MTH4100 Calculus I

Lecture notes for Week 9

Thomas' Calculus, Sections 4.6 to 5.2 except 4.7

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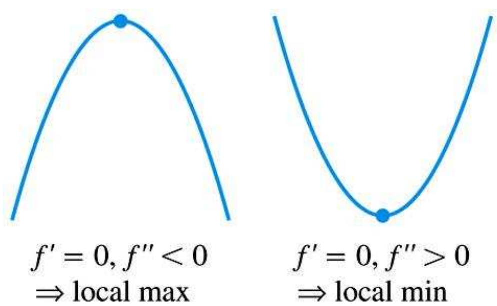
Look at second derivative instead of sign changes at critical points in order to test for local extrema:

> **THEOREM 5 Second Derivative Test for Local Extrema**
> Suppose f'' is continuous on an open interval that contains $x = c$.
> 1. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$.
> 2. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$.
> 3. If $f'(c) = 0$ and $f''(c) = 0$, then the test fails. The function f may have a local maximum, a local minimum, or neither.

proof of 1. and 2.:



proof of 3.:
Consider $y = -x^4$, $y = x^4$ and $y = x^3$ as examples. In this case use the first derivative test to identify local extrema.

> **Strategy for Graphing $y = f(x)$**
> 1. Identify the domain of f and any symmetries the curve may have.
> 2. Find y' and y''.
> 3. Find the critical points of f, and identify the function's behavior at each one.
> 4. Find where the curve is increasing and where it is decreasing.
> 5. Find the points of inflection, if any occur, and determine the concavity of the curve.
> 6. Identify any asymptotes.
> 7. Plot key points, such as the intercepts and the points found in Steps 3–5, and sketch the curve.

example: Sketch the graph of $f(x) = \dfrac{(x+1)^2}{1+x^2}$.

1. The natural **domain** of f is $(-\infty, \infty)$; no **symmetries** about any axis.

2. calculate **derivatives:**

$$f'(x) = [\text{calculation on whiteboard}] = \frac{2(1-x^2)}{(1+x^2)^2}$$

$$f''(x) = [\text{exercise}] = \frac{4x(x^2-3)}{(1+x^2)^3}$$

3. **critical points**: f' exists on $(-\infty, \infty)$ with $f'(\pm 1) = 0$ and $f''(-1) = 1 > 0$, $f''(1) = -1 < 0$ $(-1, 0)$ is a local **minimum** and $(1, 2)$ a local **maximum**.

4. On $(-\infty, -1)$ it is $f'(x) < 0$: curve **decreasing**; on $(-1, 1)$ it is $f'(x) > 0$: curve **increasing**; on $(1, \infty)$ it is $f'(x) < 0$: curve **decreasing**

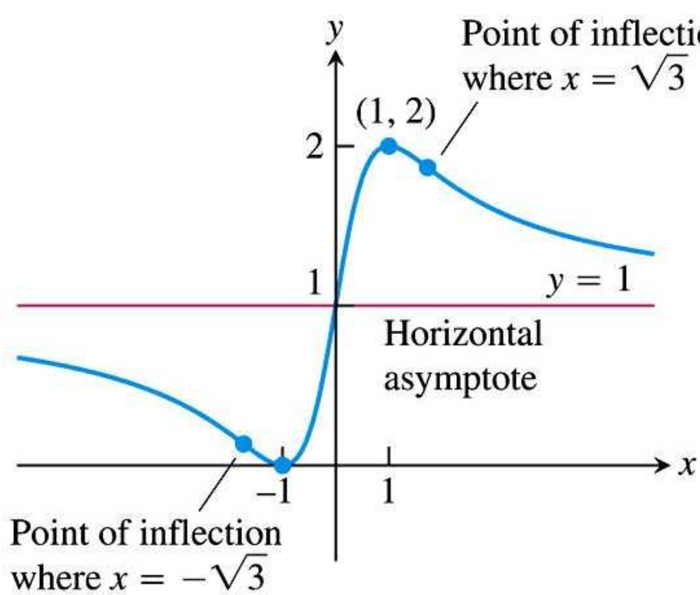
5. $f''(x) = 0$ if $x = \pm\sqrt{3}$ or 0; $f'' < 0$ on $(-\infty, -\sqrt{3})$: **concave down**; $f'' > 0$ on $(-\sqrt{3}, 0)$: **concave up**; $f'' < 0$ on $(0, \sqrt{3})$: **concave down**; $f'' > 0$ on $(\sqrt{3}, \infty)$: **concave up**. Each point is a **point of inflection**.

6. calculate asymptotes:

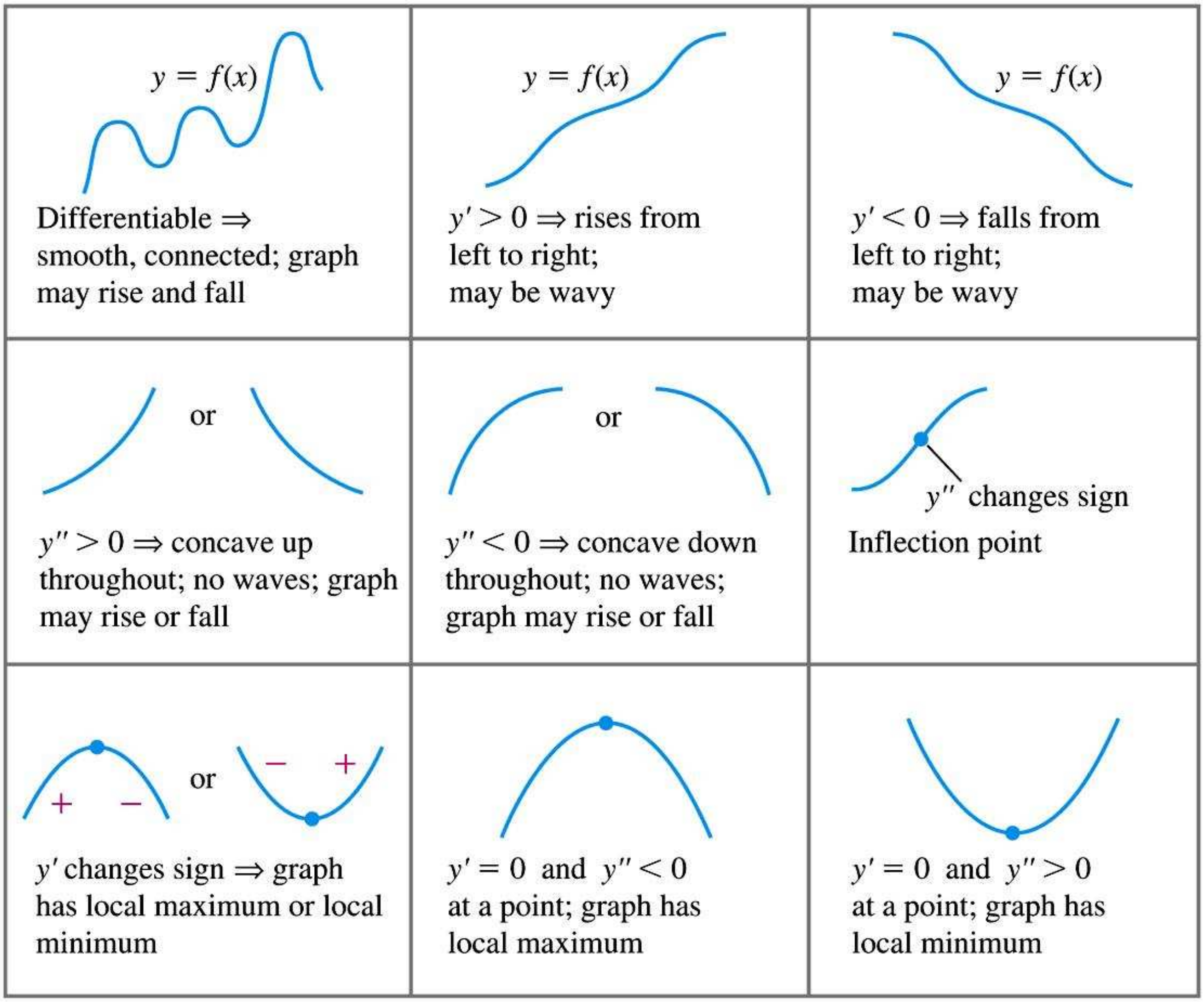
$$f(x) = \frac{(x+1)^2}{1+x^2} = \frac{x^2+2x+1}{1+x^2} = \frac{1+2/x+1/x^2}{1/x^2+1}$$

   $f(x) \to 1^+$ as $x \to \infty$ and $f(x) \to 1^-$ as $x \to -\infty$: $y = 1$ is a **horizontal asymptote**. No **vertical asymptotes**.

7. **sketch the curve:**



Summary: Learning about functions from derivatives



Indeterminate Forms and L'Hôpital's Rule

If $f(a) = g(a) = 0$, $f(a)/g(a)$ is a meaningless *indeterminate form*: $\lim_{x\to a} \dfrac{f(x)}{g(x)}$ cannot be found by substituting $x = a$. Under certain conditions, we can nevertheless calculate it:

Theorem 1 (L'Hôpital's Rule (First Form)) *Suppose that $f(a) = g(a) = 0$, that $f'(a)$ and $g'(a)$ exist and that $g'(a) \neq 0$. Then*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}\ .$$

Proof: Proceed right hand side $\rightarrow$ left hand side:

$$\begin{aligned}
& \frac{f'(a)}{g'(a)} = \\
\textit{(definition of } f', g'\textit{)} &= \frac{\lim_{x\to a} \frac{f(x)-f(a)}{x-a}}{\lim_{x\to a} \frac{g(x)-g(a)}{x-a}} \\
\textit{(limit laws)} &= \lim_{x\to a} \frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}} \\
\textit{(simplify)} &= \lim_{x\to a} \frac{f(x)-f(a)}{g(x)-g(a)} \\
\textit{(hypothesis theorem)} &= \lim_{x\to a} \frac{f(x)}{g(x)}
\end{aligned}$$

q.e.d.

WARNING:

- Always check for "0/0", i.e., $f(a) = g(a) = 0$, before using l'Hôpital!
- Do **not** compute $\left(\frac{f}{g}\right)'(x)$ but $\frac{f'(x)}{g'(x)}$!

examples: (1) $\lim_{x\to 0} \frac{5x - \sin x}{x} = \left.\frac{5 - \cos x}{1}\right|_{x=0} = 4.$

(2) $\lim_{x\to 0} \frac{1+\sin x}{1-x} = \ldots$ *(This does* **not** *fulfill the assumptions of l'Hôpital's rule!)* $\ldots = \frac{1}{1} = 1$ by substitution.

(3) $\lim_{x\to 0} \frac{x - \sin x}{x^3} = \left.\frac{1-\cos x}{3x^2}\right|_{x=0} = \frac{0}{0}$: Doesn't work! But can be handled with

Theorem 2 (L'Hôpital's Rule (Stronger Form)) *Suppose that $f(a) = g(a) = 0$, that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I* **if** *$x \neq a$. Then*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} ,$$

assuming that the limit on the right side exists.

proof: See textbook Section 4.6, via a generalized Mean Value Theorem.

example: Finish up case (3) above,

$$\lim_{x\to 0} \frac{x-\sin x}{x^3} = \lim_{x\to 0} \frac{1-\cos x}{3x^2} = \lim_{x\to 0} \frac{\sin x}{6x} = \lim_{x\to 0} \frac{\cos x}{6} = \frac{1}{6} .$$

summary:

Using L'Hôpital's Rule

To find

$$\lim_{x\to a} \frac{f(x)}{g(x)}$$

by l'Hôpital's Rule, continue to differentiate f and g, so long as we still get the form $0/0$ at $x = a$. But as soon as one or the other of these derivatives is different from zero at $x = a$ we stop differentiating. L'Hôpital's Rule does not apply when either the numerator or denominator has a finite nonzero limit.

remark: L'Hôpital also applies to one-sided limits (see proof of previous theorem).

example:

$$\lim_{x\to 0^{\pm}} \frac{\sin x}{x^2} = \lim_{x\to 0^{\pm}} \frac{\cos x}{2x} = \pm\infty$$

What's about limits involving other indeterminate forms like ∞/∞, $\infty \cdot 0$ or $\infty - \infty$?

(1) ∞/∞: Can be proved that if $f(x), g(x) \to \pm\infty$ as $x \to a$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

So use L'Hôpital same way as before also for "∞/∞".

example:

$$\lim_{x\to\infty} \frac{x - x^2}{x^2 + 7x} = \lim_{x\to\infty} \frac{1 - 2x}{2x + 7} = \lim_{x\to\infty} \frac{-2}{2} = -1$$

(2) $\infty \cdot 0$: Use

$$\lim_{x\to a}(f(x)g(x)) = \lim_{x\to a} \frac{g(x)}{1/f(x)}$$

example:

$$\lim_{x\to\infty} x\sin(1/x) = \lim_{x\to\infty} \frac{\sin(1/x)}{1/x} = \lim_{h\to 0^+} \frac{\sin h}{h} = \lim_{h\to 0^+} \frac{\cos h}{1} = 1$$

(3) $\infty - \infty$: Best demonstrated by an

example:

$$\lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right) = \lim_{x\to 0} \frac{x - \sin x}{x \sin x} = \lim_{x\to 0} \frac{1 - \cos x}{\sin x + x\cos x} = \lim_{x\to 0} \frac{\sin x}{2\cos x - x\sin x} = 0$$

Antiderivatives

Aim: Given $f(x)$ and $f(x) = F'(x)$, find $F(x)$.

DEFINITION Antiderivative

A function F is an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

examples: (1) $f(x) = 2x \Rightarrow F(x) = x^2$

(2) $h(x) = \sin x \Rightarrow H(x) = -\cos x$

But these are not the only solutions:

Corollary 1 (of the Mean Value Theorem) *If $G'(x) = F'(x)$ on (a, b) then $G(x) = F(x) + C$ for all $x \in (a, b)$.*

which implies:

If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is

$$F(x) + C$$

where C is an arbitrary constant.

Some antiderivative formulas are shown in the following table:

TABLE 4.2 Antiderivative formulas

	Function	General antiderivative
1.	x^n	$\dfrac{x^{n+1}}{n+1} + C, \quad n \neq -1, n$ rational
2.	$\sin kx$	$-\dfrac{\cos kx}{k} + C, \quad k$ a constant, $k \neq 0$
3.	$\cos kx$	$\dfrac{\sin kx}{k} + C, \quad k$ a constant, $k \neq 0$
4.	$\sec^2 x$	$\tan x + C$
5.	$\csc^2 x$	$-\cot x + C$
6.	$\sec x \tan x$	$\sec x + C$
7.	$\csc x \cot x$	$-\csc x + C$

examples: (1) $f(x) = x^4 \Rightarrow F(x) = \frac{x^5}{5} + C$

(2) $h(x) = \cos 5x \Rightarrow H(x) = \frac{\sin 5x}{5} + C$

The rules shown in the following table are easily proved by differentiation:

TABLE 4.3 Antiderivative linearity rules

		Function	General antiderivative
1.	*Constant Multiple Rule:*	$kf(x)$	$kF(x) + C$, $\quad k$ a constant
2.	*Negative Rule:*	$-f(x)$	$-F(x) + C$,
3.	*Sum or Difference Rule:*	$f(x) \pm g(x)$	$F(x) \pm G(x) + C$

More advanced techniques will come later.

example: Find the general antiderivative of $h(x) = \dfrac{5}{\sqrt{x}} + \sin 3x$.

- Function is of the form $h(x) = 5f(x) + g(x)$ with $f(x) = x^{-1/2}$ and $g(x) = \sin 3x$.
- $F(x) = 2\sqrt{x} + C_1$, which satisfies $F'(x) = f(x)$.
- $G(x) = -\frac{1}{3}\cos 3x + C_2$, which satisfies $G'(x) = g(x)$.
- Therefore

$$H(x) = 10\sqrt{x} - \frac{1}{3}\cos 3x + C \ , \ C = C_1 + C_2 \ .$$

A special symbol is used to denote the collection of all antiderivatives of f:

DEFINITION Indefinite Integral, Integrand

The set of all antiderivatives of f is the **indefinite integral** of f with respect to x, denoted by

$$\int f(x)\, dx.$$

The symbol $\int$ is an **integral sign**. The function f is the **integrand** of the integral, and x is the **variable of integration**.

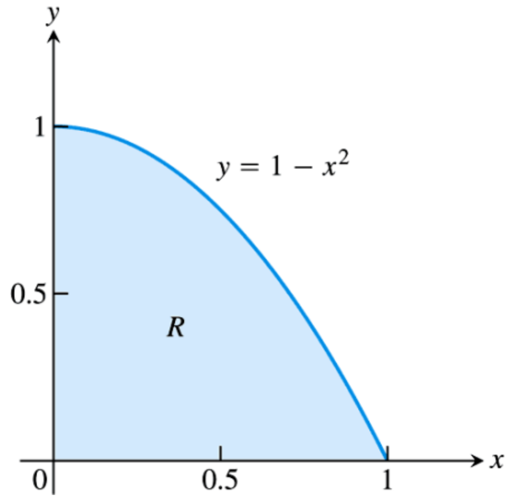
examples:

1. $\int 4x\, dx = 2x^2 + C$
2. $\int \cos x\, dx = \sin x + C$

Integration

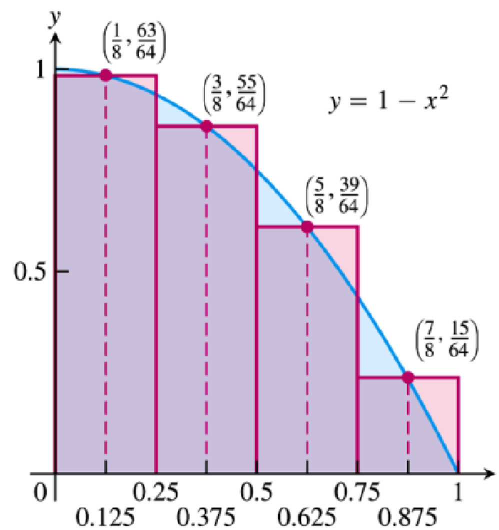
Estimating with finite sums

example: See first animation in MML Multimedia Library Section 5.1.



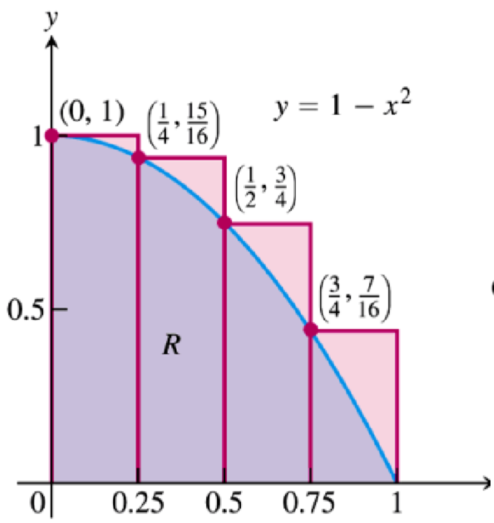
How can we compute the shaded area R?

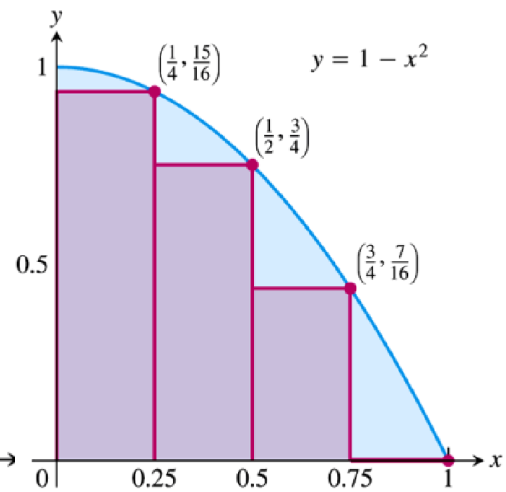
algorithm ("recipe"):



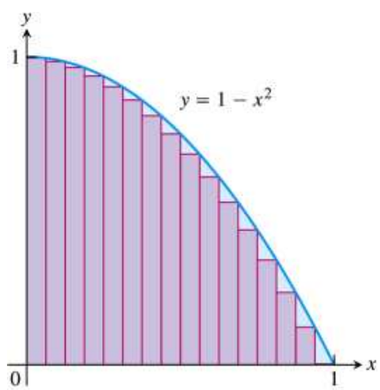
- **Subdivide** the interval $[a, b]$ into n subintervals of equal width $\Delta x = \frac{b-a}{n}$.
- Choose **point** c_k in the $k-th$ subinterval.
- Construct **rectangles:**

  1. *midpoint rule:* Choose c_k in the *middle* of the $k-th$ subinterval.
  2. *upper sum:* Choose c_k such that $f(c_k)$is *maximal*.
  3. *lower sum:* choose c_k such that $f(c_k)$ is *minimal*.

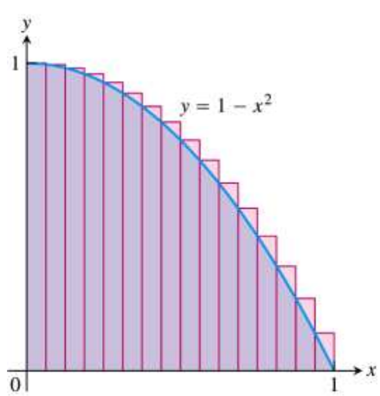




- Form the **sum** $f(c_1)\Delta x + f(c_2)\Delta x + \ldots + f(c_n)\Delta x$.
- **Refine** your approximation by choosing **more rectangles**:



(a)



(b)

TABLE 5.1 Finite approximations for the area of R

Number of subintervals	Lower sum	Midpoint rule	Upper sum
2	.375	.6875	.875
4	.53125	.671875	.78125
16	.634765625	.6669921875	.697265625
50	.6566	.6667	.6766
100	.66165	.666675	.67165
1000	.6661665	.66666675	.6671665

To handle sums with many terms, we need a better notation:

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + \ldots + a_n$$

with

The index k ends at $k = n$.

The summation symbol (Greek letter sigma) $\displaystyle\sum_{k=1}^{n} a_k$ a_k is a formula for the kth term.

The index k starts at $k = 1$.

examples: (1) $f(c_1)\Delta x + f(c_2)\Delta x + \ldots + f(c_n)\Delta x = \sum_{k=1}^{n} f(c_k)\Delta x$

(2) $\sum_{k=1}^{3}(-1)^k k = (-1)^1 \cdot 1 + (-1)^2 \cdot 2 + (-1)^3 \cdot 3 = -1 + 2 - 3 = -2$

(3)

$$\begin{aligned}
1+3+5+7+9 &= \sum_{k=1}^{5}(2k-1) \\
(k = n+1) &= \sum_{n=0}^{4}(2n+1) \\
(n = x+3) &= \sum_{x=-3}^{1}(2x+7) = 25
\end{aligned}$$

Algebra Rules for Finite Sums

1. *Sum Rule*: $\displaystyle\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$
2. *Difference Rule*: $\displaystyle\sum_{k=1}^{n}(a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$
3. *Constant Multiple Rule*: $\displaystyle\sum_{k=1}^{n} ca_k = c \cdot \sum_{k=1}^{n} a_k$ (Any number c)
4. *Constant Value Rule*: $\displaystyle\sum_{k=1}^{n} c = n \cdot c$ (c is any constant value.)

example: $\sum_{k=1}^{n}(5k - k^3) = 5\sum_{k=1}^{n} k - \sum_{k=1}^{n} k^3$ (with rules 1 and 2)

Can we calculate these sums?