MAS424/MTHM021 **Introduction to Dynamical Systems**

Exercise Sheet 4 **Rainer Klages**

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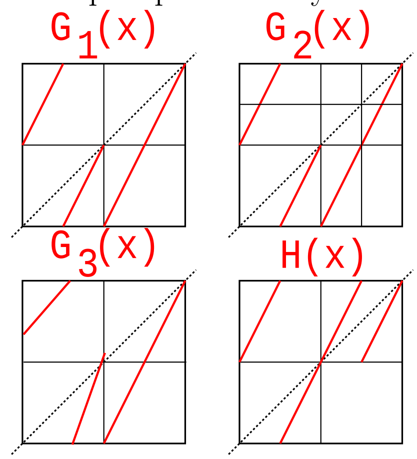
1. *Markov partitions and topological transition matrices*
   Consider the following four maps represented by the sloping bold lines below:



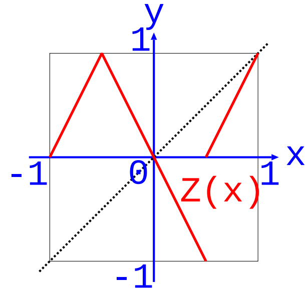
   The grids included in these figures define specific partitions for these maps, the dotted diagonals are guides to the eye.

   (a) Which of the four partitions shown are Markov partitions? Verify your answer by applying the formal definition of a Markov partition given in the lecture.

   (b) For the map $G_1(x)$, construct yourself another Markov partition (but not $G_2(x)$) and a partition that is not Markov. By using the verbal definition of a Markov partition given in the lecture, argue why they are Markov and not Markov.

   (c) If there exist Markov partitions in 1.(a), write down the topological transition matrices $\underline{\underline{T}}$ for the respective maps. Verify their construction by applying the *formal* definition of a topological transition matrix given in the lecture.

2. *Calculating invariant probability densities*
   Consider the map $Z(x)$ represented by the sloping bold lines below:



The quadratic grid included in this figure defines a Markov partition, the dotted diagonal is a guide to the eye.

(a) With respect to this partition, write down the topological transition matrix $\underline{\underline{T}}_Z$ for $Z(x)$. State the equivalent of the Frobenius-Perron equation for $\underline{\underline{T}}_Z$.

(b) By solving the eigenvalue problem of $\underline{\underline{T}}_Z$, calculate the invariant density $\varrho^*(x)$ for $Z(x)$, which is a piecewise constant function that you may write as a column vector $\underline{\varrho}^*$ defined over the Markov partition.

3. *σ-algebra and measures*

(a) Let $I \subset \mathbb{R}$ be an interval. Does $\mathcal{I} = \{I, \emptyset\}$, where $\emptyset$ stands for the empty set, define a σ-algebra?

(b) Consider the σ-algebra $\mathcal{A} = \{A_i\}_{i=1,\ldots,4} = \{[0,1], \emptyset, [0,1/2), [1/2,1],\}$ of the unit interval as discussed in the lecture. For given $a \in [0,1]$ let the Dirac measure be

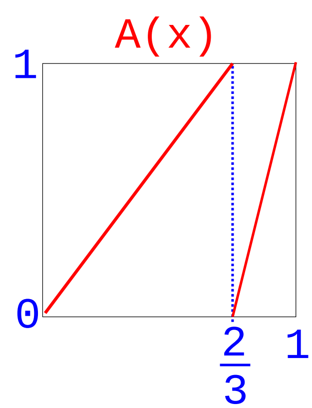
$$\delta_a(A) := \begin{cases} 1 & , \quad a \in A \\ 0 & , \quad a \notin A \end{cases}$$

Show that $\delta_a(A)$ defines a probability measure on $\mathcal{A}$.

4. *Ljapunov exponents and invariant densities*

(a) By using the definition of the Ljapunov exponent λ in terms of an invariant probability density $\underline{\varrho}^*(x)$, calculate λ for $Z(x)$.

(b) According to the same definition, calculate the Ljapunov exponent for the map $A(x)$ defined by the figure included below. Note that the invariant probability density for this map is uniform.



Model solutions will be on the course webpage starting from Thursday, December 13th, 2007.