Weak chaos, infinite ergodic theory, and anomalous diffusion

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Motivation

MAX-PLANCK-INSTITUT FÜR PHYSIK KOMPLEXER SYSTEME
DRESDEN, GERMANY

International Seminar and Workshop on
Weak Chaos, Infinite Ergodic Theory, and Anomalous Dynamics

Seminar: July 25 - 29 and August 8 - 12, 2011
Workshop: August 1 - 5, 2011

Scientific coordination:

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Organisation:
Katrin Lantsch

(deadline for applications has passed - sorry)

this talk: focus on anomalous diffusion

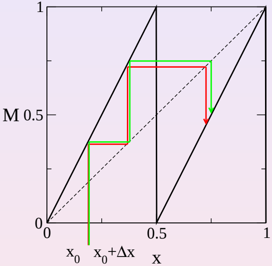
Outline

1. setting the scene: **chaos** in a simple map
2. from chaos to **weak chaos** and to **infinite ergodic theory**
3. **anomalous diffusion** in a simple map and **continuous time random walk theory**
4. **fractional diffusion equations** and anomalous biological **cell migration**

Bernoulli shift and Ljapunov exponent

warmup: deterministic chaos modeled by a simple 1d map

Bernoulli shift $M(x) = 2x \bmod 1$ with $x_{n+1} = M(x_n)$:



apply small perturbation $\Delta x_0 := \tilde{x}_0 - x_0 \ll 1$ and iterate:

$$\Delta x_n = 2\Delta x_{n-1} = 2^n \Delta x_0 = e^{n \ln 2} \Delta x_0$$

$\Rightarrow$ exponential dynamical instability with Ljapunov exponent $\lambda := \ln 2 > 0$: **Ljapunov chaos**

Ljapunov exponent and ergodicity

local definition for one-dimensional maps via *time average*:

$$\lambda(x) := \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln \left| M'(x_i) \right| \ , \ x = x_0$$

if map is ergodic: time average = ensemble average,

$$\lambda = \langle \ln |M'(x)| \rangle_\mu$$, cf. **Birkhoff's theorem**

with $< \ldots >_\mu = \int_I dx \, \varrho(x) \ldots$ average over the invariant probability density $\varrho(x)$ related to the map's SRB measure via $\mu(A) = \int_A dx \, \varrho(x) \ , \ A \subseteq I$

Bernoulli shift is *expanding*: $\forall x \, |M'(x)| > 1$, hence 'hyperbolic'

normalizable pdf exists, here simply $\varrho(x) = 1 \Rightarrow \lambda = \ln 2$

Pesin's theorem

Theorem

For closed C^2 Anosov systems the sum of positive Lyapunov exponents is equal to the Kolmogorov-Sinai entropy.
Pesin (1976), Ledrappier, Young (1984)

(believed to hold for a wider class of systems)

for one-dimensional hyperbolic maps: $\boxed{\lambda = h_{ks}}$ with

$$h_{ks} := \lim_{n\to\infty} -\frac{1}{n} \sum_{w\in\{W_i^n\}} \mu(w) \ln \mu(w)$$

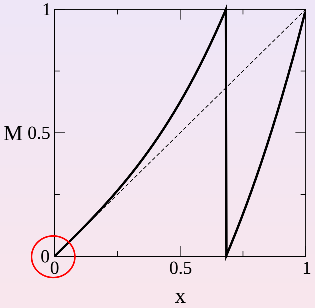
(if the partition is generating), where $\mu(w)$ is the SRB measure of an element w of the partition $\{W_i^n\}$ and n defines the level of refinement

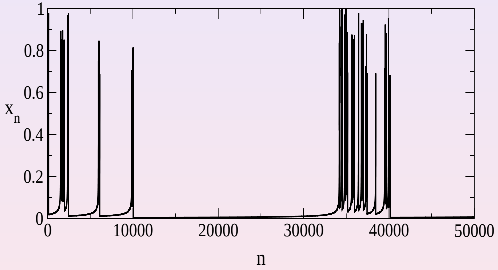
$h_{ks} > 0$: **measure-theoretic chaos**

Anomalous dynamics

consider the nonlinear Pomeau-Manneville map

$$x_{n+1} = M(x_n) = x_n + x_n^z \mod 1 \, , \, z \geq 1$$





phenomenology of **intermittency**: long periodic *laminar phases* interrupted by *chaotic bursts*; here due to an indifferent fixed point, $M'(0) = 1$ (Pomeau, Manneville, 1980)

$\Rightarrow$ map not hyperbolic ($\nexists N > 0$ s.t. $\forall x \forall n \geq N \; |(M^n)'(x)| \neq 1$)

Infinite invariant measure and dynamical instability

- **invariant density** of this map calculated to

$$\varrho(x) \sim x^{1-z} \ (x \to 0)$$

Thaler (1983)

is non-normalizable for $z \geq 2$ yielding an **infinite invariant measure** ($\rightarrow$ **infinite** ergodic theory!)

$$\mu(x) = \int_x^1 dy\, \varrho(y) \to \infty \ (x \to 0)$$

- **dispersion of nearby trajectories** calculated to

$$\Delta x_n \sim \exp\left(n^{\frac{1}{z-1}}\right) \Delta x_0 \ (z > 2)$$

Gaspard, Wang (1988)

grows weaker than exponential yielding $\lambda = 0$: **weak chaos**

Zaslavsky, Usikov (2001)

From ergodic to infinite ergodic theory

"weak ergodicity breaking:" choose a 'nice' *observable* $f(x)$

- for $1 \leq z < 2$ it is $\sum_{i=0}^{n-1} f(x_i) \sim n \ (n \to \infty)$

Birkhoff's theorem: if M is ergodic then $\frac{1}{n}\sum_{i=0}^{n-1} f(x_i) = < f >_\mu$

- but for $2 \leq z$ we have the **Aaronson-Darling-Kac theorem**,

$$\frac{1}{a_n}\sum_{i=0}^{n-1} f(x_i) \xrightarrow{d} \mathcal{M}_\alpha < f >_\mu \ (n \to \infty)$$

$\mathcal{M}_\alpha$: random variable with normalized *Mittag-Leffler* pdf

for M it is $a_n \sim n^\alpha$, $\alpha := 1/(z-1)$; integration wrt to Lebesgue measure m suggests

$$\frac{1}{n^\alpha}\sum_{i=0}^{n-1} < f(x_i) >_m \sim < f(x) >_\mu$$

note: for $z < 2 \Rightarrow \alpha = 1$ $\exists$ absolutely continuous invariant measure μ, and we have an equality; for $z \geq 2$ $\exists$ infinite invariant measure, and it remains a *proportionality*

Weak chaos quantities

This motivates to define a **generalized Ljapunov exponent**

$$\Lambda(M) := \lim_{n\to\infty} \frac{\Gamma(1+\alpha)}{n^\alpha} \sum_{i=0}^{n-1} < \ln \left| M'(x_i) \right| >_m$$

and analogously a **generalized KS entropy**

$$h_{KS}(M) := \lim_{n\to\infty} -\frac{\Gamma(1+\alpha)}{n^\alpha} \sum_{w\in\{W_i^n\}} \mu(w) \ln \mu(w)$$

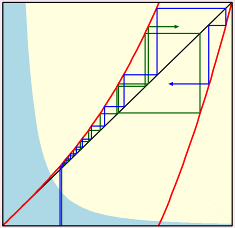
new result: Howard, RK, tbp

Shown analytically for a piecewise linearization of M that

$$\boxed{h_{KS}(M) = \Lambda(M)} ,$$

cf. **Rokhlin's formula**, generalizing *Pesin's theorem* to anomalous dynamics; see also Korabel, Barkai (2009)

summary:

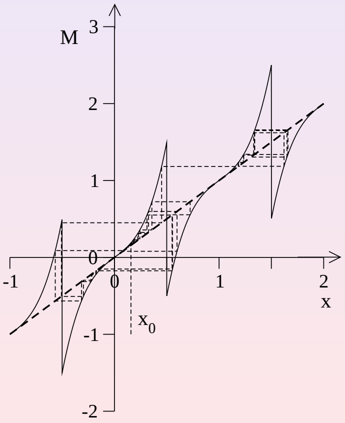


wchaos conference logo

An intermittent map with anomalous diffusion

continue map $M(x) = x + ax^z$, $0 \le x \le 1/2$, $a \ge 1$ by
$M(-x) = -M(x)$ and $M(x+1) = M(x) + 1$

Geisel, Thomae (1984); Zumofen, Klafter (1993)



deterministic random walk on the line; classify diffusion in terms of the **mean square displacement**

$$\langle x^2 \rangle = K\, n^\alpha \; (n \to \infty)$$

if $\alpha \neq 1$ one speaks of **anomalous diffusion**; here subdiffusion:

$$\alpha = \begin{cases} 1, & 1 \le z \le 2 \\ \frac{1}{z-1} < 1, & 2 < z \end{cases}$$

goal: calculate generalized diffusion coefficient $K = K(z, a)$

CTRW theory I: Montroll-Weiss equation

Montroll, Weiss, Scher, 1973:

master equation for a stochastic process defined by *waiting time distribution* $w(t)$ and *distribution of jumps* $\lambda(x)$:

$$\varrho(x,t) = \int_{-\infty}^{\infty} dx' \lambda(x-x') \int_0^t dt'\, w(t-t')\, \varrho(x',t') + \\ +(1-\int_0^t dt' w(t'))\delta(x)$$

structure: jump + no jump for particle starting at $(x,t)=(0,0)$
$\hat{\text{F}}$ourier-$\tilde{\text{L}}$aplace transform yields **Montroll-Weiss eqn (1965)**

$$\hat{\tilde{\varrho}}(k,s) = \frac{1-\tilde{w}(s)}{s} \frac{1}{1-\hat{\lambda}(k)\tilde{w}(s)}$$

with mean square displacement $\langle \tilde{x^2}(s) \rangle = -\left.\frac{\partial^2 \hat{\tilde{\varrho}}(k,s)}{\partial k^2}\right|_{k=0}$

CTRW theory II: application to maps

apply CTRW to maps: need $w(t), \lambda(x)$ (Klafter, Geisel, 1984ff)

sketch:

- $w(t)$ calulated from $w(t) \simeq \varrho(x_0) \left| \frac{dx_0}{dt} \right|$ with density of initial positions $\varrho(x_0) \simeq 1$, $x_0 = x(0)$; for waiting times $t(x_0)$ solve the continuous-time approximation of the PM-map
$x_{n+1} - x_n \simeq \frac{dx}{dt} = ax^z, \quad x \ll 1$ with $x(t) = 1$

- revised (Korabel, RK et al., 2007) ansatz for jumps:
$$\lambda(x) = \frac{p}{2}\delta(|x| - \ell) + (1-p)\delta(x)$$
with average jump length ℓ and escape probability p

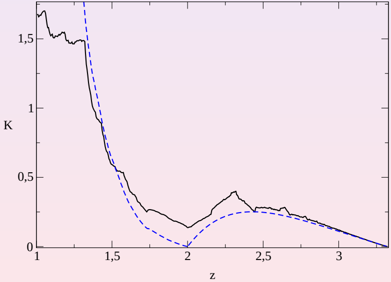
plug results into MW eqn: CTRW machinery ... yields ...

Transition from normal to anomalous diffusion

$$< x^2 > \sim \begin{cases} \frac{n}{\ln n} \ , \ n < n_{cr} \text{ and } \sim n \ , \ n > n_{cr} \ , & z < 2 \\ \frac{n}{\ln n} \ , & z = 2 \\ \frac{n^\alpha}{\ln n} \ , \ n < \tilde{n}_{cr} \text{ and } \sim n^\alpha \ , \ n > \tilde{n}_{cr} \ , & z > 2 \end{cases}$$

$\exists$ **suppression of diffusion** due to logarithmic corrections

compare CTRW approximation to simulations for $K(z,5)$:



Korabel, RK et al. (2007)

note: $\exists$ fractal structure $\rightarrow$ details see poster by G.Knight

Time-fractional equation for subdiffusion

For the lifted **PM map** $M(x) = x + ax^z \mod 1$, the MW equation in long-time and large-space asymptotic form reads

$$s^\gamma \hat{\tilde{\varrho}} - s^{\gamma-1} = -\frac{p\ell^2 a^\gamma}{2\Gamma(1-\gamma)\gamma^\gamma} k^2 \hat{\tilde{\varrho}} \ , \ \gamma := 1/(z-1)$$

LHS is the Laplace transform of the Caputo fractional derivative

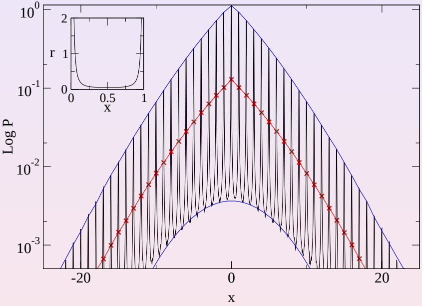
$$\frac{\partial^\gamma \varrho}{\partial t^\gamma} := \begin{cases} \frac{\partial \varrho}{\partial t} & \gamma = 1 \\ \frac{1}{\Gamma(1-\gamma)} \int_0^t dt' (t-t')^{-\gamma} \frac{\partial \varrho}{\partial t'} & 0 < \gamma < 1 \end{cases}$$

transforming the Montroll-Weiss eq. back to real space yields the time-fractional (sub)diffusion equation

$$\boxed{\frac{\partial^\gamma \varrho(x,t)}{\partial t^\gamma} = K \frac{\Gamma(1+\alpha)}{2} \frac{\partial^2 \varrho(x,t)}{\partial x^2}}$$

Deterministic vs. stochastic density

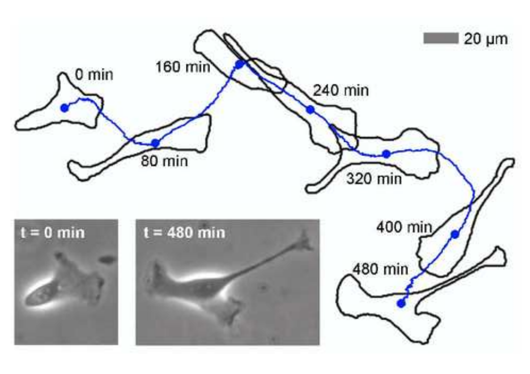
initial value problem for fractional diffusion equation can be solved exactly; compare with simulation results for $P = \varrho_n(x)$:



- fine structure due to density on the unit interval $r = \varrho_n(x)$ $(n \gg 1)$ (see inset)
- Gaussian and non-Gaussian envelopes (blue) reflect intermittency (Korabel, RK et al., 2007)

Anomalous dynamics in biological cell migration

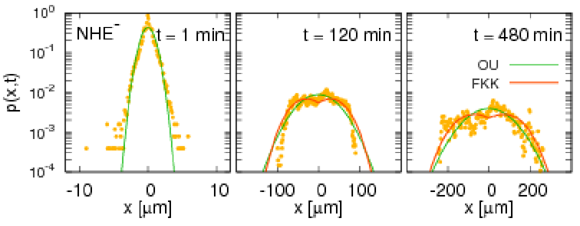
outlook: single biological cell crawling on a substrate; trajectory recorded with a video camera



Dieterich, RK et al., PNAS (2008)

Position distribution function: experiment and theory

pdf $P(x,t)$ of cell positions x (in 1d) at time t from experiment



green lines: Gaussians; red lines: solution of the fractional (super)diffusion equation (Schneider, Wyss, 1989)

$$\frac{\partial P(x,t)}{\partial t} = {}_0D_t^{1-\alpha} K_\alpha \frac{\partial^2}{\partial x^2} P(x,t)$$

with generalized diffusion coefficient K_α and Riemann-Liouville fractional derivative ${}_0D_t^{1-\alpha}$; for more evidence that this works see Dieterich, RK et al., PNAS (2008)

Summary

1. **weak chaos quantities** inspired by infinite ergodic theory
2. simple **weakly chaotic map** exhibiting complex **subdiffusion**
3. **continuous time random walk theory** and **fractional diffusion equations**
4. cross-link to **anomalous biological cell migration**

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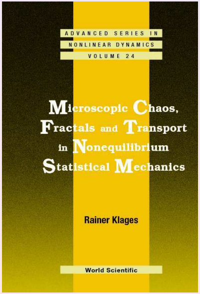
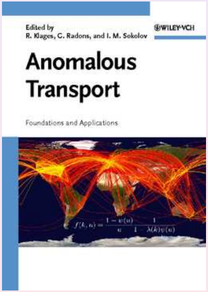
literature:





for cell migration: Dieterich et al., PNAS **105**, 459 (2008)