

## Diffusion in randomly perturbed dissipative dynamics

CHRISTIAN S. RODRIGUES<sup>1(a)</sup>, ALEKSEI V. CHECHKIN<sup>2</sup>, ALESSANDRO P. S. DE MOURA<sup>3</sup>, CELSO GREBOGI<sup>3</sup>  
and RAINER KLAGES<sup>4</sup>

<sup>1</sup> *Max Planck Institute for Mathematics in the Sciences - Inselstr. 22, 04103 Leipzig, Germany*

<sup>2</sup> *Institute for Theoretical Physics, NSC KIPT - ul. Akademicheskaya 1, UA-61108 Kharkov, Ukraine*

<sup>3</sup> *Institute for Complex Systems and Mathematical Biology, University of Aberdeen - Aberdeen AB24 3UE, UK*

<sup>4</sup> *Queen Mary University of London, School of Mathematical Sciences - Mile End Road, London E1 4NS, UK*

received 14 August 2014; accepted in final form 23 October 2014

published online 12 November 2014

PACS 05.40.-a – Fluctuation phenomena, random processes, noise, and Brownian motion

PACS 05.60.-k – Transport processes

PACS 05.45.-a – Nonlinear dynamics and chaos

**Abstract** – Dynamical systems having many coexisting attractors present interesting properties from both fundamental theoretical and modelling points of view. When such dynamics is under bounded random perturbations, the basins of attraction are no longer invariant and there is the possibility of transport among them. Here we introduce a basic theoretical setting which enables us to study this hopping process from the perspective of anomalous transport using the concept of a random dynamical system with holes. We apply it to a simple model by investigating the role of hyperbolicity for the transport among basins. We show numerically that our system exhibits non-Gaussian position distributions, power-law escape times, and subdiffusion. Our simulation results are reproduced consistently from stochastic continuous time random walk theory.

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**Introduction.** – Understanding the dynamics of systems exhibiting coexisting attractors is fundamental for modelling processes having many possible asymptotic states. Although not restricted to, this multi-stable dynamics is particularly important in systems experiencing very weak dissipation [1,2]. In contrast to strongly dissipative ones, these are typically not dominated by one or few attractors. There are many areas from which we could pick up such examples. For instance, if one considers finite-size particle in advection dynamics, the low dissipative interaction between the advected particles and the fluid can be characterised by the presence of multiple attractors trapping advected particles even in open flows [3]. Another example is found in the dynamics of space dust and its role in the formation of planetesimals [4], among others. Even when most of the attracting sets are periodic, a chaotic component may be present in the form of a fractal boundary separating the basins of attraction [1,2]. If the dynamics is fully deterministic, the attractors are invariant structures. Hence, once a particle or trajectory is trapped in one of the basins of attraction, it remains there indefinitely. However, since most natural processes

are not realistically isolated from external random perturbations, it is natural to study their impact.

The presence of random noise dramatically changes the dynamics. In contrast to deterministic systems, for randomly perturbed dynamics the invariance of attractors may not be true. If the considered perturbation is set to be unbounded Gaussian noise, the whole phase space may be the support of a unique invariant measure [5]. When bounded perturbations are used, on the other hand, there might be many coexisting invariant measures. In particular, depending on the amplitude of the noise orbits can escape from the attracting domains [6] creating the possibility of transport across their basins. This sort of hopping process has been reported before [1,7–9], yet there is a lack of understanding of its statistical properties, in particular from the anomalous transport perspective [10,11].

In this paper we analyse the statistical properties of systems lying on the border between dissipative and conservative dynamics which evolve under random perturbations and their similarities to Hamiltonian dynamics. We start by introducing what we call *effective attractors*. Below a certain level of dissipation the dynamics naturally gives rise to these attracting sets, which defined under finite resolution are indistinguishable from topological attractors.

<sup>(a)</sup>E-mail: christian.rodriques@mis.mpg.de

We then extend the description of escape in terms of a closed systems with a hole [6] to the case of coexisting attractors and establish the conditions allowing a hopping dynamics among them. We show that it is possible to characterise the hopping process by a distribution of first recurrence times to an appropriately chosen non-zero measure set. We find that such a recurrence (or escape time) distribution approaches the one expected for non-hyperbolic dynamics as the dissipation is decreased and the dynamics approaches the non-hyperbolic limit. This effect is similar to stickiness in Hamiltonian non-hyperbolic dynamics [10,12]. We verify our arguments by computer simulations for the single rotor or dissipative standard map [1,13]. The results match well to analytical predictions from stochastic continuous time random walk theory [11,14,15]. Our discussion is based on general arguments and not restricted to this particular model.

**Dynamics and effective attractors.** – We are interested in adding bounded random noise to our deterministic dynamics. More precisely, suppose our deterministic dynamics is given by the iteration of a smooth function  $f : M \rightarrow M$  with differentiable inverse in our phase space  $M$ , for example<sup>1</sup>,  $M \subset \mathbb{R}^n$ . An orbit  $(x_n)_{n \geq 1}$  is the sequence generated by the dynamical system  $x_{n+1} = f(x_n)$  from a given initial condition  $x_0 \in M$ . We will be concerned with subsets of  $M$  to which most orbits in their neighbourhood converge for sufficiently long but *finite time*, what we shall call *effective attractors* or attracting sets. In other words, those are  $f$ -invariant subsets of  $M$  contained in basins of attraction, which are open sets of initial conditions with positive Lebesgue (volume) measure converging to the attracting sets. Note that our requirements on convergence demand this to happen within finite time, which is very important for numerical/experimental investigations. In these cases, contrary to a rigorous mathematical framework and due to physical limitations one cannot ask for time going to infinity or infinitely small length intervals. By making such *finite-size* assumptions on the dynamics one may include among the detected invariant sets homoclinic tangencies and Newhouse attractors which support some invariant measure at least within finite scales, thus being indistinguishable under finite resolution from more general “real” attractors [2,16].

We will focus on the case where there is only a finite number of coexisting attractors. This is not a restriction, because for compact spaces the finiteness of the number of effective attractors follows. Indeed, it is only possible to fit a finite number of non-overlapping balls of radii bounded from below in a compact space. Furthermore, for the case of randomly perturbed dynamics we shall deal with it can be proven that the system has only a finite number of invariant *physical* measures [17]. Therefore, we represent the set of coexisting effective attractors by  $\{\Lambda_i\}_{i=1}^N$ , a family of *pairwise disjoint* compact sets, *i.e.*  $\Lambda_i \cap \Lambda_j = \emptyset$ , for  $i \neq j$ . Another important fact is that we also assume that

the union of the basins of attraction covers every point of the whole phase space, up to a zero Lebesgue measure set. So we write

$$m \left( M \setminus \bigcup_{i=1}^N W^s(\Lambda_i) \right) = 0, \quad (1)$$

where  $m$  denotes Lebesgue measure and  $W^s(\Lambda_i)$  the basin of attraction of  $\Lambda_i$ . This plays a very important role in the definition of the hopping process between different attractors, because the trajectories are always expected to converge to some attractor. The boundary between basins of attraction is a zero Lebesgue measure component, the so-called *basin boundary*, which we denote by  $\partial$ . The basin boundary plays a fundamental role in the hopping process, as we shall see in what follows.

**Random perturbations.** – We now perturb the dynamics exhibiting multiple attractors by assuming *physical random perturbation*; see [17] and appendix D of [18] for a formal definition. Roughly speaking we add bounded random uniformly distributed noise to the dynamics. That is, given the deterministic system  $f$  defined as before, we consider the dynamical system

$$F(x_j) = f(x_j) + \varepsilon_j, \quad (2)$$

with  $\|\varepsilon_j\| < \xi$ , where  $\varepsilon_j$  is the random vector of noise added to the deterministic dynamics at the iteration  $j$ , and  $\xi$  is its maximum amplitude. We require the noise to asymptotically cover uniformly a ball around the unperturbed dynamics, representing the idea that the perturbation has no preferential direction and amplitude. The orbit thus jumps from  $x$  to  $f(x)$  but misses the point at random with the conditional probability of finding the perturbed orbit in an  $\xi$ -neighbourhood of  $f(x)$  given  $x$ , see [19] for a comprehensive treatment of this topic.

**Escape.** – If the amplitude of the perturbations is small enough, an orbit in the domain of attraction approaches the attracting set, wanders around without escaping and is expected to be trapped there forever. Although the trajectory may seem very intricate, it is actually well described from a statistical perspective. In these cases, one has a unique invariant ergodic probability distribution representing a given attracting set [17]. If the system is *stochastically* stable, such distributions for the randomly perturbed system approach those of the deterministic one as the amplitude of the perturbations decreases to zero. The dynamics inside the basin can be described as that of a closed system if the amplitude of the perturbations is small enough [6]. When the amplitude of the noise increases beyond a threshold  $\xi_0$  the attracting sets lose their stability. This effect can be seen as the introduction of a hole  $I_\partial = I_\partial(\xi)$  in the basin by which the orbits can escape from the domain of attraction; see [6] and further references therein for the general setting. Under some assumptions it is possible to estimate

<sup>1</sup>More generally,  $M$  is a Riemannian manifold.

the size of such a hole, or its measure  $\mu(I_\partial) > 0$  [6]. For one-dimensional systems rigorous results in this direction have been obtained with a different approach [9].

**Hopping process.** – Now we are ready to translate the problem of noise induced escape from *pseudo attractors* into that of a closed system with a hole  $I_\partial$ , or a recurrence problem. We call *pseudo attractors* the sets where the orbits remain trapped for some amount of time before escaping due to noise. Rigorously speaking they are not attractors or attracting sets, since the invariance condition is not fulfilled. In our context, a pseudo attractor  $A$  is a *quasi-invariant set* when the amplitude of the random perturbations is increased beyond  $\xi_0$ . With the assumption above we can describe our dynamics and the escape from a single attractor as  $x_{j+1} = F(x_j)$  if  $x_j \in A$  or *escape* if  $x_j \in I_\partial$ . We do not define the dynamics in  $I_\partial$  as it is irrelevant to our discussion, hence when the orbit falls into  $I_\partial$  we stop considering it. However, we allow the trajectory to come back from the hole to  $A$ . If so, we restart the process of counting the time in  $A$  by neglecting the number of iterations that it had spent in  $I_\partial$ .

Similar arguments apply to systems with many coexisting pseudo attractors  $A_i$  for which eq. (1) holds. In such dynamics, when a trajectory falls into the  $i$ -th hole  $I_{\partial_i}$  there is the possibility of swapping basins. Using a Markov assumption we argue this to be equivalent to restarting the process. Although for the  $i$ th hole there is a distinct measure  $\mu_i(I_{\partial_i}) > 0$ , according to our assumption we treat all holes qualitatively in the same way. Ignoring the dependence on  $i$  we simplify the recurrence in probability space to the  $i$ th interval by dropping the index  $i$ . We are thus characterising the dynamics in terms of a representative hole  $I_\partial$  with average measure  $\mu(I_\partial)$ . Correspondingly we reduce the sojourn time distribution of the hopping process to the statistics of the time intervals that a random orbit takes to access the representative hole  $I_\partial$ . Furthermore, we assume the general *basin property* to hold, which tells us that up to a set of zero Lebesgue measure the time averages of orbits in the basins of attraction converge to the space average with respect to the invariant measures supported on the attractors; see Chapt. 1.6 in [18].

**Pseudo stickiness.** – Let us now look further at the microscopic dynamics in order to understand the overall statistical behaviour of the noise induced hopping process between different attractors. In particular we shall explore its analogy with non-hyperbolic Hamiltonian dynamics where *stickiness* plays a fundamental role for explaining the statistical dynamics.

To set the scene let us forget about the noise for the moment. Recall that Hamiltonian non-hyperbolic dynamics is characterised by elliptic orbits, whose eigenvalues are purely imaginary. These orbits are surrounded by complex structures formed by marginally stable periodic orbits, known as Kolmogorov-Arnold-Moser (KAM) invariant tori or islands, as well as regions of chaotic motion. Large islands are surrounded by smaller ones which,

on the other hand, are surrounded by even smaller ones, repeating this pattern on smaller scales *ad infinitum*. Trajectories starting in the chaotic region exhibit *intermittent dynamics*: they spend long sporadic periods of time performing almost regular motion near the borders of the islands before escaping to the chaotic sea again. Even small islands can have a great impact on the dynamics of an orbit. Given the hierarchical structure of the phase space, when an orbit eventually escapes from the neighbourhood of an island it may spend some time wandering in the chaotic sea before it gets *trapped* once more by the same or another island. This effect, generally known as *stickiness* [12], slows down the dynamics. Among its statistical signatures one typically observes power-law decay of correlations and anomalous diffusion [10].

*Uniformly hyperbolic* dynamics, on the other hand, is characterised by exponential-like laws. Roughly speaking a system is called hyperbolic if at each point on the attracting set distances are contracted or expanded with exponential rate. If the rate of convergence does not depend on the point, the system is called uniformly hyperbolic [18]. In what follows we argue that, from a statistical point of view, in our case *the presence of random perturbations destroys uniformly hyperbolic behaviour*. That is, the perturbations destroy uniform contraction and expansion rates, therefore exponential statistical signatures are lost. Furthermore, when the noise amplitude is set above a threshold, the orbits can escape from the attracting sets as explained in the previous section “Escape”. The general statistical effect is similar to that observed in non-hyperbolic Hamiltonian systems. Namely, the pseudo attractors behave in a manner similar to the KAM islands, where the orbits perform an almost regular motion for a limited time interval. The presence of noise furthermore washes out fine-scale structures of the phase space. Thus, the trapping regions of small attractors have less but non-negligible importance, since the orbits might stay inside them only for a short time by performing almost regular motion before escaping again. Once an orbit escapes from a pseudo attractor, it undergoes an erratic motion until it falls again into the same or another trapping region. Although some of the trapping regions may be very small, yet they have great influence on the statistical characterisation of the dynamics because, just like small KAM islands in the case of non-hyperbolic dynamics, every pseudo attractor has a stickiness-like effect. An important difference nevertheless is that for the dissipative case, the attractiveness to a nearly invariant sets determines the type of diffusion. The mean square displacement is thus expected to show a slower diffusive dynamics compared to Hamiltonian systems.

**Sojourn time distribution and hyperbolicity.** –

In the previous sections we focused on the connection between a hopping process and escape in a dynamical system with holes. As a consequence, the sojourn time distribution for the hopping process given by the distribution of

escape times  $P(t)$  for a system with holes depends on the dynamics in the pseudo attractors (*i.e.* the sets  $A_i$ ) governed by their hyperbolic properties. We consider two “extreme types” of dynamics: on the one side, the escape of orbits from sets in uniformly hyperbolic dynamical systems has been shown to follow an exponential time distribution. On the other side, escape in Hamiltonian systems with mixed phase space yields power-law tails [12,20–22].

Now suppose that in a given dynamical system we could somehow control “how hyperbolic” it is. We might then switch the escape time distribution between  $P(t) \approx ae^{-\alpha t}$  and  $P(t) \approx bt^{-\beta}$ , where the parameters  $a$  and  $b$  depend on the hyperbolicity of the dynamics. They are determined by the dynamics in the pseudo attractors, or more generally, in the set with a hole from where the trajectories escape. For uniformly hyperbolic systems the parameter  $a$  is large and the dynamics in the pseudo attractor has hyperbolic characteristics. Therefore, we have a hyperbolic recurrence time distribution to  $I_\partial$ , and the asymptotic decay of the corresponding escape times is exponential. On the other hand, when the non-hyperbolic component of the dynamics is increased, the parameter  $b$  gains importance and the diffusion of the random orbit in the support of the conditionally invariant measure<sup>2</sup> experiences a stickiness effect, resulting in a slower distribution of recurrence times to  $I_\partial$  with a power-law tail. Such an increase of non-hyperbolic characteristics under parameter change may be the result of homoclinic tangencies with highly non-uniformly hyperbolic properties [16,18]. Since we deal with dynamics under finite resolution, we cannot distinguish them from the other attractors. Note that this behaviour should be independent of the noise amplitude within some range of it, because its amplitude will control the number of pseudo attractors, but the type of escape should be controlled by the hyperbolicity of the system. In the next section we present numerical evidence supporting our arguments, showing that for systems close to the non-hyperbolic regime the escape time distribution indeed has the power-law signature of non-hyperbolicity rather than being exponential as expected for uniformly hyperbolic dynamics.

**Numerical results.** – We illustrate our results by simulations of the perturbed system defined by  $F(x_j, y_j) = f(x_j, y_j) + (\varepsilon_{x,j}, \varepsilon_{y,j})$  with uniformly distributed *i.i.d.* random noise. For  $f$  we choose the single rotor map [13]

$$f \begin{pmatrix} x_j \\ y_j \end{pmatrix} = \begin{pmatrix} x_j + y_j \pmod{2\pi} \\ (1 - \nu)y_j + f_0 \sin(x_j + y_j) \end{pmatrix}, \quad (3)$$

with  $x \in [0, 2\pi]$ ,  $y \in \mathbb{R}$  and damping parameter  $\nu \in [0, 1]$ . When  $\nu \neq 0$  the dynamics is dissipative. In the strongly dissipative limit  $\nu \rightarrow 1$  this model shows uniformly hyperbolic statistical properties, at least from the perspective

<sup>2</sup>The conditional measure is defined such that, for each iteration, when the set  $A$  loses a fraction of its orbits to the hole, we renormalise its measure by what remains in  $A$ ; see [6] for details.

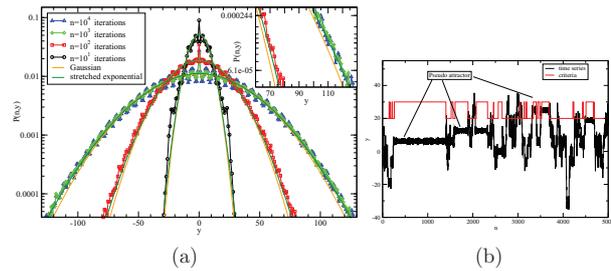


Fig. 1: (Colour on-line) (a) Probability density function  $P(n, y)$  at position  $y$  for different iteration numbers  $n$ . An ensemble of  $10^6$  random initial conditions uniformly distributed around  $x = y = 0$  was iterated by the map eq. (3) randomly perturbed by noise of level  $\xi = 0.06$  and dissipation  $\nu = 0.002$ . The lower (orange) lines display fits with Gaussian distributions for the three smaller  $n$ , the upper (dark green) lines are stretched exponential fits with eq. (4). The inset shows a blowup of two tails. (b) The black graph depicts a representative time series of the noisy system for  $\nu = 0.02$  and  $\xi = 0.2$ . The corresponding result by our eigenvalue criterion to identify pseudo attractors (see text) is given by the red line. The plateaus at  $y = 30$  reveal pseudo attractors, which coincide with the visual identification of localisation in the time series.

of effective attractors [1]. Conversely, when  $\nu \rightarrow 0$  the dynamics approaches the non-hyperbolic Hamiltonian limit, and under finite-resolution dynamics there is an increase of the number of periodic attractors [1,2]. For  $\nu = 0$  we recover the area preserving standard map with Hamiltonian dynamics [23]. Therefore, we can think of  $\nu$  as a control parameter measuring how far the dynamics is away from the non-hyperbolic regime. We use  $f_0 = 4.0$ , which results in multiple attractors when  $\nu \neq 0$  [1]. At this parameter value and  $\nu = 0$  the standard map displays superdiffusion, due to the existence of accelerator modes [24].

If we evolve our system under the presence of random noise beyond a certain amplitude  $\xi \geq \xi_0$  the attracting sets lose their stability, as discussed in the section “Escape”. Note that each attractor may have a different value of minimum noise amplitude such that escape takes place, which is proportional to the size of their basins of attraction. We choose as a global  $\xi_0$  the minimum value for the escape from the largest trapping region. For  $\xi \geq \xi_0$  escape from the attracting sets consequently gives rise to diffusion of trajectories through the phase space.

Figure 1(a) shows the time dependence of the  $y$ -position probability density function of such a process. It confirms our hypothesis that diffusion of trajectories induced by random perturbations indeed takes place. While at first view the included fits to Gaussian distributions seem to match well to the simulation data, the inset shows deviations in the tails especially for long times. This deviation will be explained later on by matching the data with a stochastic theory. Note also the existence of a periodic fine structure, which reflects the spatial distribution of the attracting sets along the  $y$ -axis [1]. Analogous results have been obtained for simulations under different levels of

random noise, for different dissipation parameters  $\nu$ , and also for different values of  $f_0$ .

For general systems a rigorous investigation of the sojourn time distribution and the identification of pseudo attractors can be a very difficult task [9,17]. Even from the numerical point of view the fact that, *a priori*, neither the physical nor the conditionally invariant measures are known can represent an obstacle to the identification of pseudo attractors. A way to detect whether an orbit is trapped in the trapping region of some pseudo attractor for a period of time is given in terms of finite-time Lyapunov exponents. Equivalently, one can calculate the eigenvalues of the Jacobian matrix of  $F$  along the orbit. As a consequence of meta-stability of the pseudo attractors, while an orbit remains trapped the maximum eigenvalue of the Jacobian has, on average, magnitude less than one; see Theorem V1.1 in [25] for a rigorous discussion on characteristic exponents in the case of random transformations. Figure 1(b) illustrates our criterion for the random dynamical system eq. (3) where we have, without loss of generality, plotted  $y = 30$  when a pseudo attractor is identified and  $y = 20$  otherwise. Also without loss of generality we only consider trajectories that remain trapped for more than 20 iterations.

Once a proper identification of the different dynamical regimes, *i.e.* trapped or wandering, is obtained, we are ready to statistically analyse these different behaviours. We start by computing the probability distributions for the times an orbit stays trapped for  $n < t$  iterations in a pseudo attractor. For a range of larger values of  $\nu$  in our simulations we observe a predominantly exponential escape, as was to be expected [12,20–22]. However, when the damping is decreased below  $\nu = 0.02$  the probability distribution is roughly described by a power law, similar to the case of non-hyperbolic Hamiltonian dynamics [21]. In fig. 2(a) we show the probability distributions of escape times from pseudo attractors, or equivalently, the first recurrence time distributions to  $I_\partial$ , for fixed small dissipation  $\nu$  but different noise amplitudes  $\xi$ . Approximately up to times  $t < 300$  the escape time distributions match reasonably well to power laws with exponents around  $\beta = 1.95$  as shown in the figure. This will be justified later by matching all data consistently with a theoretical prediction. The value is in agreement with the range of exponents  $1.5 \leq \beta \leq 3$  obtained analytically for trapping regimes in bounded Hamiltonian systems [26].

Although the precise value of the noise escape threshold  $\xi_0$  depends on the parameters  $f_0$  and  $\nu$ , for amplitudes  $\xi \geq \xi_0$  the existence of a power-law decay is independent of the amplitude of the noise. This is not shown here but observed in further simulations. When we decrease  $\xi$  the orbit typically takes longer to escape, consequently the probability distributions are stretched to longer times. In fig. 2(a) we observe a cross-over to exponential laws which changes with  $\xi$ , as is highlighted by the inset. The most important result of this analysis is that when the dynamics is near the non-hyperbolic Hamiltonian limit,

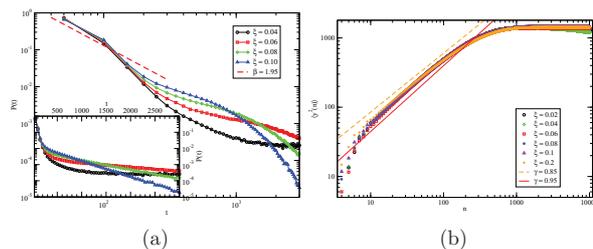


Fig. 2: (Colour on-line) (a) Double-logarithmic plot of the probability distribution  $P(t)$  of escape times  $t$  for an orbit to stay trapped in a pseudo attractor for  $n < t$ . The map eq. (3) was iterated  $10^9$  times for dissipation  $\nu = 0.002$  and different values of the noise amplitude  $\xi$ . The dashed line represents a power-law decay with exponent  $\beta = 1.95$ . The inset shows the corresponding semi-logarithmic plot. (b) Mean square displacement  $\langle y^2(n) \rangle$  for the coordinate  $y$  as a function of time  $n$ . An ensemble of  $10^6$  initial conditions was iterated by the map eq. (3) for different amplitudes  $\xi$  of random noise and fixed small dissipation  $\nu = 0.002$ . The lower bold line corresponds to a power law with exponent  $\gamma = 0.95$ , the upper dashed line to an exponent  $\gamma = 0.85$ .

*i.e.* for small dissipation parameters  $\nu$ , the behaviour of diffusive trajectories indeed has, from the statistical point of view, non-hyperbolic characteristics. This is what we shall address next.

In order to understand the type of diffusion process we are dealing with, we computed the mean square displacement  $\langle y^2(n) \rangle$  for the coordinate  $y$ , the relevant one for diffusion, as a function of time  $n$ . The two lines shown in fig. 2(b) represent power laws  $\langle y^2(n) \rangle \sim n^\gamma$  with exponents  $\gamma < 1$ . They reveal power-law behaviour for the data up to approximately  $t < 300$  by providing upper and lower bounds for the exponents. For the corresponding subdiffusive hopping process among the different basins the power laws persist independently of  $\xi$  but with a slightly varying exponent. Changing other parameters such as  $\nu$  typically generates the same behaviour. This finding is in agreement with our analogy to non-hyperbolic Hamiltonian dynamics generating stickiness to pseudo attractors as discussed in the section “Pseudo Stickiness”. Note that for  $t > 1000$  all power-law exponents of  $\langle y^2(n) \rangle$  are close to zero. This is due to the fact that the fastest particles have reached the region in phase space where the pseudo attractors of the map cease to exist [1] meaning they cannot move any further, and trivial localization sets in.

In the area preserving standard map superdiffusion has successfully been modeled by stochastic *Continuous Time Random Walk (CTRW) theory* [27]. As our randomly perturbed dissipative model displays subdiffusion, here we test the subdiffusive CTRW version put forward in refs. [11,14,15] to explain our simulation results. This theory predicts that if the mean square displacement exhibits a power law with exponent  $\langle y^2(n) \rangle \sim n^\gamma$ , the respective escape (or waiting) time distribution must be  $P(t) \sim t^{-(\gamma+1)}$  on corresponding time scales. It furthermore predicts that the position distribution function of the subdiffusive

process must approximately be of the stretched exponential form<sup>3</sup>:

$$P(n, y) \sim \exp\left(-c(n)y^{2/(2-\gamma)}\right). \quad (4)$$

The lower straight line in fig. 2(b) representing a power law with exponent  $\gamma = 0.95$  matches well to the mean square displacement of  $\xi = 0.06$ . The dashed line in fig. 2(a) yields the corresponding power law with exponent  $\gamma + 1 = 1.95$  as predicted by CTRW theory, which matches well to the numerical result for the escape time distribution for the same  $\xi = 0.06$  in the regime of  $t < 300$  where the system is subdiffusive. Finally, the stretched exponential fits for  $\xi = 0.06$  in fig. 1(a) have all been performed with eq. (4) by using the very same value of  $\gamma$ . Evidently, these fits match much better to the numerical results in the tails than the corresponding Gaussian distributions, at least for long enough times. We thus conclude that the subdiffusive CTRW of refs. [11,14,15] consistently explains our numerical findings, thus confirming theoretically that our randomly perturbed dissipative dynamics generates a subdiffusive process that is well-known in stochastic theory. This is quite surprising, as we did not take the strongly non-uniform distribution of pseudo attractors along the  $y$ -axis into account but just averaged over all of them by performing a kind of mean-field approximation.

**Conclusion.** – We have investigated the hopping process of points generated by randomly perturbed dissipative dynamics. We have set up a theoretical framework that describes escape in terms of a closed system with a hole. Escape occurs when the support of the conditional invariant measure of one pseudo attractor overlaps with the neighbourhood of another basin boundary. In this setting the sojourn time distribution becomes the recurrence time distribution of the orbit wandering to a hole. We then showed by simulations that for the randomly perturbed weakly dissipative single rotor map the distribution of sojourn times is described by a power law up to relevant time scales, in contrast to an exponential distribution for strong dissipation. We found that the hopping process among different basins is subdiffusive for a wide range of perturbation strengths. Using only the subdiffusive power-law exponent as a fit parameter, we showed that stochastic CTRW theory consistently explains all of our simulation data by revealing stretched exponential tails in the position distribution function. We conclude that bounded random perturbations generate a kind of non-hyperbolic stickiness in the diffusion process for the considered dissipative dynamics which leads to non-Gaussian position distributions, power laws in the escape time distributions, and subdiffusion. It would be interesting to investigate whether similar phenomena occur in other diffusive randomly perturbed deterministic dynamical systems.

<sup>3</sup>In detail the asymptotic CTRW results for  $P(n, y)$  look a bit different, cf. eq. (51) in ref. [15]. But we have checked that the stretched exponential dominates the expression for our  $\gamma$  and at least large  $y$ .

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The research leading to these results has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP7/2007-2013)/ERC grant agreement No. 267087. RK thanks the MPIPES Dresden for hospitality.

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