Queen Mary University of London

MAS424/MTHM021 **Introduction to Dynamical Systems**

Exercise Sheet 2 **Rainer Klages**

If not said otherwise, in the following we consider one-dimensional maps $F : \mathbb{R} \to \mathbb{R}$, $x_{n+1} = F(x_n)$, $n \in \mathbb{N}_0$.

1. *Periodic points and stability*

   (a) For the map $C(x)$ from 3.(a) of exercise sheet 1, classify analytically the stability of all fixed points and of all prime period two orbits.

   (b) Consider the map defined by the function $D(x) := 2x^2 - 5x$. Calculate the set $Per_2(D)$ of all period 2 points for this map and classify analytically the stability of all fixed points and of all prime period two orbits.

   (c) Suppose F is of class C^1. State a criterion from which one can calculate the stability of a fixed point p for this map. Illustrate the following types of stability for p by drawing cobweb plots for three maps of your choice:
   (i) $F'(p) = 0$, (ii) $0 < F'(p) < 1$, (iii) $-1 < F'(p) < 0$.

2. *The Bernoulli shift*

   Consider the map $B_2(x) := 2x \bmod 1$, $x \in [0, 1)$.

   (a) Show that the number of periodic orbits of period n is $|Per_n(B_2)| = 2^n - 1$, as follows: Draw $B_2(x)$ and at least its second iterate. Identify $Fix(B_2)$ and at least $Per_2(B_2)$ in your drawings and calculate the corresponding periodic points analytically. Along these lines, calculate the result for general n. A proof by induction is not required.

   (b) Is $B_2(x)$ (piecewise) expanding and/or (piecewise) hyperbolic?

3. *Expanding, hyperbolicity and sensitivity*

   (a) Consider the map defined by the function

$$E(x) = \begin{cases} 2x+1 & , \quad -1 \le x < -1/2 \\ 2x & , \quad -1/2 \le x < 1/2 \\ 2x-1 & , \quad 1/2 \le x \le 1 \, . \end{cases}$$

   Draw the graph of this map. (i) Is $E(x)$ expanding or piecewise expanding? (ii) Is it hyperbolic or piecewise hyperbolic? (iii) Is it sensitive? (iv) Has it a dense set of periodic orbits? (v) Is it topologically transitive or minimal? Justify your answers (for (iii) and (iv) you may argue with results shown in the lectures).

(b) Consider the map defined by $R(x) := |x|$, $x \in \mathbb{R}$. Draw the graph of $R(x)$. (i) Is $R(x)$ expanding or piecewise expanding? (ii) Is it hyperbolic or piecewise hyperbolic? (iii) Has it a dense set of periodic orbits? (iv) Is it topologically transitive or minimal? Justify your answers.

(c) Prove that $R(x)$ is not sensitive (hint: proof by contradiction).

(d) Consider the quadratic map defined by $Q(x) = 1 - \mu x^2$, $x \in [-1, 1]$, where $\mu \in [0, 2]$ is a parameter. Is this map expanding? For which values of μ is it contracting?

(e) Suppose F is of class C^1. Prove that if F is expanding or contracting it is hyperbolic.

Model solutions will be on the course webpage starting from Wednesday, November 15th, 2007.