Anomalous Diffusion in Random Dynamical Systems

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Consider a chaotic dynamical system generating diffusionlike Brownian motion. Consider a second, nonchaotic system in which all particles localize. Let a particle experience a random combination of both systems by sampling between them in time. What type of diffusion is exhibited by this random dynamical system? We show that the resulting dynamics can generate anomalous diffusion, where in contrast to Brownian normal diffusion the mean square displacement of an ensemble of particles increases nonlinearly in time. Randomly mixing simple deterministic walks on the line, we find anomalous dynamics characterized by aging, weak ergodicity breaking, breaking of self-averaging, and infinite invariant densities. This result holds for general types of noise and for perturbing nonlinear dynamics in bifurcation scenarios.

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Many diffusion processes in nature and society were found to behave profoundly different from Brownian motion, which describes the random-looking flickering of a tracer particle in a fluid [1–8]. Brownian dynamics provided a long-standing powerful paradigm to understand spreading in terms of normal diffusion, where the mean square displacement (MSD) of an ensemble of particles increases linearly in the long time limit, $\langle x^2 \rangle \sim t^\alpha$ with $\alpha = 1$. Anomalous diffusion is characterized by an exponent $\alpha \neq 1$ [1–4]. Subdiffusion with $\alpha < 1$ is commonly encountered in crowded environments as, e.g., for organelles moving in biological cells and single-file diffusion in nanoporous material [5,6]. Superdiffusion with $\alpha > 1$ is displayed by a variety of other systems, like animals searching for food and light propagating through disordered matter [7,8].

Experimental data exhibiting anomalous diffusion are often modeled successfully by advanced concepts of stochastic theory, most notably subdiffusive continuous time random walks (CTRW), superdiffusive Lévy walks, generalized Langevin equations, or fractional Fokker-Planck equations [1–8]. In these stochastic models the mechanisms generating anomalous diffusion are put in by hand on a coarse-grained level, either via non-Gaussian probability distributions or via power law memory kernels. While this stochastic approach to anomalous diffusion has matured impressively, anomalous diffusion in deterministic dynamical systems is yet poorly understood. In nonlinear deterministic equations of motion there are only a few mechanisms known to generate anomalous diffusion [3]: stickiness of orbits to Kolmogorov-Arnold-Moser tori in Hamiltonian dynamics [1,2,9,10], marginally unstable fixed points in dissipative Pomeau-Manneville-like maps [11–15], and nontrivial topologies exhibited by polygonal billiards [16]. In this Letter we introduce a simple hybrid system at the interface between deterministic and stochastic dynamics. We show that it yields another generic mechanism for anomalous diffusion based on stochastic chaos in random dynamical systems [17,18]. This sheds new light on the microscopic origin of anomalous dynamics. Similar models have been used to understand the convection of particles in flowing fluids [19], including fractal clustering [20] and path coalescence [21], the localization transition in continuum percolation problems [22], intermittency in nonlinear electronic circuits [23], and random attractors in stochastic climate dynamics [24]. Accordingly, we expect fruitful applications of our approach to these problems.

Figure 1 gives our recipe to combine two deterministic dynamical systems D and L randomly in time. Here, D generates normal diffusion while L yields localization of particles. We sample randomly between both systems with probability p of choosing L at discrete time step $t \in \mathbb{N}$, respectively, probability $1 - p$ of choosing D. For $p = 0$, we thus recover the dynamics of D, while for $p = 1$, we obtain the dynamics of L. This implies that there must exist a transition between these two different dynamics under variation of p. Our central question is, for $0 < p < 1$, what type of diffusive dynamics emerges in the resulting random dynamical system R? Here we model deterministic diffusion by chaotic random walks on the line [16,25–27] defined by the equation of motion $x_{t+1} = M_a(x_t)$, where

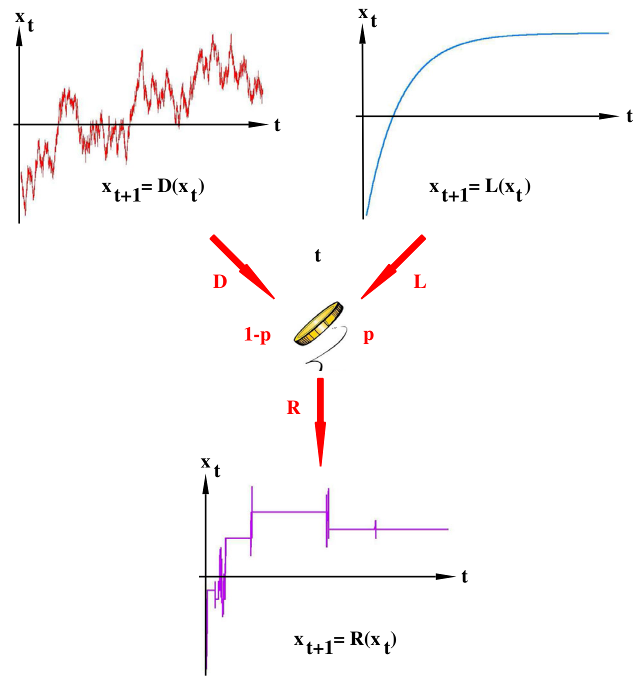

FIG. 1. Diffusion generated by a random dynamical system. The three time series in space-time plots display the position $x_t \in \mathbb{R}$ of a point particle at discrete time $t \in \mathbb{N}$. The trajectory in the upper left is generated by the equation of motion $x_{t+1} = D(x_t)$ using a deterministic dynamical system D that yields normal diffusion. The trajectory in the upper right is from $x_{t+1} = L(x_t)$ for a deterministic dynamical system L where all particles localize in space. The random dynamical system R mixes these two types of dynamics at time t based on flipping a biased coin: The position x_{t+1} of the particle at the next time $t+1$ is determined by choosing with probability $1-p$ the diffusive system D while L is picked with probability p. The trajectory generated by $x_{t+1} = R(x_t)$ displays intermittency, where long regular phases alternate randomly with irregular-looking, chaotic motion.

$$M_a(x) = \begin{cases} ax & 0 \le x < \frac{1}{2} \\ ax + 1 - a & \frac{1}{2} \le x < 1, \end{cases} \qquad a > 0, \quad (1)$$

is a piecewise linear map lifted onto $\mathbb{R}$ by $M_a(x+1) = M_a(x) + 1$, cf. the inset in Fig. 2(a). For $a > 2$, this model exhibits normal diffusion with a Lyapunov exponent calculated to (see Sec. I in Supplemental Material [28]) $\lambda(a) = \ln a$ [16,40–42]. The sample trajectory in the upper left-hand part of Fig. 1 was obtained from $D = M_4(x)$, where the dynamics is chaotic according to $\lambda(4) = \ln 4 > 0$. The trajectory in the upper right-hand part of Fig. 1 corresponds to $L = M_{1/2}(x)$, where the dynamics is nonchaotic due to $\lambda(1/2) = -\ln 2 < 0$. Here all particles contract onto stable fixed points at integer positions $x \in \mathbb{Z}$. For defining the random map R, the slope a becomes an independent and identically distributed, multiplicative random variable: At any time step t we choose for our map $R = M_a(x)$, with probability $p \in [0, 1]$ the slope $a = 1/2$ while with probability $1-p$ we pick $a = 4$. The sequence of random slopes may or may not depend on the individual particle if we consider an ensemble of them [43], as we explore below. Random maps of this type are also called iterated function systems [44,45]. They have been studied by both mathematicians and physicists in view of their measure-theoretic [45–47] and statistical physical properties [19,43,48,49].

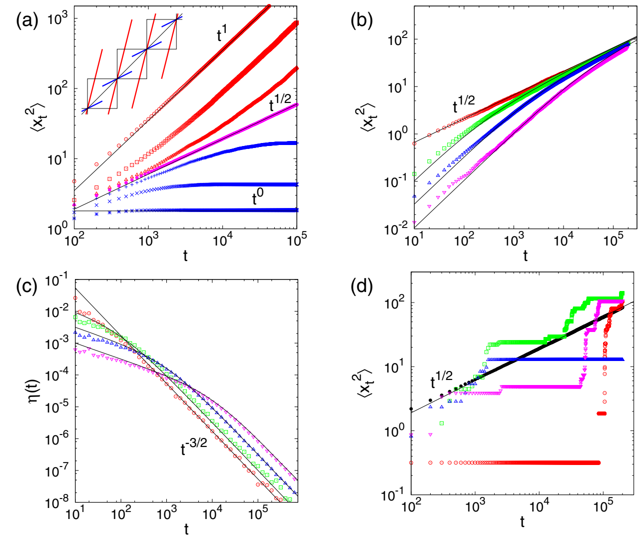

FIG. 2. Mean square displacement (MSD) and waiting time distribution (WTD) for randomized deterministic diffusion. The two deterministic dynamical systems that are randomly sampled in time with probability p by applying the recipe of Fig. 1 are illustrated in the inset of (a). All symbols are generated from computer simulations. (a) MSD $\langle x_t^2 \rangle$ for $p = 0.6$, 0.63, 0.663, 2/3, 0.669, 0.68, 0.7 (top to bottom) for an ensemble of particles, where each particle experiences a different random sequence. There is a characteristic transition between normal diffusion and localization via subdiffusion at a critical $p_c = 2/3$. (b) MSD at p_c by starting the computations after different aging times $t_a = 0$, 10^2, 10^3, 10^4 (top to bottom). The MSD displays aging similar to analytical results from continuous time random walk (CTRW) theory [13] (bold lines). (c) WTD $\eta(t)$ at p_c for particles leaving a unit interval at the same aging times t_a as in (b). The bold lines are again analytical results from CTRW theory [13]. (d) MSD at the critical probability p_c for different types of averaging over the random variable. For the straight black line with matching symbols each particle experiences a different random sequence (called uncommon noise), cf. Fig. 2(a). The other four lines depict MSDs obtained from applying the same sequence of random variables to all particles (called common noise). In these four cases the MSD becomes a random variable breaking self-averaging.

One can show straightforwardly that the Lyapunov exponent $\lambda(p)$ of the random map R is zero at probability $p_c = 2/3$ [28]. Since $\lambda(p) > 0$ for $p < p_c$, the map R should generate normal diffusion in this regime while $p > p_c$ with $\lambda(p) < 0$ should lead to localization for long times. In Fig. 2 we test this conjecture by comparing numerical with analytical results. For our simulations we

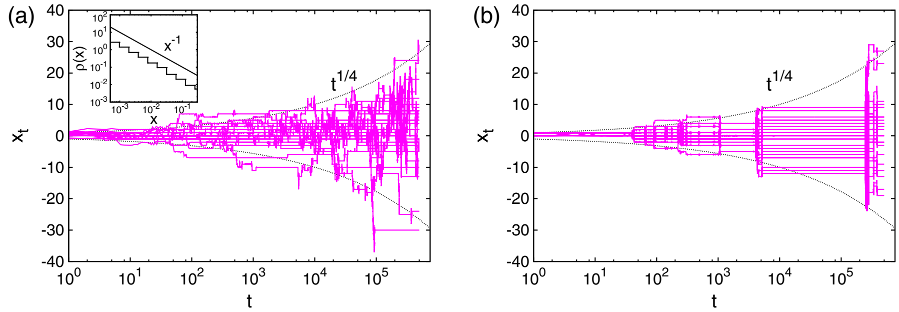

FIG. 3. Subdiffusion for different types of randomness. (a) Sample trajectories at p_c corresponding to 30 different initial conditions with uncommon noise. (b) Same as (a) with common noise. The envelopes (smooth black curves $\sim t^{1/4}$) in (a) and (b) correspond to the subdiffusive spreading for uncommon noise shown in Fig. 2(d). Panel (b) displays jump time synchronization of all particles. The inset in (a) yields in double-logarithmic plot the infinite invariant density within a unit cell for uncommon noise for the map R mod1.

used $\sim 10^5$ iterations of R with $\sim 10^5$ initial points, which were distributed randomly and uniformly in the unit interval [0, 1). Here each particle experienced a different sequence of random slopes. We used an arbitrary precision algorithm with up to $10^{10\,000}$ decimal digits. Figure 2(a) depicts the MSD $\langle x_t^2 \rangle$ under variation of p by confirming the diffusion scenario conjectured above. However, passing through p_c the dynamics displays a subtle transition: Right at p_c we obtain long-time subdiffusion, $\langle x^2(t) \rangle \sim t^{1/2}$, while around p_c this dynamics survives for long transient times. Figures 2(b) and 2(c) reveal that right at p_c R exhibits aging [50,51] in both the MSD and the waiting time distribution (WTD). The latter is the probability distribution $\eta(t)$ of the times t it takes a particle to escape from a unit interval of R. In both Figs. 2(b) and 2(c) there is good agreement with analytical results from CTRW theory for long times, $\langle x_t^2 \rangle \sim (t + t_a)^\alpha - t_a^\alpha$ and $\eta(t) \sim t_a^\alpha / [(t + t_a) t^\alpha]$, where t_a is the aging time [13]. This theory furthermore predicts that for long times a WTD of $\eta(t) \sim t^{-\gamma}$ implies a MSD of $\langle x^2(t) \rangle \sim t^{\gamma - 1}$ [11–15]. For R this relation is fulfilled with $\gamma = 3/2$. An exponent of the WTD of $3/2$ yields a diverging mean waiting time. This as well as the existence of aging imply weak ergodicity breaking of the dynamics [50–52].

However, our map R generates dynamics that goes beyond conventional CTRW theory. This becomes apparent by looking at different types of averaging over the random variables shown in Fig. 2(d): While in Figs. 2(a)–2(c) each particle experienced a different sequence of random slopes, as reproduced by the straight line with matching symbols for the MSD in Fig. 2(d), for all the other MSDs in 2(d) the corresponding random sequences are the same for all particles. Accordingly, we call the former type of randomness uncommon noise, the latter common noise. Crucially, while in Fig. 2(a), based on uncommon noise, the MSD is well defined for all p, Fig. 2(d) shows that for common noise sequences it becomes a random variable at p_c in the long time limit that completely depends on the random sequence chosen. This bears strong similarity to what is called breaking of self-averaging for random walks in quenched disordered environments [53], which also implies weak ergodicity breaking [54].

Figure 3 displays space-time plots of 30 trajectories starting at different initial points for Fig. 3(a) uncommon noise and Fig. 3(b) common noise. While in Fig. 3(a) the different trajectories look rather irregular, yielding a spreading front that matches to the subdiffusion depicted in Fig. 2(d) for uncommon noise, Fig. 3(b) shows "temporal clustering" in the form of jump time synchronization; i.e., all particles eventually jump from unit cell to unit cell at the same time step. This matches to the fact that the MSD does not converge for common noise, as seen in Fig. 2(d). The inset in Fig. 3(a) represents the invariant density of the map R mod 1, i.e., within a unit cell, with uncommon noise [55]. We see that it decays on average like $\rho(x) \sim x^{-1}$. This result and the stepwise structure of $\rho(x)$ are in agreement with analytical calculations [46,47]. At zero Lyapunov exponent, uncommon noise thus leads to a weak spatial clustering [20] and path coalescence [21] of particles at integer positions $x \in \mathbb{Z}$. In contrast, for common noise an invariant density does not exist, and we do not find any spatial clustering.

We now explore the origin of this type of anomalous dynamics in terms of dynamical systems theory. As exemplified by the trajectory shown at the bottom of Fig. 1, around p_c the dynamics of R is intermittent [55], meaning long regular phases alternate randomly with irregular-looking, chaotic motion. A paradigmatic intermittent dynamical system is the Pomeau-Manneville map, $P_{z,b}(x) = x + bx^z,\ b \geq 1,\ x \in [-1/2, 1/2)$. Defining its equation of motion in the same way as for M_a above, it generates subdiffusion characterized by a MSD and a WTD that in suitable scaling limits match to the predictions of CTRW theory with $\gamma = z/(z-1)$ [11,12,15]. As shown above, for the map R, CTRW theory correctly predicts the relation between the long-time MSD and the WTD by using $\gamma = 3/2$. Trying to understand the random map R in terms of $P_{z,b}$ thus suggests to choose $z = 3$. One should now compare the invariant densities $\rho(x)$ of the two maps mod1: For $P_{z,b}$ mod 1, it is known that $\rho(x) \sim x^{1-z}$, $x \gg 0$, which for $z \geq 2$ becomes a non-normalizable, infinite invariant density [56,57]. But for $z = 3$, this yields $\rho(x) \sim x^{-2}$, while for R mod 1, we have $\rho(x) \sim x^{-1}$; see the inset of Fig. 3(a). Hence, the intermittency displayed by R is not of Pomeau-Manneville type but of a fundamentally different microscopic dynamical origin. This might relate to deviations between CTRW theory, which on a coarse-grained level works well for the Pomeau-Manneville map, and our numerical results for R on short timescales in the MSD and the WTD of Fig. 2. It would be interesting to further explore such differences, e.g., by the approach outlined in Ref. [58].

However, there is another type of intermittency in dynamical systems that is profoundly different from Pomeau-Manneville dynamics, called on-off intermittency [59–64]. It was first reported for two-dimensional coupled maps,

$$\begin{aligned} x_{t+1} &= (1-\epsilon)f(x_t) + \epsilon f(y_t), \\ y_{t+1} &= (1-\epsilon)f(y_t) + \epsilon f(x_t), \end{aligned} \qquad (2)$$

where $x_{t+1} = f(x_t)$ is chaotic with positive Lyapunov exponent and $\epsilon \in [0,1]$ [59,60]. When ϵ is large, the possibly chaotic dynamics is trapped on the synchronization manifold $x_t = y_t$. By decreasing ϵ to a critical parameter $\epsilon = \epsilon^*$, trajectories start to escape from this manifold into the full two-dimensional space. This is called a blowout bifurcation and the associated intermittency on-off intermittency [65]. In subsequent works Eqs. (2) were boiled down to more specific two-dimensional maps [19,47,59,61–64,66]. The simplest ones are piecewise linear [47,63,66], such as [47]

$$\begin{aligned} x_{t+1} &= \begin{cases} a x_t & (x_t < 1, 0 \leq y_t \leq p) \\ \frac{1}{a} x_t & (x_t < 1, p < y_t \leq 1) \\ 1 + b(1 - x_t) & (x_t \geq 1), \end{cases} \\ y_{t+1} &= \begin{cases} \frac{y_t}{p} & (0 \leq y_t \leq p) \\ \frac{y_t - p}{1-p} & (p < y_t \leq 1), \end{cases} \end{aligned} \qquad (3)$$

with symmetry $y \to -y$ and parameters $a > 0$, $b \in \mathbb{R}$, $p \in (0,1)$. Because of its skew product form this system can be understood as a one-dimensional map $x_{t+1} = f(x_t)$ with multiplicative randomness generated by $y_{t+1} = g(y_t)$ [19,59,61,62,64,66]. In a next step one might replace the deterministic chaotic dynamics of y_t by stochastic noise. If we now consider the dynamics of x_t in Eqs. (3) on the unit interval only by choosing $a = 2$, taking the map mod1, and choosing dichotomic noise, we obtain a simple piecewise linear map with multiplicative randomness that is qualitatively identical to our model R mod 1 [19,46,47]. For this class of systems it has been shown numerically and analytically that at a critical p_c the invariant density of $x = x_t$ decays like $\rho(x) \sim x^{-1}$ [47,60,62,64,66–68] and that a suitably defined waiting time distribution between chaotic "bursts" obeys $\eta(t) \sim t^{-3/2}$ [47,64,66–68]. In Refs. [67,68] different diffusive models driven by on-off intermittency have been studied, and for two of them [68] subdiffusion with a MSD of $\langle x^2(t) \rangle \sim t^{1/2}$ has been obtained by matching simulation results to CTRW theory. We thus conclude that our model R exhibits anomalous diffusion generated by on-off intermittency. We emphasize, however, that the mechanism underlying our model depicted in Fig. 1 is more general than this particular type of intermittent dynamics.

In order to check for the generality of our results, in the Supplemental Material [28] we first replace the dichotomic noise by physically more realistic continuous noise distributions choosing (1) uniform noise on a bounded interval and (2) a nonuniform unbounded log-normal distribution. Figures 1 and 2 in Secs. 2 and 3 of Ref. [28], respectively, show that our mechanism is very robust under variation of the type of noise. We may thus conjecture that our scenario of subdiffusion generated by random maps holds for any generic type of noise. We also tested whether the strong localization due to contraction onto a stable fixed point can be replaced by a weaker chaotic localization to a subregion in phase space. However, in this case the transition between diffusive and localized dynamics is entirely different without displaying any subdiffusion, cf. Fig. 3 in Sec. 4 of Ref. [28]. As a general principle, one must thus mix expansion with contraction to generate anomalous dynamics. Finally, in Sec. 5 of Ref. [28] we study a simple nonlinear map that exhibits different types of diffusion in different parameter regions of a bifurcation scenario generating chaotic and periodic windows. Randomizing this map according to Fig. 1 yields again subdiffusion with a MSD of $\langle x_t^2 \rangle \sim t^{1/2}$ and a WTD of $\eta(t) \sim t^{-3/2}$, cf. Fig. 4 in Ref. [28]. This demonstrates that the basic mechanism generating anomalous diffusion which we propose is also robust in a nonlinear setting.

In summary, we have shown that anomalous dynamics emerges if we randomly mix chaotic diffusion and nonchaotic localization with a sampling probability yielding a zero Lyapunov exponent of the randomized dynamics. Interestingly, our basic mechanism bears similarity with the famous problem of a protein searching for a target at a DNA strand [69]: Here the protein randomly switches between (normal) diffusion in the bulk of the cell and moving along the DNA. This is called facilitated diffusion, as the random switching between different modes may decrease the average time to find a target [69–71]. We are not aware, however, that for this problem any emergence of anomalous diffusion as an effective dynamics representing the whole diffusion process has been discussed. Along these lines, one might speculate that using our framework for combining normal diffusion with constant velocity scanning [70] could yield a kind of Lévy walk [8], which poses an interesting open problem.

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[1] M. F. Shlesinger, G. M. Zaslavsky, and J. Klafter, Nature (London) **363**, 31 (1993).
[2] J. Klafter, M. F. Shlesinger, and G. Zumofen, Phys. Today **49**, No. 2, 33 (1996).
[3] *Anomalous Transport: Foundations and Applications*, edited by R. Klages, G. Radons, and I. M. Sokolov (Wiley-VCH, Berlin, 2008).
[4] R. Metzler, J.-H. Jeon, A. G. Cherstvy, and E. Barkai, Phys. Chem. Chem. Phys. **16**, 24128 (2014).
[5] F. Höfling and T. Franosch, Rep. Prog. Phys. **76**, 046602 (2013).
[6] Y. Meroz and I. M. Sokolov, Phys. Rep. **573**, 1 (2015).
[7] G. M. Viswanathan *et al.*, *The Physics of Foraging* (Cambridge University Press, Cambridge, England, 2011).
[8] V. Zaburdaev, S. Denisov, and J. Klafter, Rev. Mod. Phys. **87**, 483 (2015).
[9] A. Zacherl, T. Geisel, J. Nierwetberg, and G. Radons Phys. Lett. **114A**, 317 (1986).
[10] G. M. Zaslavsky, Phys. Rep. **371**, 461 (2002).
[11] T. Geisel and S. Thomae, Phys. Rev. Lett. **52**, 1936 (1984).
[12] G. Zumofen and J. Klafter, Phys. Rev. E **47**, 851 (1993).
[13] E. Barkai, Phys. Rev. Lett. **90**, 104101 (2003).
[14] N. Korabel, A. V. Chechkin, R. Klages, I. M. Sokolov, and V. Y. Gonchar, Europhys. Lett. **70**, 63 (2005).
[15] N. Korabel, R. Klages, A. V. Chechkin, I. M. Sokolov, and V. Y. Gonchar, Phys. Rev. E **75**, 036213 (2007).
[16] R. Klages, *Microscopic Chaos, Fractals and Transport in Nonequilibrium Statistical Mechanics* (World Scientific, Singapore, 2007).
[17] D. Faranda, Y. Sato, B. Saint-Michel, C. Wiertel, V. Padilla, B. Dubrulle, and F. Daviaud, Phys. Rev. Lett. **119**, 014502 (2017).
[18] Y. Sato *et al.*, arXiv:1811.03994.
[19] L. Yu, E. Ott, and Q. Chen, Phys. Rev. Lett. **65**, 2935 (1990).
[20] M. Wilkinson, B. Mehlig, K. Gustavsson, and E. Werner, Eur. Phys. J. B **85**, 18 (2012).
[21] M. Wilkinson and B. Mehlig, Phys. Rev. E **68**, 040101(R) (2003).
[22] F. Höfling, T. Franosch, and E. Frey, Phys. Rev. Lett. **96**, 165901 (2006).
[23] P. W. Hammer, N. Platt, S. M. Hammel, J. F. Heagy, and B. D. Lee, Phys. Rev. Lett. **73**, 1095 (1994).
[24] M. Chekroun, E. Simonnet, and M. Ghil, Physica (Amsterdam) **240D**, 1685 (2011).
[25] H. Fujisaka and S. Grossmann, Z. Phys. B **48**, 261 (1982).
[26] T. Geisel and J. Nierwetberg, Phys. Rev. Lett. **48**, 7 (1982).
[27] M. Schell, S. Fraser, and R. Kapral, Phys. Rev. A **26**, 504 (1982).
[28] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.122.174101 for details, which includes Refs. [29–39].
[29] E. Ott, *Chaos in Dynamical Systems* (Cambridge University Press, Cambridge, England, 1993).
[30] C. Robinson, *Dynamical Systems* (CRC Press, London, 1995).
[31] K. T. Alligood, T. S. Sauer, and J. A. Yorke, *Chaos—An Introduction to Dynamical Systems* (Springer, New York, 1997).
[32] J. R. Dorfman, *An Introduction to Chaos in Nonequilibrium Statistical Mechanics* (Cambridge University Press, Cambridge, England, 1999).
[33] R. Klages, http://www.maths.qmul.ac.uk/~klages/teaching/mas424.
[34] R. Klages, in *Dynamical and Complex Systems*, edited by S. Bullet, T. Fearn, and F. Smith, LTCC Advanced Mathematics Series (World Scientific, Singapore, 2017), Vol. 5, pp. 1–40.
[35] P. Bak, T. Bohr, and M. H. Jensen, Phys. Scr. **T9**, 50 (1985).
[36] N. Korabel and R. Klages, Phys. Rev. Lett. **89**, 214102 (2002).
[37] N. Korabel and R. Klages, Physica (Amsterdam) **187D**, 66 (2004).
[38] W. Bauer and G. F. Bertsch, Phys. Rev. Lett. **65**, 2213 (1990).
[39] P. Gaspard, *Chaos, Scattering, and Statistical Mechanics* (Cambridge University Press, Cambridge, England, 1998).
[40] R. Klages and J. R. Dorfman, Phys. Rev. Lett. **74**, 387 (1995).
[41] R. Klages and J. R. Dorfman, Phys. Rev. E **59**, 5361 (1999).
[42] J. Groeneveld and R. Klages, J. Stat. Phys. **109**, 821 (2002).
[43] T. Bódai, E. G. Altmann, and A. Endler, Phys. Rev. E **87**, 042902 (2013).
[44] M. Barnsley, *Fractals Everywhere* (Academic Press Professional, Inc., San Diego, CA, 1988).
[45] N. Abbasi, M. Gharaei, and A. J. Homburg, Nonlinearity **31**, 3880 (2018).
[46] S Pelikan, Trans. Am. Math. Soc. **281**, 813 (1984).
[47] P. Ashwin, P. J. Aston, and M. Nicol, Physica (Amsterdam) **111D**, 81 (1998).
[48] R. Klages, Europhys. Lett. **57**, 796 (2002).
[49] A. Lipowski, I. Bena, M. Droz, and A. L. Ferreira, Physica (Amsterdam) **339A**, 237 (2004).
[50] J.-P. Bouchaud, J. Phys. I (France) **2**, 1705 (1992).
[51] R. Metzler, Int. J. Mod. Phys. Conf. Ser. **36**, 1560007 (2015).
[52] G. Bel and E. Barkai, Phys. Rev. Lett. **94**, 240602 (2005).
[53] J.-P. Bouchaud and A. Georges, Phys. Rep. **195**, 127 (1990).
[54] M. Dentz, A. Russian, and P. Gouze, Phys. Rev. E **93**, 010101(R) (2016).
[55] O. Bénichou, C. Loverdo, M. Moreau, and R. Voituriez, Rev. Mod. Phys. **83**, 81 (2011).
[56] N. Korabel and E. Barkai, Phys. Rev. Lett. **102**, 050601 (2009).
[57] We have computed the infinite invariant density starting from a normalized uniform initial distribution of $N = 10^5$

points by iterating for $n = 10^5$ time steps, which yields the right scaling in x [56], by not showing the cutoff at small x.

[58] T. Albers and G. Radons, Europhys. Lett. **102**, 40006 (2013).

[59] A. S. Pikovsky, Z. Phys. B **55**, 149 (1984).

[60] H. Fujisaka and T. Yamada, Prog. Theor. Phys. **74**, 918 (1985).

[61] H. Fujisaka and T. Yamada, Prog. Theor. Phys. **75**, 1087 (1986).

[62] A. S. Pikovsky and P. Grassberger, J. Phys. A **24**, 4587 (1991).

[63] N. Platt, E. A. Spiegel, and C. Tresser, Phys. Rev. Lett. **70**, 279 (1993).

[64] J. F. Heagy, N. Platt, and S. M. Hammel, Phys. Rev. E **49**, 1140 (1994).

[65] E. Ott and J. C. Sommerer, Phys. Lett. A **188**, 39 (1994).

[66] H. Hata and S. Miyazaki, Phys. Rev. E **55**, 5311 (1997).

[67] T. Harada, H. Hata, and H. Fujisaka, J. Phys. A **32**, 1557 (1999).

[68] S. Miyazaki, T. Harada, and A. Budiyono, Prog. Theor. Phys. **106**, 1051 (2001).

[69] O. G. Berg, R. B. Winter, and P. H. von Hippel, Biochemistry **20**, 6929 (1981).

[70] Y. Meroz, I. Eliazar, and J. Klafter, J. Phys. A **42**, 434012 (2009).

[71] M. Bauer and R. Metzler, Biophys. J. **102**, 2321 (2012).