more general method of how to solve the FPE?

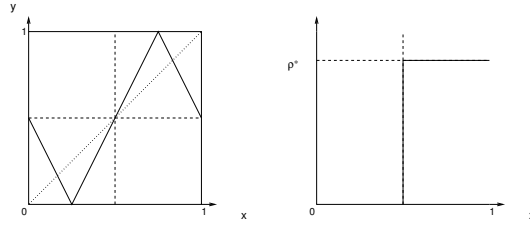


Figure 38: Example of a map with no unique invariant density.

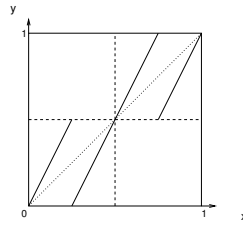


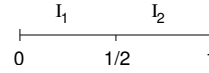
Figure 39: Another example of a map with no unique invariant density.

## 7 Partitions and transition matrices

Definition: A partition of the interval  $I \subset \mathbb{R}$  is a set  $\{I_i\}$ ,  $i = 1 \dots m$  of subintervals  $I_i$  such that

1.  $I = \bigcup_{i=1}^m I_i$ .
2.  $\text{int} I_i \cap \text{int} I_j = \emptyset (i \neq j)$ .

Example:  $I = [0, 1]$  unit interval,  $\{I_i\}_{i=1,2} = \{I_1, I_2\} = \{[0, \frac{1}{2}], [\frac{1}{2}, 1]\}$  is a partition.



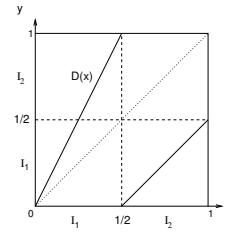
Question: If we apply a map onto a partition, what happens to the partition parts?

Example:

$$D(I_1) = I = I_1 \cup I_2.$$

$$D(I_2) = I_1.$$

Here: partition parts (or unions of them) are recovered exactly under the action of the map: Markov property.



Definition (verbal): For one-dimensional maps, a partition is called Markov iff parts of the partition get mapped again onto parts of the partition, or onto unions of parts of the partition.

Definition (formal): Let  $\{I_i\}_{i=1\dots m}$  be a partition of some interval  $I \subset \mathbb{R}$  and let  $M$  be a map acting on  $I$ ,  $M : I \rightarrow I$ . If  $\forall i, j [\text{int} M(I_j) \cap \text{int} I_i \neq \emptyset \Rightarrow \text{int} I_i \subseteq \text{int} M(I_j)]$ , then the partition is Markov.

Example:  $D(I_1) \cap I_1 \neq \emptyset \Rightarrow I_1 \subseteq D(I_1)$ . ✓

$D(I_1) \cap I_2 \neq \emptyset \Rightarrow I_2 \subseteq D(I_1)$ . ✓

$D(I_2) \cap I_1 \neq \emptyset \Rightarrow I_1 \subseteq D(I_2)$ . ✓

$D(I_2) \cap I_2 = \emptyset$ . ✓

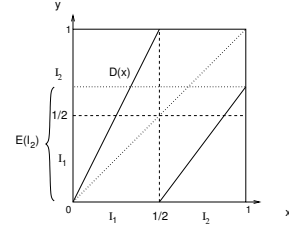
Counterexample:

$E(I_2) \cap I_1 \neq \emptyset \Rightarrow I_1 \subseteq E(I_2)$ . ✓

$E(I_2) \cap I_2 \neq \emptyset \Rightarrow I_2 \subseteq E(I_2)$ . ✗ not true!

Note:

- Loosely speaking (!), maps for which  $\exists$  Markov partition are called Markov maps.
- Markov partitions are not unique for a given map!
- Why important? Defines an invariant ‘frame’ according to which the action of a map can be described as a Markov process.



Next step: describe the dynamics of an ensemble of points with respect to Markov partition:

Example:

Let  $N_m^i$  be the number of points in partition part  $i$  at the  $n^{\text{th}}$  iterate of  $F$ .

Let us start with a uniform distribution of  $N = \sum_i N_n^i$  points on  $[0, 1]$  at  $n = 0$ .

$$N_1^1 = \frac{1}{2}N_0^1 + N_0^2 = \frac{1}{2}[N_0^1 + 2N_0^2].$$

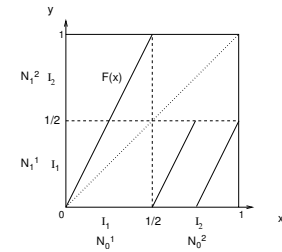


Figure 40: Map  $F$ .

$$N_1^2 = \frac{1}{2}N_0^1 + 0\dot{N}_0^2 = \frac{1}{2}N_0^1.$$

$$\begin{pmatrix} N_1^1 \\ N_1^2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} N_0^1 \\ N_0^2 \end{pmatrix}.$$

Or  $\underline{N}_{n+1} = \frac{1}{2}\underline{T}_F \underline{N}_n$  with column vector  $\underline{N} \in \mathbb{R}^2$  and the topological transition matrix (or transfer matrix)  $\underline{T}_F \in M_{2,2}$  as a linear operator.

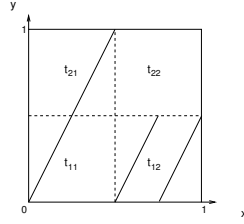


Figure 41: Illustration of where the transition matrix elements come from for map  $F$ .

Alternative representation of  $\underline{T}_M$  :

Definition: Let  $M : I \rightarrow I$ ,  $I \subset \mathbb{R}$  be a map;  $\{I_i\}$  a Markov partition and  $\underline{T}_M = ((t_{ij}))$ ,  $i, j \in \{1 \dots m\}$  be the corresponding transition matrix.

In a Markov graph  $t_{ij}$  arrows are drawn from  $j$  to  $i$ .

Example:

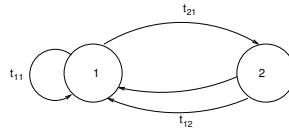


Figure 42: Markov graph for the map  $F(x)$ .

Note:

- Representation of symbolic dynamics of  $F$  defined with respect to the Markov partition in the Markov graph.
- Here  $t_{ij} \in \mathbb{N}_0$ ! in textbooks usually  $t_{ij} \in \{0, 1\}$ .

- Markov graph can be constructed without  $\underline{T}_M$  just by looking at the map  $M$  on the Markov partition. (First Markov graph, then  $\underline{T}_M$ .)

Definition: Let  $\{I_i\}$  be a Markov partition and let  $M$  be invertible on  $I_i$ . Then the topological transition matrix  $\underline{T}_M = ((t_{ij}))$ ,  $i, j \in \{1 \dots m\}$  is defined by

$$t_{ij} := \begin{cases} 1, & \text{int} I_i \subseteq \text{int} M(I_j) \\ 0 & \text{otherwise} \end{cases}$$

If  $M$  is not invertible on  $I_j$ , let  $\{I_{jk}\}, k = 1 \dots p$  (refinement).

$$\text{Then } t_{ij} := \begin{cases} \text{number of parts } I_{jk} \text{ such that } \text{int} I_i \subseteq \text{int} M(I_{jk}) \\ 0 \text{ otherwise} \end{cases}$$

Example:

$$\text{int} I_1 \subseteq \text{int} F(I_1) \Rightarrow t_{11} = 1.$$

$$\text{int} I_2 \subseteq \text{int} F(I_1) \Rightarrow t_{21} = 1.$$

$$\text{int} I_1 \subseteq \text{int} F(I_2) \text{ and } \text{int} I_1 \subseteq \text{int} F(I_2) \Rightarrow t_{12} = 2.$$

$$\text{int} I_2 \subseteq \text{int} F(I_2) \text{ not true} \Rightarrow t_{22} = 1.$$

$$\Rightarrow \underline{T}_F = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \text{ as before.}$$

Back to the action of  $\underline{T}_M$  on a Markov partition by putting everything together:  
Recall that for  $F(x)$  it was

$$\underline{N}_{n+1} = \frac{1}{2} \underline{T}_F \underline{N}_n : (N \cdot \text{diam}(I_i)) \quad (\text{diam}(I_i) = \frac{1}{2}).$$

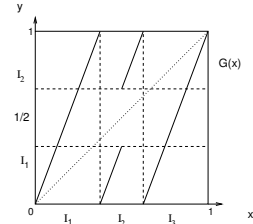
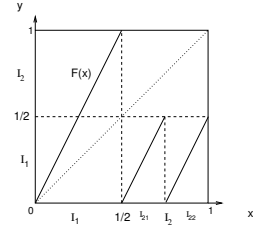
$\Rightarrow \underline{\rho}_{n+1} = \frac{1}{2} \underline{T}_F \underline{\rho}_n$ , where  $\frac{1}{2} \underline{T}_F$  is the Frobenius-Perron operator and  $\rho_n, \rho_{n+1}$  are probability density vectors whose components are piecewise constant on the Markov partition parts ('step functions').

Proposition: For piecewise linear maps  $M : I \rightarrow I$  with  $M'(x) = \text{const.} \forall x \in I$  (uniform slope, except at a countable number of points), if  $\exists$  a topological transition matrix  $\underline{T}_M$  the Frobenius-Perron equation can be written as

$$\underline{\rho}_{n+1} = \frac{1}{|M'(x)|} \underline{T}_F \underline{\rho}_n.$$

Idea of proof:

- Note that  $\frac{1}{2} \underline{T}_F$  for  $F$  is a stochastic matrix (sum of all entries of a column add up to one) - but here 'coincidence'. (Counterexample):  $G(x)$ ,



constant slope  $|G'(x)| \notin \mathbb{N} \Rightarrow \underline{T}_G$  not stochastic.

- See handout from Beck/Schlögel [3]: Construct the Frobenius-Perron operator for piecewise linear maps in terms of stochastic matrices.
- For  $|M'(x)| = \text{const.} \forall x \in I$ , simplify to topological transition matrices.

Note: That way, we have transformed the FP-equation onto a Markov chain.

Here: 
$$\underbrace{\rho_{n+1}^i}_{P_i} = C \cdot \sum_j \underbrace{t_{ij}}_{\tilde{P}_{ij}} \underbrace{\rho_n^j}_{P_j}.$$

Advantage of FP operator as a topological transition matrix:

The FP matrix equation can be used to calculate  $\rho^*$ !

Algorithm:

1. For a given map  $M$  defined on a Markov partition, construct  $\underline{T}_M$ .
2. Solve the eigenvalue problem of  $\underline{T}_M$ ,  $\underline{T}_M \rho_\tau = \tau \rho_\tau$  ( $\dagger$ ) (where  $\tau$  is an eigenvalue and  $\rho_\tau$  a corresponding eigenvector).
3. The largest  $\tau$  will always be equal to  $|M'(x)|$ ,  $\tau_{\max} = |M'(x)|$  ( $\star$ ), with eigenvector  $\rho_{\tau_{\max}}$ .
4. It follows:  $\rho_n = C \cdot \rho_{\tau_{\max}}$  ( $\times$ )

$$\begin{aligned} \Rightarrow \rho_{n+1} &\stackrel{FPE}{=} \frac{C}{|M'|} \underline{T}_M \rho_{\tau_{\max}} \stackrel{(\dagger)}{=} C \frac{\tau_{\max}}{|M'|} \rho_{\tau_{\max}} \stackrel{(\star)}{=} C \rho_{\tau_{\max}} \stackrel{(\times)}{=} \rho_n \\ \Rightarrow \boxed{C \cdot \rho_{\tau_{\max}} = \rho^*}! \end{aligned}$$

That is, the largest eigenvector  $\rho_{\tau_{\max}} = \rho^*$ .

5. Determine  $C$  by normalisation:  $\sum_{i=1}^m C \rho_{\tau_{\max}}^i \cdot \text{diam}(I_i) = 1$ .

Proposition: Under ‘certain assumptions’ on  $\underline{T}_M$ ,  $\exists$  unique  $\tau_{\max} > 0$  and a corresponding unique  $\rho_{\tau_{\max}}$  with  $\rho_{\tau_{\max}}^i \approx \rho^{*i} > 0$ !

Proof: Perron-Frobenius theorem for nonnegative matrices, see handout from Robinson [12].

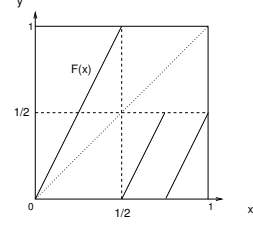
Example: Map  $F(x)$

- First part of algorithm see notes (around figure 40).

$\Rightarrow$  FP matrix equation for  $F(x)$  reads

$$\underline{\rho}_{n+1} = \frac{1}{2} \underline{T}_F \underline{\rho}_n \text{ with } \underline{T}_F = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}.$$

$\underline{\rho}_n = \begin{pmatrix} \rho_n^1 \\ \rho_n^2 \end{pmatrix} \in \mathbb{R}^2$ , where  $\rho_n^i = \text{const.}$  on the  $i^{\text{th}}$  Markov partition part.



- Solve eigenvalue problem:  $\underline{T}_F \underline{\rho}_\tau = \tau \underline{\rho}_\tau$ ;  $\det(\underline{T}_F - \tau \underline{I}) = 0$  ( $\underline{I}$  = identity  $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ).

$$\det \begin{pmatrix} 1-\tau & 2 \\ 1 & -\tau \end{pmatrix} = 0, \quad (1-\tau)(-\tau) - 2 = 0$$

$$-\tau + \tau^2 - 2 = 0 \Rightarrow \tau = \frac{1}{2} \pm \sqrt{\frac{1}{4} + 2} = \frac{1}{2} \pm \frac{3}{2} = \underline{2, -1}.$$

$\Rightarrow \tau_{\max} = |F'(x)|$  : ok, see step 3 of algorithm.

- Calculate largest eigenvector:  $\begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \rho_\tau^1 \\ \rho_\tau^2 \end{pmatrix} = 2 \begin{pmatrix} \rho_\tau^1 \\ \rho_\tau^2 \end{pmatrix} \Rightarrow \begin{pmatrix} \rho_\tau^1 + 2\rho_\tau^2 = 2\rho_\tau^1 \\ \rho_\tau^1 = 2\rho_\tau^2 \end{pmatrix}.$

$$\text{Assume } \rho_\tau^1 = 1 \Rightarrow \underline{\rho_\tau^2} = \frac{1}{2} \rho_\tau^1 = \frac{1}{2} \Rightarrow \underline{\rho_{\tau_{\max}}} = \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix}.$$

- Last step: Calculate  $\underline{\rho}^*$  by normalisation.

$\underline{\rho}^* = C \underline{\rho_{\tau_{\max}}}$  and normalisation:

$$1 = \sum_{i=1}^2 \rho^{*i} \cdot \text{diam}(I_i) = C \sum_{i=1}^2 \rho_\tau^i \cdot \text{diam}(I_i) = C(1 \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2}) = C \frac{3}{4}$$

$$\Rightarrow C = \frac{4}{3}.$$

$$\underline{\rho}^* = \frac{4}{3} \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{4}{3} \\ \frac{2}{3} \end{pmatrix}.$$

Cross-check with respect to the FP-equation.

$$\begin{pmatrix} \frac{4}{3} \\ \frac{2}{3} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{4}{3} \\ \frac{2}{3} \end{pmatrix} = \begin{pmatrix} \frac{2}{3} + \frac{2}{3} \\ \frac{2}{3} \end{pmatrix}. \quad \checkmark$$

## 8 Probability Measures

Motivation: Let  $\rho(x)$ ,  $x \in I$ , be a probability density; if  $\exists \rho(x)$  and if it is integrable on a subinterval  $A \subseteq I$ , then  $\mu(A) := \int_A dx \rho(x)$  is the probability of finding a point in  $A$ . In this case,  $\mu(A)$  is called a measure of  $A$ , that is, ‘assign a positive number to a set  $A$ ’.

Example:  $I = [0, 1]$ ,  $A = [0, \frac{1}{2}]$ ,  $\rho(x) = 1$  (e.g. Bernoulli shift).

$\Rightarrow \mu(A) = \int_0^{\frac{1}{2}} dx 1 = x|_0^{\frac{1}{2}} = \frac{1}{2}$  is the probability of finding a point in  $A$ .

Definition: A nonempty collection  $\mathcal{A}$  of subsets of a set  $I$  is called a  $\sigma$ -algebra if

1.  $A \in \mathcal{A} \Rightarrow I \setminus A \in \mathcal{A}$  complementarity.
2.  $A_1, A_2, \dots, A_n \in \mathcal{A} \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$  additivity.

Example:  $I = [0, 1]$ ,  $A_1 = [0, \frac{1}{2})$ ,  $A_2 = [\frac{1}{2}, 1]$ .

Is  $\mathcal{A} = \{A_1, A_2\}$   $\sigma$ -algebra?

1.  $A_1 \in \mathcal{A} \Rightarrow I \setminus A_1 = A_2 \in \mathcal{A}$  ✓.
2.  $A_1, A_2 \in \mathcal{A} \Rightarrow A_1 \cup A_2 = I \in \mathcal{A}$  is not the case!

But with  $I \in \mathcal{A} \Rightarrow I \setminus I = \emptyset \notin \mathcal{A}$ ,

so we find that  $\mathcal{A} = \{A_1, A_2, I, \emptyset\}$  is a  $\sigma$ -algebra.

Definition: A function  $\mu : \mathcal{A} \rightarrow \mathbb{R}_+ \cup \infty$  defined on a  $\sigma$ -algebra  $\mathcal{A}$  is called a measure if

1.  $\mu(A) \geq 0 \forall A \in \mathcal{A}$ .
2.  $\mu(\emptyset) = 0$ .
3.  $A_1, A_2, \dots, A_n \in \mathcal{A}$ ;  $A_k \cap A_l = \emptyset (k \neq l) \Rightarrow \mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$ ,  $\sigma$ -additivity.

Example:  $\mathcal{A} = \{A_1, A_2, I, \emptyset\}$  as before.

Let  $\mu(A) :=$  the length of the interval  $A$ ,

$A = [a, b) \Rightarrow \mu(A) = b - a$ ; is it a measure?

1.  $\mu(A) \geq 0 \forall A \in \mathcal{A}$  ✓ length never  $< 0$ .
2.  $\mu(\emptyset) = 0$ . ✓
3.  $A_1 \cap A_2 = \emptyset \Rightarrow \mu(A_1 \cup A_2) = \mu(I) = 1 = \mu(A_1) + \mu(A_2) = \frac{1}{2} + \frac{1}{2}$ . ✓  
 $A_1 \cup I \neq \emptyset$ . ✓  
 $A_2 \cup I \neq \emptyset$ . ✓  
 $A_1 \cup \emptyset = \emptyset \Rightarrow \mu(A_1 \cup \emptyset) = \mu(A_1) = \frac{1}{2} = \mu(A_1) + \mu(\emptyset)$  ( $\mu(\emptyset) = 0$ ). ✓  
 Same for  $A_2 \cap \emptyset$ ,  $I \cap \emptyset$ .

$\Rightarrow \mu(A)$  is a measure.  $\square$

This measure  $\mu_L$  is called the Lebesgue measure.

Definition: A measure space is a triple  $(I, \mathcal{A}, \mu)$  of a set  $I$ , a  $\sigma$ -algebra  $\mathcal{A}$  of subsets of  $I$  and a measure  $\mu$ .

In the following we will work on measure spaces (mostly without saying it).

For the example above,  $(I, \mathcal{A}, \mu_L)$  defines a measure space.

Definition:  $\mu$  is called a probability measure on  $I$  if  $\mu(I) = 1$  ('normalisation').

Example: For  $\mu_L$  on  $I = [0, 1]$  then  $\mu(I) = 1 \Rightarrow$  probability measure.

But not, say, for  $\tilde{I} = [0, 2]$ .

Note:  $\mu_L$  is only one example of a measure; another one is, e.g., the Dirac measure - see exercise!

Back to motivation: We had  $\mu(A) := \int_A dx \rho(x)$ .

Let  $\rho$  be a probability density and  $\mathcal{A}$  be a  $\sigma$ -algebra on  $I$ ,  $A \subseteq I$ . Then  $\mu(A)$  is a probability measure (check yourself).

Let us now assume that  $\rho(x)$  is generated by a map such as, e.g., the Bernoulli shift  $B(x) = 2x \bmod 1$ . We have that  $\exists$  Frobenius-Perron equation,

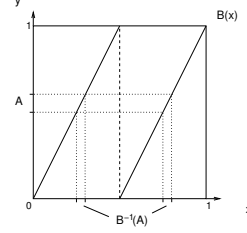
$\rho_{n+1}(x) = P_M \rho_n(x)$  with Frobenius-Perron operator  $P_M$  for a map  $M$  governing the time evolution of  $\rho_n(x)$ .

Question:  $\exists$  analogous equation of associated measure?

Answer: Check for  $B(x)$ .

Start again from ‘conservation of points’:

- Number of points  $x_n$  in  $B^{-1}(A)$  = number of points  $x_n + 1$  in  $A$  (because  $A = B \circ B^{-1}(A)$ ).
- Implies that the corresponding probability measures  $\mu(A) = \int_A dx \rho(x)$  are the same under iteration.



$$\boxed{\mu_{n+1}(A) = \mu_n(M^{-1}(A))} \text{ for a general map } M.$$

‘ $\mu$ -version of the Frobenius-Perron equation for  $\rho_n(x)$ .’

Note: Can also be proven by integrating the FP-equation.

Recall: The density,  $\rho^*(x) = P\rho^*(x)$  was called the invariant density.

Definition:  $\mu_{n+1}(A) = \mu_n(A) =: \mu^*(A)$  is called an invariant measure, i.e., the measure does not change with time.

Hence, for our ‘ $\mu$ -FP-equation’ it must hold that:

$$\mu^*(A) = \mu^*(M^{-1}(A)).$$

Definition: Consider a measure space  $(I, \mathcal{A}, \mu_L)$ , where  $I \subseteq \mathbb{R}$  and  $\mu_L$  is the Lebesgue measure. Let  $\rho$  be a probability density on  $I$ .

If  $\exists$  continuous  $\mu(A) := \int_A dx \rho(x) \quad \forall A \in \mathcal{A}$ ,  $\mu$  is called absolutely continuous with respect to  $\mu_L$ .

Note:

- Loosely speaking,  $\exists$  a.c. measure  $\mu$  means that  $\exists \mu \in C^0$  by integrating a density.
- $\rho \in C^0$  and bounded  $\Rightarrow \frac{d\mu}{dx} = \rho(x)$  and  $\mu$  is  $C^1$ -smooth corresponding to  $\rho$  (cf. fundamental theorem of calculus).

Example:  $B(x) = 2x \mod 1$  on  $[0, 1)$  with  $\rho^*(x) = 1$  :

$$\exists \text{ a.c. } \underline{\text{invariant}} \text{ measure } \mu^*(x) = \int_0^x d\tilde{x} \rho^*(\tilde{x}) = x.$$

If  $\rho$  is continuous except at finitely many points and bounded, then  $\exists \mu$ , but  $\mu \in C^0$ ?

Example:

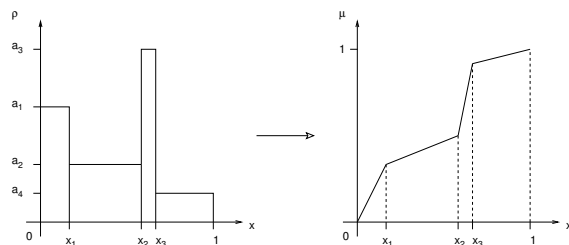


Figure 43: A step function density implies  $\mu(x) = \int_0^x d\tilde{x} \rho(\tilde{x}) \in C^0$ .

Definition: If  $\mu$  is not absolutely continuous, it is called singular.

Example: For  $B(x)$  let  $\tilde{\rho}^*(x) := \frac{1}{2}[\delta(x - \frac{1}{3}) + \delta(x - \frac{2}{3})]$  be the invariant density defined on the period 2 orbit  $\{\frac{1}{3}, \frac{2}{3}\}$  only!

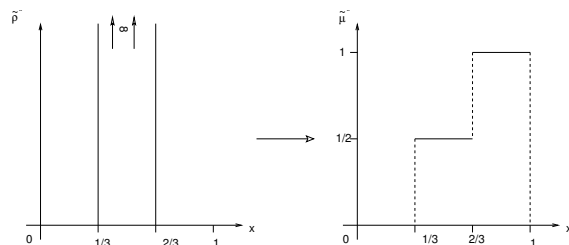


Figure 44: A delta function density implies  $\tilde{\mu}(x)$  is singular.

Definition: A measure  $\mu$  is called atomic if all of the measure is contained in a finite or countably infinite set of points.

Example:  $\tilde{\mu}^*$  is atomic.

For higher-dimensional dynamical systems we commonly have a combination of both types of measures:

Example: 1. 2-dimensional baker map.

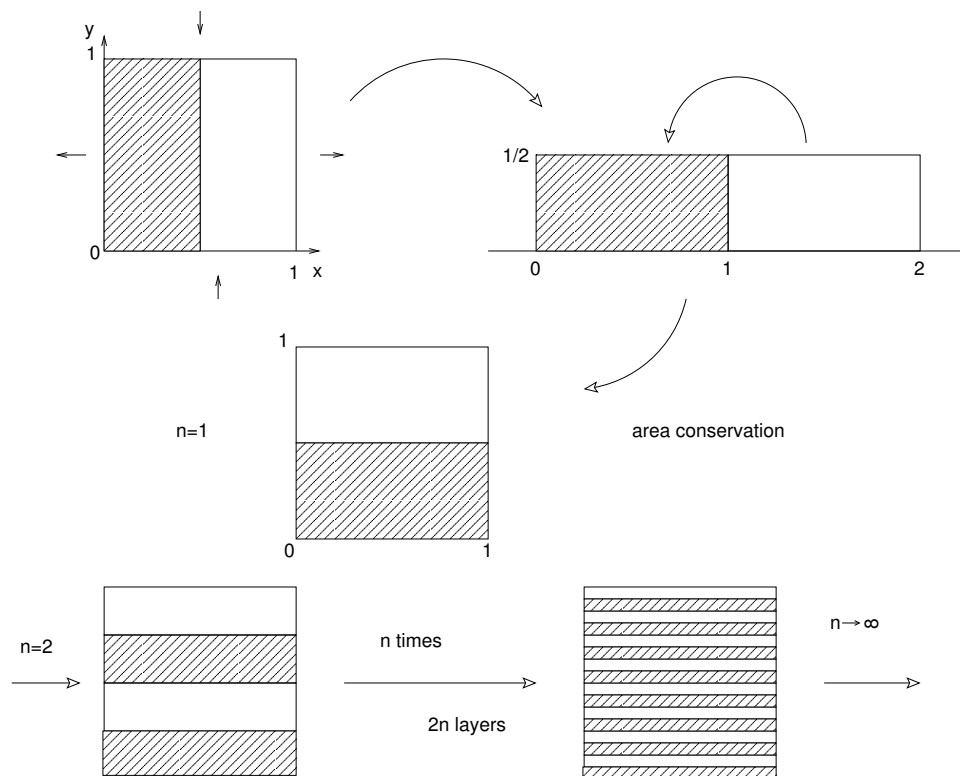


Figure 45: 2-dimensional baker map.

2. ‘Dissipative baker map’.

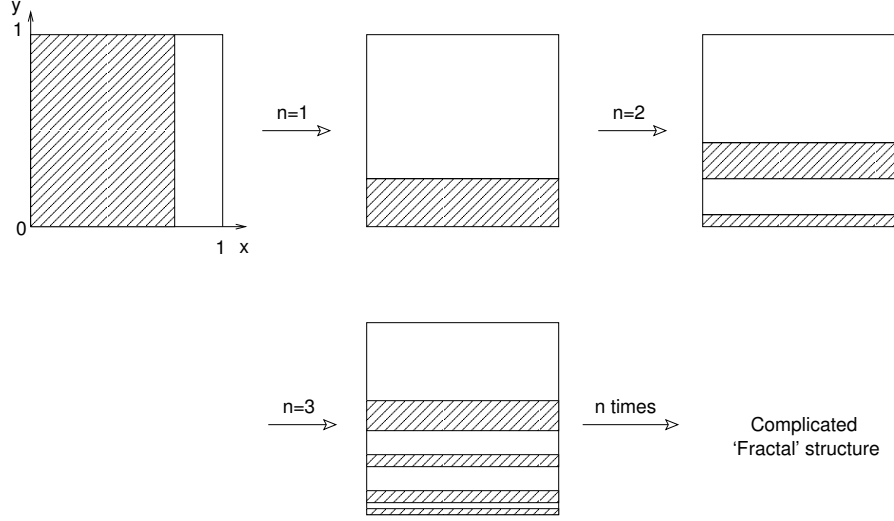


Figure 46: The dissipative baker map eventually generates a fractal structure from a step function density.

$$\mu^*(x) \in C^1.$$

$$\mu^*(y) \in C^0 \text{ but } \underline{\text{not}} C^1/\text{absolutely continuous!}$$

For this example, strictly speaking an invariant density  $\rho^*(x, y)$  does not exist, because  $\exists$  ‘fractal’ structure parallel to the  $y$ -axis, but there still exists an invariant measure  $\mu^*(x, y)$ !

Also  $\mu^*$  is smooth parallel to the  $x$ -axis but singular parallel to the  $y$ -axis.

Definition: An invariant probability measure  $\mu^*$  is called an SRB measure (Sinai, Ruelle, Bowen) if it is absolutely continuous along unstable manifolds.

(Unstable manifold = ‘set of initial conditions whose orbits are repelled from a source’, characterised by  $\lambda(x) > 0$ .)

Example: Bernoulli shift  $B(x) = 2x \bmod 1$  is unstable  $\forall x \in [0, 1) \Leftrightarrow B'(x) = 2$  expanding  $\mu^*(x) = \int_0^x d\tilde{x} \rho^*(\tilde{x}) = \int_0^x d\tilde{x} 1 = \underline{x}$  is SRB corresponding to 1!

## 9 Basics of ergodic theory

Let  $M : I \rightarrow I$ ,  $x_{n+1} = M(x_n)$ ,  $n \in \mathbb{N}_0$  be a one-dimensional map acting on the set  $I \subseteq \mathbb{R}$  and  $\mu^*$  be a corresponding invariant probability measure.

Let  $g : I \rightarrow \mathbb{R}$ ,  $g \in C^0$ , be some function ('observable').

Definition:

- $\overline{g(x)} := \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} g(x_k)$  is called the time average of  $g$  with respect to  $M(x)$  (Birkhoff average).

Example:  $g(x) = \ln |M'(x)| \Rightarrow \overline{g(x)}$  defines the Ljapunov exponent (see section (5.3)).

- $\langle g(x) \rangle := \int_I d\mu^* g(x)$ , where, if  $\exists$ ,  $d\mu^* = \underline{\rho^*(x)} dx$ .

This object is called the ensemble (or space) average of  $g$  with respect to  $M$ .

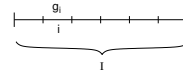
Example:  $\langle \ln |M'(x)| \rangle := \int_I dx \rho^*(x) \ln |M'(x)|$  for example,  $B(x) = 2x \mod 1$ ,  $B'(x) = 2$ :

- $\overline{\ln |B'(x)|} = \frac{1}{n} \sum_{k=0}^{n-1} \ln 2 = \underline{\underline{\ln 2}}$ .
- $\langle \ln |B'(x)| \rangle = \int_0^1 dx \rho^*(x) \ln |B'(x)| = \int_0^1 dx 1 \cdot \ln 2 = \underline{\underline{\ln 2}}$ .

Coincidence?

Step 1: Define partition on  $I$ : (uniform),

with  $g_i$  being the (local) average of  $g(x)$  with respect to the  $i$ -th partition part. The total number of partition parts in  $N$ .



Take a long trajectory  $\{x_0, x_1, \dots, x_{n-1}\}$  of  $M$  and let  $n_i$  be the number of occasions when the trajectory hits partition part  $i$ .

$$\Rightarrow \frac{1}{n} \sum_{k=0}^{n-1} g(x_k) \simeq \frac{1}{n} \sum_{i=1}^N n_i g_i = \sum_{i=1}^N \frac{n_i}{n} g_i \quad (\sum_{i=1}^N n_i = n).$$

Boltzmann's ergodic hypothesis (1871):

*The trajectory of a 'typical' point on  $I$  (i.e.,  $\forall$  points except a set of Lebesgue-measure zero) spends equal time in regions of equal (invariant probability) measure.*

$$\Rightarrow \frac{n_i}{n} = \frac{\mu_i^*}{\mu^*(I)} = \mu^*(I).$$

$\mu_i^*$  is the measure of region  $i$ !

$$\Rightarrow \overline{g(x)} = \sum_{i=1}^N \mu_i^* g_i \xrightarrow{(N \rightarrow \infty)} \int d\mu^* g(x) = \langle g(x) \rangle.$$

Step 2: Birkhoff's ergodic theorem (1931):

If  $\exists$  invariant probability measure  $\mu^*(I)$  with  $\int_I d\mu^* |g(x)| < \infty$ , then for ' $\mu^*$  almost every'  $x \in I$   $\overline{g(x)}$ .

Note: Only  $\overline{g(x)}$ ! Might be different for different  $x \in I$ !

Definition: A dynamical system  $M(x)$  is called ergodic if  $\overline{g(x)} = \langle g(x) \rangle$   $\mu^*$  almost everywhere. (= const.!)

Implies:  $\overline{g(x)}$  does not depend on  $x$ !

Definition: Assume  $I$  is the disjoint union of two sets  $I_1, I_2$  of positive measure, each of which is invariant under  $M : M(I) : M(I_1) = I_1, M(I_2) = I_2$ .

Such a system is called decomposable.

Proposition A dynamical system  $M(x)$  is ergodic  $\Leftrightarrow M$  on  $I$  is indecomposable. (Sometimes also used to define ergodicity.)

Example: 1. Decomposable  $\rightarrow$  not ergodic.

2. Bernoulli shift  $B(x)$  is ergodic/'not decomposable'.

Now back to Ljapunov exponent  $\lambda(x)$  :

If the map  $M$  is ergodic, then  $\overline{\ln |M'(x)|} = \langle \ln |M'(x)| \rangle = \lambda(x) = \lambda$ .

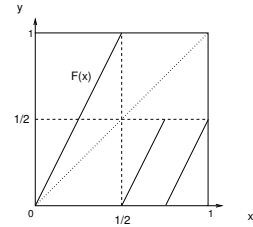
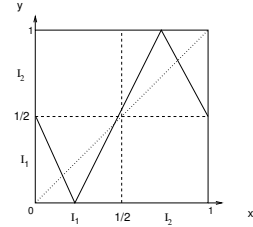
$\Rightarrow$  global quantity characterising  $M(x)$  irrespective of  $x$ !

This facilitates the calculation:

Example:  $\rho^*(x) = \begin{cases} \frac{4}{3}, & 0 \leq x < \frac{1}{2} \\ \frac{2}{3}, & \frac{1}{2} \leq x < 1 \end{cases}$

$\lambda$  with respect to the invariant probability density of  $F$  :

$$\lambda = \int_0^1 dx \rho^*(x) \ln |F'(x)| = \ln 2 \left[ \int_0^{\frac{1}{2}} dx \frac{4}{3} + \int_{\frac{1}{2}}^1 dx \frac{2}{3} \right].$$



$$= \ln 2(\frac{4}{3} \cdot \frac{1}{2} + \frac{2}{3} \cdot \frac{1}{2}) = \underline{\underline{\ln 2}}.$$

Definition: Let  $M : I \rightarrow I$  be a map and  $\mu^*$  be an invariant probability measure.

$\mu^*$  is called a physical or natural measure if for a function  $g : I \rightarrow \mathbb{R}$ ,  $g \in C^0$  (some observable)  $g(x) = \langle g(x) \rangle$  for  $\mu^*$  almost every  $x \in I$ .

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