Diffusion in randomly perturbed dissipative dynamics

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Outline

1. **Motivation:**
   standard map, diffusion, and dissipation
2. **Randomly perturbed dissipative dynamics:**
   from invariant attractors over escape from pseudo attractors to hopping between attractors
3. **Randomly perturbed dissipative standard map:**
   simulation results for escape and diffusive spreading
4. **Continuous Time Random Walk theory:**
   match simulation results to analytical predictions from stochastic theory

The standard map and diffusion

- paradigmatic Hamiltonian dynamical system:

standard map

$$x_{n+1} = x_n + y_n \mod 2\pi$$
$$y_{n+1} = y_n + K \sin x_{n+1}$$

derived from kicked rot(at)or where $x_n \in \mathbb{R}$ is an angle, $y_n \in \mathbb{R}$ the angular velocity with $n \in \mathbb{N}$ and $K > 0$ the kick strength

- define **diffusion coefficient** as

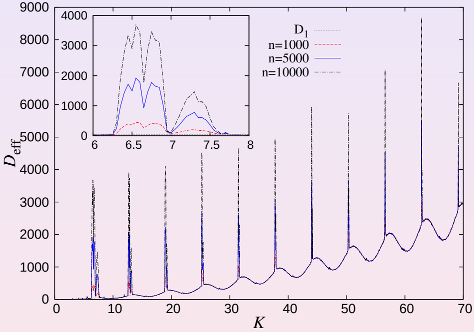
$$D(K) = \lim_{n\to\infty} \frac{1}{n} < (y_n - y_0)^2 >$$

with ensemble average over the initial density

$< \ldots >= \int dx\, dy\ \varrho(x,y) \ldots\ , x \in [0, 2\pi)\ ,\ y = y_0 \in [0, 2\pi)$

Diffusion in the standard map

analytical (Rechester, White, 1980) and numerical studies of parameter-dependent diffusion $D(K)$:



Manos, Robnik, PRE (2014)

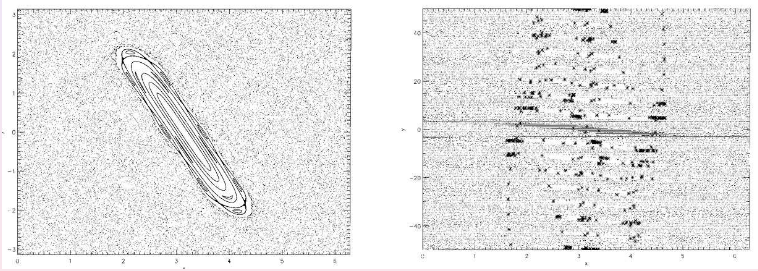
- $D(K)$ is highly irregular
- for some K there is superdiffusion with mean square displacement $< y_n^2 > \sim n^\gamma$, $\gamma > 1$ due to accelerator modes

The dissipative standard map

model damping in the standard map by

$$x_{n+1} = x_n + y_n \mod 2\pi$$
$$y_{n+1} = (1-\nu)y_n + f_0 \sin x_{n+1}$$

with $\nu \in [0, 1]$:



Feudel, Grebogi, Hunt, Yorke, PRE (1996)

- islands in phase space for $\nu = 0$ (left) become coexisting attractors (right): 150 found for $\nu = 0.02$, $f_0 = 4$
- no long-time diffusion: points converge onto attractors

Dissipative dynamics and random perturbations

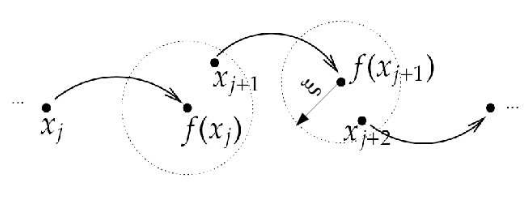
Question: What happens to dissipative deterministic dynamics under random perturbations?

Consider the dissipative standard map with additive noise:

$$\begin{aligned} x_{n+1} &= x_n + y_n + \epsilon_{x,n} \mod 2\pi \\ y_{n+1} &= (1-\nu)y_n + f_0 \sin x_{n+1} + \epsilon_{y,n} \end{aligned}$$

with iid random variables $\epsilon_n = (\epsilon_{x,n}, \epsilon_{y,n})$ drawn from uniform distribution bounded by $||\epsilon_n|| < \xi$ of noise amplitude ξ

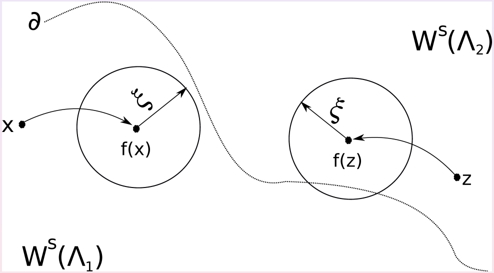
perturbed dynamics:



From attractors to hopping on pseudo attractors

Consequences of the random perturbations:

• beyond a noise threshold $\xi \geq \xi_0$ the attracting sets $W^S(\Lambda_i)$ lose their stability due to holes

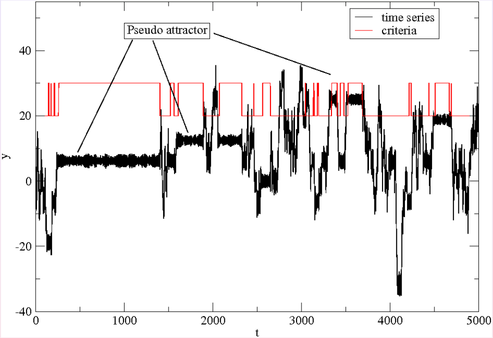


• the (invariant) attractors become (quasi-invariant) pseudo attractors from which there is noise-induced escape

• the noise induces a hopping process between all coexisting pseudo attractors

Intermittency and stickiness

the resulting perturbed dissipative dynamics is intermittent:

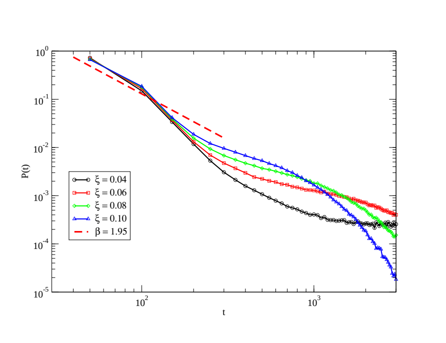


$$f_0 = 4 \ , \ \xi = 0.06 \ , \ \nu = 0.002$$

• stickiness to pseudo attractors measured by criterium that maximal eigenvalue of the Jacobian matrix along orbit < 1

Escape time distribution

probability distributions $P(t)$ of escape times t from pseudo attractors computed by using eigenvalue criterium (and a Markov assumption by averaging over all non-uniform pseudo attractors):

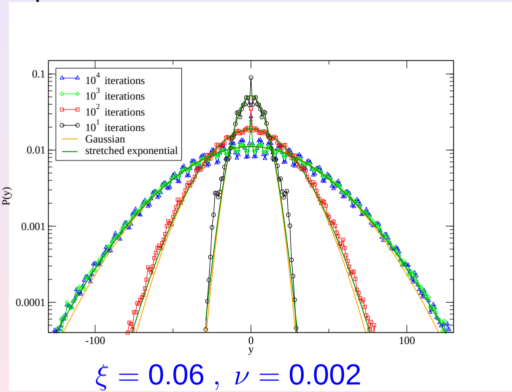


dissipation $\nu = 0.002$ with different noise strength ξ

- transition from power law (stickiness) to exponential
- transition takes longer when $\xi \to 0$

Diffusion

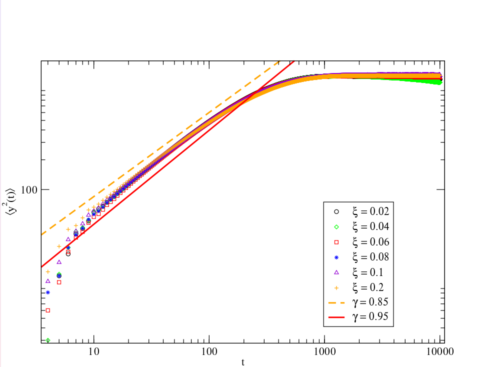
probability distribution function $P_n(y)$ for position y_n at different time steps n:



- there is Gaussian-like diffusive spreading up to $n < 1000$
- localization trivvially due to boundedness of pseudo attractors

Mean square displacement

mean square displacement $< y_n^2 >$ for position y_n and different noise amplitudes ξ at $\nu = 0.002$:



- transient subdiffusion $< y_n^2 > \sim n^\gamma$ up to $n < 1000$
- only small variation of the subdiffusive exponent $0.85 < \gamma < 0.95$ for different ξ

Continuous time random walk theory

reproduce simulation results by **CTRW theory** (Montroll, Weiss, Scher, 1973): define stochastic process by master equation with *waiting time distribution* $w(t)$ and *jump distribution* $\lambda(x)$

$$\varrho(x,t) = \int_{-\infty}^{\infty} dx' \lambda(x-x') \int_0^t dt' \, w(t-t') \, \varrho(x',t') + \\ +(1-\int_0^t dt' w(t'))\delta(x)$$

structure: jump + no jump for points starting at $(x,t)=(0,0)$

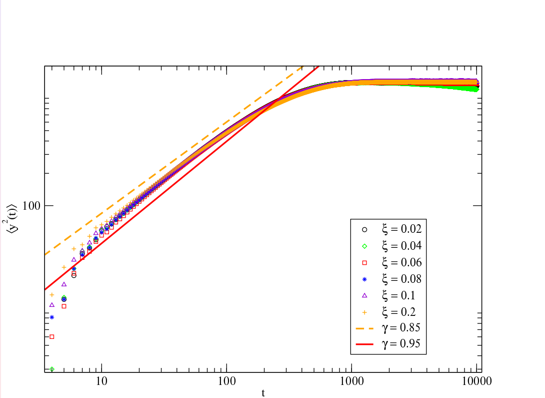
$\hat{\text{F}}$ourier-$\tilde{\text{L}}$aplace transform yields **Montroll-Weiss eqn (1965)**

$$\boxed{\hat{\tilde{\varrho}}(k,s) = \frac{1-\tilde{w}(s)}{s} \frac{1}{1-\hat{\lambda}(k)\tilde{w}(s)}}$$

with mean square displacement $\langle \tilde{x^2}(s) \rangle = - \left. \frac{\partial^2 \hat{\tilde{\varrho}}(k,s)}{\partial k^2} \right|_{k=0}$

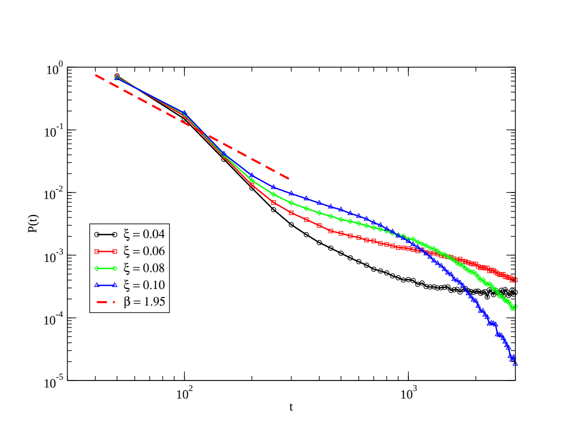
CTRW theory and mean square displacement

CTRW theory predicts that solving the MW eqn. for a power law waiting time distribution $w(t) \sim t^{-(\gamma+1)}$ with jump distribution $\lambda(x) = \delta(|x| - 1)$ yields $< x^2(t) > \sim t^\gamma$



for $\nu = 0.002$, $\xi = 0.06$ we have $< y_n^2 > \sim n^\gamma$ with $\gamma \simeq 0.95$

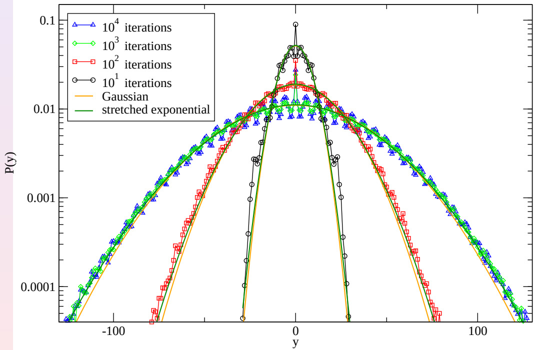
CTRW theory and escape time distribution



the dashed red line represents the CTRW theory prediction of $P(t) \sim t^{-1.95}$ corresponding to $< y_n^2 > \sim n^{0.95}$

CTRW theory and position pdf

CTRW theory also predicts a stretched exponential position pdf, here: $P_n(y) \sim \exp\left(-cx^{2/(2-\gamma)}\right)$



green lines represent the CTRW theory pdf for $\gamma = 0.95$: corrects the mismatch to Gaussian in the tails

Summary

- **central theme:** study of *diffusion generated by randomly perturbing dissipative deterministic dynamics*
- **main result:** for the dissipative standard map non-hyperbolic stickiness to pseudo attractors under random perturbations generates
  - *power law escape time distributions* and
  - *stretched exponential position distributions* leading to
  - *subdiffusion*

  simulation results consistently explained by *CTRW theory*
- **outlook:** similar phenomena in other randomly perturbed deterministic dynamical systems?

reference:
Christian S. Rodrigues et al., submitted (soon on arXiv)